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Notes on Semi-Simplicity of a Lie Algebra of Isometries

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Abstract

The Lie algebra of isometries of dimension superior than or equal to three is semi-simple if and only if the horizontal nullity space of the Nijenhuis tensor of the canonical connection is reduced to zero and the derivative ideal coincides with algebra.

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Introduction

This paper is the complement of our study in [1], the notations and notions are those of [1]. Let M be a paracompact manifold of dimension n (n≥2) and of class C^∞, TM the tangent bundle to M. On an open set U of M, $(x^i, y^j)_{i,j\in\{1,\dots,n\}}$, the natural coordinate system of TU, the energy function E is written:

$$E=\frac{1}{2}g_{ij}(x^1,\dots,x^n)y^iy^j$$

Where $g_{ij}(x^1,\dots, x^n)$ are positive functions such that the symmetric matrix $(g_{ij}(x^1,\dots,x^n))$ is invertible. The function E defines a symplectic scalar 2-form $\Omega=dd_JE$ which gives a vertical metric on the tangent bundle to M, with $d_J=[i_J, d]$, J being the natural tangent structure to M. The canonical spray S is defined by:

$$i_S dd_J E=-dE.$$

i_S being the inner product with respect to S and, the canonical connection is $\Gamma=[J,S]$. If χ (M) denotes the set of vector fields on M, $\overline{\chi(M)}$ the complete lift of and $\chi(\mathrm{M})$, $\overline{A_\Gamma}=\{\overline{X}\in\overline{\chi(\mathrm{M})}\ such\ that\ L_{\overline{X}}\Gamma=0\}$ $\overline{A_g}=\{\overline{X}\in\overline{\chi(M)}\ such\ that\ L_{\overline{X}}\Omega=0\}$, $L_{\overline{X}}$ being the Lie derivative with respect to $\overline{X}$.

Case of the Commutative Ideal of $\overline{A_g}$

Example 2 of [1] show that the commutative ideal of $\overline{A_\Gamma}$ come from the horizontal nullity space of the curvature of Γ and $\frac{\partial}{\partial x^3}$ such that $\frac{\partial E}{\partial x^3}=0$.

If $\frac{\partial E}{\partial x^i}=0$ for all $i\in\{1,\dots,n\}$. This means that the function E is independent of $x^i, i\in\{1,\dots,n\}$. In this case, the curvature R of Γ is zero and the horizontal nullity space of the curvature R of $\overline{A_\Gamma}$ provides a commutative ideal of $\overline{A_\Gamma}$, hence an commutative ideal of $\overline{A_g}$.

We assume that $1\le i\le p(p<n)$, coordinates such that $\frac{\partial E}{\partial x^i}\neq 0$. By Proposition 19 of [1], $1\le i\le p, \frac{\partial}{\partial x^i}\notin\overline{A_g}$ and $\frac{\partial}{\partial x^{p+1}},\dots,\frac{\partial}{\partial x^n}\in\overline{A_g}$. We will study the case where the Lie subalgebra generated by $\left\{\frac{\partial}{\partial x^{p+1}},\dots,\frac{\partial}{\partial x^n}\right\}$ noted *I* forms a commutative ideal of $\overline{A_g}$. An element $\overline{X}\in\overline{\chi(M)}$ is written:

$$\overline{X}=X^i\frac{\partial}{\partial x^i}+y^j\frac{\partial X^i}{\partial x^j}\frac{\partial}{\partial y^i}\qquad 1\le i,j\le n.$$

$$=\sum_{l=1}^{p}X^l\frac{\partial}{\partial x^l}+y^j\frac{\partial X^l}{\partial x^j}\frac{\partial}{\partial y^l}+\sum_{r=p+1}^{n}X^r\frac{\partial}{\partial x^r}+y^j\frac{\partial X^r}{\partial x^j}\frac{\partial}{\partial y^r},1\le j\le n$$

For I to be an ideal, we must have $\left[\frac{\partial}{\partial x^k},\overline{X}\right]\in I$ for all, $k, p+1\le k\le n.$

This is explained by:

$$\begin{cases}\frac{\partial X^l}{\partial x^k}=0,\ for\ all\ l\ such\ that\ 1\le l\le p\ and\ for\ all\ k\ such\ that\ p+1\le k\le n,\\ X^r=a^r_jx^j+b^r, p+1\le r, j\le n, a^r_j, b^r\end{cases}$$

are constants

By asking

$$\overline{X_1}=\sum_{l=1}^{p}X^l\frac{\partial}{\partial x^l}+y^j\frac{\partial X^l}{\partial x^j}\frac{\partial}{\partial y^l},1\le j\le p$$

$$\overline{X_2}=\sum_{r=p+1}^{n}(a^r_kx^k+b^r)\frac{\partial}{\partial x^r}+a^r_ky^k\frac{\partial}{\partial y^r}, p+1\le k\le n, a^r_k, b^r$$ are constants

and $\overline{X}=\overline{X_1}+\overline{X_2}$, we have $[\overline{X_1},\overline{X_2}]=0$. To get $\overline{X}\in\overline{A_g}$, it is necessary and sufficient that $L_{\overline{X}}E=0$ or even $L_{\overline{X_1}}E+L_{\overline{X_2}}E=0$. Such a decomposition of $\overline{X}\in\overline{A_g}$ clearly indicates that the derivative ideal from $\overline{A_g}$ cannot coincide with $\overline{A_g}$ if $\overline{X_2}\neq 0$. It is the same if $\overline{X_2}$ is an expression of a non-zero part of I.

In the case where the indices are not ordered, in this way, we take, $x_{i_1}, x_{i_2},\dots,x_{i_q} i_1<i_2<\dots<i_q\ such\ that\ \frac{\partial E}{\partial x_{i_j}}=0, 1\le j\le q$. We arrive at the same result. Taking into account the studies made in [1], we then have:

Theorem

The Lie algebra $\overline{A_g}$ of the Killing fields contained in $\overline{A_\Gamma}$ of dimension superior than or equal to three is semi-simple if and only if the horizontal nullity space of the Nijenhuis tensor of Γ is reduced to zero and that the ideal derived from $\overline{A_g}$ coincides with $\overline{A_g}$.

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Corollary

The Lie algebra $\overline{A_g}$ of finite dimension superior than or equal to three is semi-simple if and only if the derivative ideal of $\overline{A_g}$ coincides with $\overline{A_g}$ and any derivation of A_g is inner.

Proof: From the above theorem, it remains to prove that the horizontal nullity space hN_R of the curvature R of the connection Γ is reduced to zero. Otherwise, by Theorem 1 and Proposition 19 of [1], there would exist vector fields (e_i) $i \in \{1,...,q\}, q \le n$ such that $(e_i) \in \overline{A_g}$ and that $(e_i) \in hN_R$ .In this case the derivation D of $\overline{A_g}$ defined by $D(e_i) = e_i, 1 \le i \le p$, is outer. Hence the result.

Some examples

Example 1

We take $M = \mathbf{R}^4$ and the energy function written

$$E = \frac{1}{2}(e^{x^3}(y^1)^2 + (y^2)^2 + (y^3)^2 + e^{x^2}(y^4)^2)$$

The coordinates x^1, x^4 do not appear on the function *E*

The non-zero coefficients of Γ are:

$$\Gamma_1^1 = \frac{y^3}{2}, \Gamma_3^1 = \frac{y^1}{2}, \Gamma_4^2 = -\frac{e^{x^2}y^4}{2}, \Gamma_1^3 = -\frac{e^{x^3}y^1}{2}, \Gamma_2^4 = \frac{y^4}{2}, \Gamma_4^4 = \frac{y^2}{2}.$$

The horizontal nullity space of the curvature $hN_R = \{0\}$.

The Lie algebra $\overline{A_\Gamma}$ is generated by

$$g_1 = -\left(-e^{-x^3} + \frac{(x^1)^2}{4}\right)\frac{\partial}{\partial x^1} + x^1\frac{\partial}{\partial x^3} - \left(y^3e^{-x^3} + \frac{x^1y^1}{2}\right)\frac{\partial}{\partial y^1} + y^1\frac{\partial}{\partial y^3},$$

$$g_2 = -\frac{x^1}{2}\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^3} - \frac{y^1}{2}\frac{\partial}{\partial y^1},$$

$$g_3 = x^4\frac{\partial}{\partial x^2} - \left(\frac{(x^4)^2}{4} - e^{-x^2}\right)\frac{\partial}{\partial x^4} + y^4\frac{\partial}{\partial y^2} - \left(\frac{x^4y^4 + 2y^2e^{-x^2}}{2}\right)\frac{\partial}{\partial y^4},$$

$$g_4 = -2\frac{\partial}{\partial x^2} + x^4\frac{\partial}{\partial x^4} + y^4\frac{\partial}{\partial y^4}, g_5 = \frac{\partial}{\partial x^4}, g_6 = \frac{\partial}{\partial x^1}.$$

Multiplication table of $\overline{A_\Gamma}$

[,]	g_1	g_2	g_3	g_4	g_5	g_6
g_1	0	$\frac{g_1}{2}$	0	0	0	$-g_2$
g_2	$-\frac{g_1}{2}$	0	0	0	0	$\frac{g_6}{2}$
g_3	0	0	0	$-g_3$	$\frac{g_4}{2}$	0
g_4	0	0	g_3	0	$-g_5$	0
g_5	0	0	$-\frac{g_4}{2}$	g_5	0	0
g_6	g_2	$-\frac{g_6}{2}$	0	0	0	0

The Lie algebras $\overline{A_g} = \overline{A_{gc}} = \overline{A_\Gamma}$ and they are semi-simple.

Example 2

We take $M = \mathbf{R}^6$ and the energy function written:

$$E = \frac{1}{2}\left((y^1)^2 + e^{x^6}(y^2)^2 + e^{x^1}(y^3)^2 + e^{x^3}(y^4)^2 + e^{x^3}(y^5)^2 + (y^6)^2\right)$$

The missing coordinates are x^2, x^4 and x^5. The non-zero coefficients Γ_i^j of Γ are:

$$\Gamma_3^1 = -\frac{e^{x^1}y^3}{2}, \Gamma_2^2 = \frac{y^6}{2}, \Gamma_6^2 = \frac{y^2}{2}, \Gamma_1^3 = \frac{y^3}{2}, \Gamma_3^3 = \frac{y^1}{2}, \Gamma_4^3 = -\frac{y^4e^{x^3-x^1}}{2},$$

$$\Gamma_5^3 = -\frac{e^{x^3-x^1}y^5}{2}, \Gamma_3^4 = \frac{y^4}{2}, \Gamma_4^4 = \frac{y^3}{2}, \Gamma_3^5 = \frac{y^5}{2}, \Gamma_5^5 = \frac{y^3}{2}, \Gamma_2^6 = -\frac{y^2e^{x^6}}{2}.$$

The horizontal nullity space of the curvature is reduced to zero.

The Lie algebra $\overline{A_\Gamma}$ is generated by:

$$g_1 = -2\frac{\partial}{\partial x^3} + x^4\frac{\partial}{\partial x^4} + x^5\frac{\partial}{\partial x^5} + y^4\frac{\partial}{\partial y^4} + y^5\frac{\partial}{\partial y^5},$$

$$g_2 = x^5\frac{\partial}{\partial x^4} - x^4\frac{\partial}{\partial x^5} + y^5\frac{\partial}{\partial y^4} - y^4\frac{\partial}{\partial y^5},$$

$$g_3 = \frac{\partial}{\partial x^4}, g_4 = \frac{\partial}{\partial x^5},$$

$$g_5 = -\left(\frac{(x^2)^2}{4} - e^{-x^6}\right)\frac{\partial}{\partial x^2} + x^2\frac{\partial}{\partial x^6} - \left(y^6e^{-x^6} + \frac{x^2y^2}{2}\right)\frac{\partial}{\partial y^2} + y^2\frac{\partial}{\partial y^6},$$

$$g_6 = x^2\frac{\partial}{\partial x^2} - 2\frac{\partial}{\partial x^6} + y^2\frac{\partial}{\partial y^2}, g_7 = \frac{\partial}{\partial x^2}.$$

Multiplication table of $\overline{A_\Gamma}$

[,]	g_1	g_2	g_3	g_4	g_5	g_6	g_7
g_1	0	0	$-g_3$	$-g_4$	0	0	0
g_2	0	0	g_4	$-g_3$	0	0	0
g_3	g_3	$-g_4$	0	0	0	0	0
g_4	g_4	g_3	0	0	0	0	0
g_5	0	0	0	0	0	$-g_5$	$\frac{g_6}{2}$
g_6	0	0	0	0	g_5	0	$-g_7$
g_7	0	0	0	0	$-\frac{g_6}{2}$	g_7	0

The commutative ideal is generated by g_3 and g_4. The derivative ideal of $\overline{A_g}$ does not coincide with $\overline{A_g}$.

The Lie algebras $\overline{A_g}$, $\overline{A_{gc}}$ and $\overline{A_g}$ are identical.

The derivations are all inner.

Remark 1: The two examples above show that the relation in $\overline{A_g}$, $\overline{A_{gc}}$ and $\overline{A_g}$ do not allow knowing a priori the nature of $\overline{A_g}$, although $\overline{A_g}$ is an ideal of $\overline{A_g}$. They confirm our decomposition of the elements of $\overline{A_g}$.

Remark 2: In [2], we have studied some Lie algebras of countable vector fields where the derivative ideal coincides with the algebra and all derivation is inner, yet this algebra contains a commutative ideal. The above corollary gives a view in the case of a finite dimensional Lie algebra.

References

1. Anona, M. “Semi-simplicity of a Lie Algebra of Isometries.” *J Generalized Lie Theory Appl* 14 (2020): 1-10.
2. H. S. G. Ravelonirina, P. Randriambololondrantomalala, and M. Anona. ‘’Sur les algèbres de Lie des champs de vecteurs polynomiaux.’’ *Afr Diaspora J Math* 10 (2010): 87-95.

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