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Note on 2d binary operadic harmonic oscillator [1]

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Abstract

It is explained how the time evolution of the operadic variables may be introduced. As an example, a 2-dimensional binary operadic Lax representation for the harmonic oscillator is constructed.

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1 Introduction

It is well known that quantum mechanical observables are linear *operators*, i.e the linear maps $V \to V$ of a vector space V and their time evolution is given by the Heisenberg equation. As a variation of this one can pose the following question [7]: how to describe the time evolution of the linear algebraic operations (multiplications) $V^{\otimes n} \to V$. The algebraic operations (multiplications) can be seen as an example of the *operadic* variables [2, 3, 4, 5].

When an operadic system depends on time one can speak about *operadic dynamics* [7]. The latter may be introduced by simple and natural analogy with the Hamiltonian dynamics. In particular, the time evolution of operadic variables may be given by operadic Lax equation. In [8] it was shown how the dynamics may be introduced in a 2-dimensional Lie algebra. In the present paper, an operadic Lax representation for the harmonic oscillator is constructed in general 2-dimensional binary algebras.

2 Operad

Let K be a unital associative commutative ring, and let C^n $(n \in \mathbb{N})$ be unital K-modules. For $f \in C^n$, we refer to n as the *degree* of f and often write (when it does not cause confusion) f instead of $\deg f$. For example, $(-1)^f \doteq (-1)^n$, $C^f \doteq C^n$ and $\circ_f \doteq \circ_n$. Also, it is convenient to use the *reduced* degree $|f| \doteq n-1$. Throughout this paper, we assume that $\otimes \doteq \otimes_K$.

Definition 2.1 (operad (e.g [2, 3])). A linear (non-symmetric) *operad* with coefficients in K is a sequence $C \doteq \{C^n\}_{n\in\mathbb{N}}$ of unital K-modules (an $\mathbb{N}$-graded K-module), such that the following conditions are satisfied.

(1) For $0 \le i \le m-1$ there exist *partial compositions*

$$\circ_i \in \operatorname{Hom}(C^m \otimes C^n, C^{m+n-1}), \quad |\circ_i| = 0$$

(2) For all $h \otimes f \otimes g \in C^h \otimes C^f \otimes C^g$, the *composition (associativity) relations* hold,

$$(h \circ_i f) \circ_j g = \begin{cases} (-1)^{|f||g|}(h \circ_j g) \circ_{i+|g|} f & \text{if } 0 \le j \le i-1 \\ h \circ_i (f \circ_{j-i} g) & \text{if } i \le j \le i+|f| \\ (-1)^{|f||g|}(h \circ_{j-|f|} g) \circ_i f & \text{if } i+f \le j \le |h|+|f| \end{cases}$$

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(3) Unit $\mathrm{I} \in C^1$ exists such that

$$\mathrm{I} \circ_0 f = f = f \circ_i \mathrm{I}, \quad 0 \leq i \leq |f|$$

In the second item, the *first* and *third* parts of the defining relations turn out to be equivalent.

Example 2.2 (endomorphism operad [2]). Let V be a unital K-module and $\mathcal{E}_V^n \doteq \mathcal{E}nd_V^n \doteq \mathrm{Hom}(V^{\otimes n}, V)$. Define the partial compositions for $f \otimes g \in \mathcal{E}_V^f \otimes \mathcal{E}_V^g$ as

$$f \circ_i g \doteq (-1)^{i|g|} f \circ (\mathrm{id}_V^{\otimes i} \otimes g \otimes \mathrm{id}_V^{\otimes(|f|-i)}), \quad 0 \leq i \leq |f|$$

Then $\mathcal{E}_V \doteq \{\mathcal{E}_V^n\}_{n \in \mathbb{N}}$ is an operad (with the unit $\mathrm{id}_V \in \mathcal{E}_V^1$) called the *endomorphism operad* of V.

Thus, the algebraic operations can be seen as elements of an endomorphism operad. Just as elements of a vector space are called *vectors*, it is natural to call elements of an abstract operad *operations*.

3 Gerstenhaber brackets and operadic Lax pair

Definition 3.1 (total composition [2, 3]). The *total composition* $\bullet\colon C^f \otimes C^g \to C^{f+|g|}$ is defined by

$$f \bullet g \doteq \sum_{i=0}^{|f|} f \circ_i g \in C^{f+|g|}, \quad |\bullet| = 0$$

The pair $\mathrm{Com}\, C \doteq \{C, \bullet\}$ is called the *composition algebra* of C.

Definition 3.2 (Gerstenhaber brackets [2, 3]). The *Gerstenhaber brackets* $[\cdot,\cdot]$ are defined in $\mathrm{Com}\, C$ as a graded commutator by

$$[f, g] \doteq f \bullet g - (-1)^{|f||g|} g \bullet f = -(-1)^{|f||g|}[g, f], \quad |[\cdot,\cdot]| = 0$$

The *commutator algebra* of $\mathrm{Com}\, C$ is denoted as $\mathrm{Com}^- C \doteq \{C, [\cdot,\cdot]\}$. One can prove that $\mathrm{Com}^- C$ is a *graded Lie algebra*. The Jacobi identity reads

$$(-1)^{|f||h|}[[f, g], h] + (-1)^{|g||f|}[[g, h], f] + (-1)^{|h||g|}[[h, f], g] = 0$$

Assume that $K \doteq \mathbb{R}$ and operations are differentiable. The dynamics in operadic systems (operadic dynamics) may be introduced by

Definition 3.3 (operadic Lax pair [7]). Allow a classical dynamical system to be described by the evolution equations

$$\frac{dx_i}{dt} = f_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n$$

An *operadic Lax pair* is a pair (L, M) of homogeneous operations $L, M \in C$, such that the above system of evolution equations is equivalent to the *operadic Lax equation*

$$\frac{dL}{dt} = [M, L] \doteq M \bullet L - (-1)^{|M||L|} L \bullet M$$

Evidently, the degree constraints $|M| = |L| = 0$ give rise to ordinary Lax pair [6, 1].

4 Operadic harmonic oscillator

Consider the Lax pair for the harmonic oscillator:

$$L = \begin{pmatrix} p & \omega q \\ \omega q & -p \end{pmatrix}, \quad M = \frac{\omega}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Since the Hamiltonian is

$$H(q,p) = \frac{1}{2}(p^2 + \omega^2 q^2)$$

it is easy to check that the Lax equation

$$\dot{L} = [M, L] \doteq ML - LM$$

represents the Hamiltonian system

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} = p, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q} = -\omega^2 q$$

If μ is a homogeneous operadic variable one can use the above Hamilton's equations to obtain

$$\frac{d\mu}{dt} = \frac{\partial \mu}{\partial q}\frac{dq}{dt} + \frac{\partial \mu}{\partial p}\frac{dp}{dt} = p\frac{\partial \mu}{\partial q} - \omega^2 q \frac{\partial \mu}{\partial p}$$

Therefore, the linear partial differential equation for the operadic variable $\mu(q,p)$ reads

$$p\frac{\partial \mu}{\partial q} - \omega^2 q \frac{\partial \mu}{\partial p} = M \bullet \mu - \mu \bullet M$$

By integrating one gains sequences of operations called the *operadic (Lax representations of) harmonic oscillator.*

5 Example

Let $A \doteq \{V, \mu\}$ be a binary algebra with operation $xy \doteq \mu(x \otimes y)$. We require that $\mu = \mu(q,p)$ so that (μ, M) is an operadic Lax pair, i.e the operadic Lax equation

$$\dot{\mu} = [M, \mu] \doteq M \bullet \mu - \mu \bullet M, \quad |\mu| = 1, \quad |M| = 0$$

represents the Hamiltonian system of the harmonic oscillator.

Let $x, y \in V$. Assuming that $|M| = 0$ and $|\mu| = 1$, one has

$$M \bullet \mu = \sum_{i=0}^{0} (-1)^{i|\mu|} M \circ_i \mu = M \circ_0 \mu = M \circ \mu$$

$$\mu \bullet M = \sum_{i=0}^{1} (-1)^{i|M|} \mu \circ_i M = \mu \circ_0 M + \mu \circ_1 M = \mu \circ (M \otimes \mathrm{id}_V) + \mu \circ (\mathrm{id}_V \otimes M)$$

Therefore, one has

$$\frac{d}{dt}(xy) = M(xy) - (Mx)y - x(My)$$

Let $\dim V = n$. In a basis $\{e_1, \ldots, e_n\}$ of V, the structure constants μ^i_{jk} of A are defined by

$$\mu(e_j \otimes e_k) \doteq \mu^i_{jk} e_i, \quad j, k = 1, \ldots, n$$

In particular,

$$\frac{d}{dt}(e_j e_k) = M(e_j e_k) - (Me_j)e_k - e_j(Me_k)$$

By denoting $Me_i \doteq M_i^s e_s$, it follows that

$$\dot{\mu}_{jk}^i = \mu_{jk}^s M_s^i - M_j^s \mu_{sk}^i - M_k^s \mu_{js}^i, \quad i, j, k = 1, \ldots, n$$

In particular, one has

Lemma 5.1. *Let* $\dim V = 2$ *and* $M \doteq (M_j^i) \doteq \frac{\omega}{2}\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. *Then the* 2*-dimensional binary operadic Lax equations read*

$$\begin{cases}
\dot{\mu}_{11}^1 = -\frac{\omega}{2}\left(\mu_{11}^2 + \mu_{21}^1 + \mu_{12}^1\right), & \dot{\mu}_{11}^2 = \frac{\omega}{2}\left(\mu_{11}^1 - \mu_{21}^2 - \mu_{12}^2\right) \\
\dot{\mu}_{12}^1 = -\frac{\omega}{2}\left(\mu_{12}^2 + \mu_{22}^1 - \mu_{11}^1\right), & \dot{\mu}_{12}^2 = \frac{\omega}{2}\left(\mu_{12}^1 - \mu_{22}^2 + \mu_{11}^2\right) \\
\dot{\mu}_{21}^1 = -\frac{\omega}{2}\left(\mu_{21}^2 - \mu_{11}^1 + \mu_{22}^1\right), & \dot{\mu}_{21}^2 = \frac{\omega}{2}\left(\mu_{21}^1 + \mu_{11}^2 - \mu_{22}^2\right) \\
\dot{\mu}_{22}^1 = -\frac{\omega}{2}\left(\mu_{22}^2 - \mu_{12}^1 - \mu_{21}^1\right), & \dot{\mu}_{22}^2 = \frac{\omega}{2}\left(\mu_{22}^1 + \mu_{12}^2 + \mu_{21}^2\right)
\end{cases}$$

For the harmonic oscillator, define its auxiliary functions $A_\pm$ and $D_\pm$ by

$$\begin{cases}
A_+^2 + A_-^2 = 2\sqrt{2H} \\
A_+^2 - A_-^2 = 2p \\
A_+ A_- = \omega q
\end{cases}, \qquad
\begin{cases}
D_+ \doteq \frac{A_+}{2}(A_+^2 - 3A_-^2) \\
D_- \doteq \frac{A_-}{2}(3A_+^2 - A_-^2)
\end{cases}$$

Then one has the following

Theorem 5.2. *Let* $C_\beta \in \mathbb{R}$ $(\beta = 1, \ldots, 8)$ *be arbitrary real–valued parameters,* $M \doteq \frac{\omega}{2}\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ *and*

$$\begin{cases}
\mu_{11}^1(q,p) = C_5 A_- + C_6 A_+ + C_7 D_- + C_8 D_+ \\
\mu_{12}^1(q,p) = C_1 A_+ + C_2 A_- - C_7 D_+ + C_8 D_- \\
\mu_{21}^1(q,p) = -C_1 A_+ - C_2 A_- - C_3 A_+ - C_4 A_- - C_5 A_+ + C_6 A_- - C_7 D_+ + C_8 D_- \\
\mu_{22}^1(q,p) = -C_3 A_- + C_4 A_+ - C_7 D_- - C_8 D_+ \\
\mu_{11}^2(q,p) = C_3 A_+ + C_4 A_- - C_7 D_+ + C_8 D_- \\
\mu_{12}^2(q,p) = C_1 A_- - C_2 A_+ + C_3 A_- - C_4 A_+ + C_5 A_- + C_6 A_+ - C_7 D_- - C_8 D_+ \\
\mu_{21}^2(q,p) = -C_1 A_- + C_2 A_+ - C_7 D_- - C_8 D_+ \\
\mu_{22}^2(q,p) = -C_5 A_+ + C_6 A_- + C_7 D_+ - C_8 D_-
\end{cases}$$

Then (μ, M) *is a* 2*-dimensional binary operadic Lax pair of the harmonic oscillator.*

Idea of proof. Denote

$$\begin{cases}
G_\pm^{\omega/2} & \doteq \dot{A}_\pm \pm \frac{\omega}{2} A_\mp \\
G_\pm^{3\omega/2} & \doteq \dot{D}_\pm \pm \frac{3\omega}{2} D_\mp
\end{cases}$$

Define the matrix

$$\Gamma = (\Gamma_\alpha^\beta) \doteq \begin{pmatrix}
0 & G_+^{\omega/2} & -G_+^{\omega/2} & 0 & 0 & G_-^{\omega/2} & -G_-^{\omega/2} & 0 \\
0 & G_-^{\omega/2} & -G_-^{\omega/2} & 0 & 0 & -G_+^{\omega/2} & G_+^{\omega/2} & 0 \\
0 & 0 & -G_+^{\omega/2} & -G_-^{\omega/2} & G_+^{\omega/2} & G_-^{\omega/2} & 0 & 0 \\
0 & 0 & -G_-^{\omega/2} & G_+^{\omega/2} & G_-^{\omega/2} & -G_+^{\omega/2} & 0 & 0 \\
G_-^{\omega/2} & 0 & -G_+^{\omega/2} & 0 & 0 & G_-^{\omega/2} & 0 & -G_+^{\omega/2} \\
G_+^{\omega/2} & 0 & G_-^{\omega/2} & 0 & 0 & G_+^{\omega/2} & 0 & G_-^{\omega/2} \\
G_-^{3\omega/2} & -G_+^{3\omega/2} & -G_+^{3\omega/2} & -G_-^{3\omega/2} & -G_+^{3\omega/2} & -G_-^{3\omega/2} & -G_-^{3\omega/2} & G_+^{3\omega/2} \\
G_+^{3\omega/2} & G_-^{3\omega/2} & G_-^{3\omega/2} & -G_+^{3\omega/2} & G_-^{3\omega/2} & -G_+^{3\omega/2} & -G_+^{3\omega/2} & -G_-^{3\omega/2}
\end{pmatrix}$$

Then it follows from Lemma 5.1 that the 2-dimensional binary operadic Lax equations read

$$C_\beta \Gamma^\beta_\alpha = 0, \quad \alpha = 1, \dots, 8$$

Since the parameters C_β are arbitrary, the latter constraints imply $\Gamma = 0$. Thus one has to consider the following differential equations

$$G^{\omega/2}_\pm = 0 = G^{3\omega/2}_\pm$$

By direct calculations [9] one can show that

$$\begin{cases} \dot{p} = -\omega^2 q \\ \dot{q} = p \end{cases} \quad \Longleftrightarrow \quad G^{\omega/2}_\pm = 0 \quad \Longleftrightarrow \quad G^{3\omega/2}_\pm = 0 \qquad \square$$

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