Notes on Semi-Simplicity of a Lie Algebra of Isometries

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Abstract
The Lie algebra of isometries of dimension superior to three is semi-simple if and only if the horizontal nullity space of the Nijenhuis tensor of the canonical connection is reduced to zero and the derivative ideal coincides with algebra.

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Introduction
This paper is the complement of our study in [1], the notations and notions are those of [1]. Let M be a paracompact manifold of dimension n (n≥2) and of class $C^\infty$, TM the tangent bundle to M. On an open set U of M, $(x^1, y^1, \ldots, x^n, y^n)$ the natural coordinate system of TU, the energy function E is written:

$$E = \frac{1}{2} g_{ij}(x^i, y^i) y^i y^j$$

Where $g(x^i, y^i)$ are positive functions such that the symmetric matrix $(g(x^i, y^i))$ is invertible. The function E defines a symplectic scalar 2-form $\Omega=dd^c E$ which gives a vertical metric on the tangent bundle to M, with $d^c J=\{i J, d\}$, J being the natural tangent structure to M. The canonical spray S is defined by:

$$i_x E = -dE.$$

$i_x$ being the inner product with respect to S and, the canonical connection is $\Gamma = [J, S]$. If $\chi(M)$ denotes the set of vector fields on M, $\chi(M)$ the complete lift of and $\chi(M)$, $\Gamma = \{X \in \chi(M) \text{such that } L^\Gamma X = 0\}$, $\Gamma$ being the Lie derivative with respect to $\Gamma$.

Case of the Commutative Ideal of $\overline{\mathfrak{g}}$
Example 2 of [1] show that the commutative ideal of $\overline{\mathfrak{g}}$ come from the horizontal nullity space of the curvature of $\Gamma$ and $\frac{\partial}{\partial x^i}$ such that $\frac{\partial E}{\partial x^i} = 0$.

If $\frac{\partial E}{\partial x^i} = 0$ for all $i \in \{1, \ldots, n\}$, this means that the function E is independent of $x^i$ $i \in \{1, \ldots, n\}$. In this case, the curvature $R$ of $\Gamma$ is zero and the horizontal nullity space of the curvature $R$ of $\Gamma$ provides a commutative ideal of $\overline{\mathfrak{g}}$, hence a commutative ideal of $\Gamma$.

We assume that $1 \leq i \leq p(p<n)$, coordinates such that $\frac{\partial E}{\partial x^i} = 0$. By Proposition 19 of [1], $1 \leq i \leq p, \frac{\partial}{\partial x^i} \notin \overline{\mathfrak{g}}$ and $\frac{\partial}{\partial x^{i+p}} \notin \overline{\mathfrak{g}}$. We will study the case where the Lie subalgebra generated by $\{\frac{\partial}{\partial x^{i+p}}\}$ noted $I$ forms a commutative ideal of $\overline{\mathfrak{g}}$. An element $X \in \chi(M)$ is written:

$$X = X^i \frac{\partial}{\partial x^i} + y^i \frac{\partial X^j}{\partial x^i} \frac{\partial}{\partial y^j}, 1 \leq i, j \leq n.$$ 

$$= \sum_{r=p}^n X^r \frac{\partial}{\partial x^r} + y^r \frac{\partial X^s}{\partial x^r} \frac{\partial}{\partial y^s}, 1 \leq r \leq n.$$ 

For $I$ to be an ideal, we must have $\left[ \frac{\partial}{\partial x^i}, X \right] \in I$ for all, $k, p+1 \leq k \leq n$. This is explained by:

$$\frac{\partial X^s}{\partial x^r} = 0, \text{for all such that } 1 \leq r \leq p \text{ and for all } k \text{ such that } p+1 \leq k \leq n, X^s = a^s_1 x^1 + b^s, 1 \leq r, j \leq n, a^s_1, b^s$$

are constants

By asking

$$X^s = \sum_{r=p}^n (a^s_1 x^1 + b^s) \frac{\partial}{\partial x^r} + a^s_1 y^r \frac{\partial}{\partial y^r}, 1 \leq j \leq p$$

and $X = X^s = X_j$, we have $[X^s, X_j] = 0$. To get $X \in \mathfrak{g}$, it is necessary and sufficient that $L^\Gamma X = 0$ or even $L^\Gamma X + L^\Gamma E = 0$. Such a decomposition of $X \in \mathfrak{g}$ clearly indicates that the derivative ideal from $\mathfrak{g}$ cannot coincide with $\overline{\mathfrak{g}}$ if $X_j \neq 0$. It is the same if $X_j$ is an expression of a non-zero part of $I$.

In the case where the indices are not ordered, in this way, we take

$$x_i x_j, \ldots, x_i k, \ldots, x_i$$

such that $\frac{\partial E}{\partial x^i} = 0, 1 \leq j \leq q$. We arrive at the same result. Taking into account the studies made in [1], we then have:

Theorem
The Lie algebra $\overline{\mathfrak{g}}$ of the Killing fields contained in $\overline{\mathfrak{g}}$ of dimension superior to three is semi-simple if and only if the horizontal nullity space of the Nijenhuis tensor of $\Gamma$ is reduced to zero and that the ideal derived from $\overline{\mathfrak{g}}$ coincides with $\overline{\mathfrak{g}}$. 

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Corollary

The Lie algebra $\mathcal{A}_g$ of finite dimension superior or equal to three is semi-simple if and only if the derivative ideal of $\mathcal{A}_g$ coincides with $\mathcal{A}_g$ and any derivation of $\mathcal{A}_g$ is inner.

Proof: From the above theorem, it remains to prove that the horizontal nullity space $\mathcal{hN}_g$ of the curvature $R$ of the connection $\Gamma$ is reduced to zero. Otherwise, by Theorem 1 and Proposition 19 of [1], there would exist vector fields $(e_i)_{1 \leq i \leq n}$ such that $(e_i) \in A_g$ and that $(e_i) \in \mathcal{hN}_g$. In this case, the derivation $D$ of $\mathcal{A}_g$ defined by $D(e_i) = e_i, 1 \leq i \leq p$, is outer. Hence the result.

Some examples

Example 1

We take $M = \mathbb{R}^4$ and the energy function written

$$E = \frac{1}{2} (e^t (y^i)' + (y^j)')^2 + e^r (y^i)' y^r (y^j)')$$

The coordinates $x^i, x^j$ do not appear on the function $E$

The non-zero coefficients of $\Gamma$ are:

$$\Gamma^i_j = \Gamma^i_j = \Gamma^i_0 = \Gamma^i_1 = \frac{e^t (y^i)' - e^t (y^j)'}{2}, \quad \Gamma^i_2 = \frac{y^i}{2} \times \frac{y^j}{2}, \quad \Gamma^i_3 = \frac{y^i}{2} \times \frac{y^j}{2}.$$

The horizontal nullity space of the curvature is reduced to zero.

The Lie algebra $\mathcal{A}_g$ is generated by

$$g_1 = -\frac{e^t (y^i)' + (y^j)'}{4} \partial_x - \frac{y^i}{2} \partial_x + \frac{y^j}{2} \partial_x,$$

$$g_2 = \frac{x^i}{2} \partial_x + \frac{y^i}{2} \partial_x,$$

$$g_3 = x^i \frac{\partial}{\partial x} - \frac{e^t (y^i)'}{4} \partial_x + y^i \frac{\partial}{\partial x},$$

$$g_4 = -\frac{y^i}{2} \partial_x + x^i \frac{\partial}{\partial x} + y^i \frac{\partial}{\partial x},$$

$$g_5 = \frac{\partial}{\partial y},$$

$$g_6 = -\frac{\partial}{\partial y}.$$

Multiplication table of $\mathcal{A}_g$

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The commutative ideal is generated by $g_3$ and $g_4$. The derivative ideal of $\mathcal{A}_g$ does not coincide with $\mathcal{A}_g$.

The Lie algebras $\mathcal{A}_g, \mathcal{A}_g$ and $\mathcal{A}_g$ are identical.

The derivations are all inner.

Remark 1: The two examples above show that the relation in $\mathcal{A}_g, \mathcal{A}_g$ and $\mathcal{A}_g$ do not allow knowing a priori the nature of $\mathcal{A}_g$, although $\mathcal{A}_g$ is an ideal of $\mathcal{A}_g$. They confirm our decomposition of the elements of $\mathcal{A}_g$.
Remark 2: In [2], we have studied some Lie algebras of countable vector fields where the derivative ideal coincides with the algebra and all derivation is inner, yet this algebra contains a commutative ideal. The above corollary gives a view in the case of a finite dimensional Lie algebra.

References


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