Ashdin Publishing
Journal of Physical Mathematics
Vol. 1 (2009), Article ID S090604, 15 pages
doi:10.4303/jpm/S090604

Newton's laws for a biquaternionic model of the electro-gravimagnetic field, charges, currents, and their interactions

L. A. ALEXEYEVA

Wave Dynamics Laboratory, Institute of Mathematics, Ministry of Education and Science of the Republic of Kazakhstan, 125 Pushkin Street, Almaty 050010, Kazakhstan

E-mail: alexeeva@math.kz

Abstract

By using the Hamiltonian form of the Maxwell equations, a biquaternionic model of electro-gravimagnetic (EGM) field is proposed. The EGM-field equations, generating different charges and currents, are constructed. The field analogs of three Newton's laws are formulated for free and interacting charges and currents, as well as the total field of interaction. The Lorentz invariance of the EGM-field equations is investigated (in particular, the charge-current conservation law). It is shown that at the presence of field interaction, this law differs from the well-known one. A new modification of the Maxwell equations is proposed with the scalar resistance field in the biquaternion EGM-field tension. Relativistic transformations of mass and charge-current densities, forces, and their powers are constructed. The solution of the Cauchy problem is given for equation of charge-current transformations.

2000 MSC: 35Q60, 83C50

1 Introduction

In the present paper, a biquaternionic model of the electro-gravimagnetic (EGM) field is considered, which is called *energetic*. For this, a complex Hamiltonian form of Maxwell equations (MEs) is used, which allows to get the biquaternionic form of these equations [1, 2]. Note that Maxwell gave his equations in quaternionic form, but the modern form belongs to Heaviside [5]. Quaternionic forms of MEs were used by some authors [5, 6, 7] for their solving. Kassandrov applied similar forms for building a unified field model [8].

Here we use the scalar-vector form of biquaternion, which is very impressive and can be adapted for writing the physical variables and equations. Based on Newton's laws, the biquaternionic transformation equations of charges and currents at the presence of the EGM fields are constructed. The relation of these equations to hydrodynamics equations is considered. The energy conservation law at the presence of the field interaction is found.

The Lorentz invariance of equations of energetic field interaction is studied, and also the invariance of charge-current conservation law. It is shown that by the charge-current interaction this law differs from the well-known one for the closed electromagnetic (EM) field. The new modification of the ME is proposed for open field. One has to introduce the scalar *field of resistance*. The relativistic transformations of mass and electric charge-current densities, acting forces, and their powers are constructed.

2 Hamiltonian form of Maxwell equations

Let

$$\mathbf{M} = R^{1+3} = \big\{(\tau, x) = (ct, x) : t \in R^1,\ x = \big(x_1, x_2, x_3\big) \in R^3\big\}$$

denote the Minkowski space. In $\mathbf{M}$ the symmetric form of Maxwell equations for electromagnetic (EM) field can be written [1] as

$$\partial_\tau A + i \operatorname{rot} A + J = 0, \tag{2.1}$$

$$\rho = \operatorname{div} A, \tag{2.2}$$

where A is the complex vector of EM field:

$$A = A^E + i\, A^H = \sqrt{\varepsilon} E + i\sqrt{\mu} H. \tag{2.3}$$

Vectors E and H are the electric and magnetic fields, ε and μ are constants of electric conductivity and magnetic permeability of medium, respectively. The complex charge density ρ and current density J are expressed through the densities of electric and magnetic charges and currents by

$$\rho = \rho^E/\sqrt{\varepsilon} - i\, \rho^H/\sqrt{\mu}, \quad J = \sqrt{\mu}\, j^E - i\sqrt{\varepsilon}\, j^H, \tag{2.4}$$

$$\rho^E = \varepsilon \operatorname{div} E, \quad \rho^H = -\mu \operatorname{div} H. \tag{2.5}$$

Energy density W and the Poynting vector P of A-field read

$$W = 0,5\big(\varepsilon\|E\|^2 + \mu\|H\|^2\big) = 0,5(A, \bar{A}), \quad P = c^{-1} E \times H = 0,5i\, \big[A, \bar{A}\,\big], \tag{2.6}$$

where $\bar{A} = \sqrt{\varepsilon} E - i\sqrt{\mu} H$ and $c = 1/\sqrt{\varepsilon\mu}$ is the speed of light.

Here and hereinafter

$$(a, b) = \sum_{i=1}^{3} a_i b_i, \quad [a, b] = a \times b = \sum_{i,j,k=1}^{3} e_{ijk} e_i a_j b_k$$

are scalar and vector products of a and b, respectively, e_{ijk} is the Levi-Civita pseudotensor and e_i $(i = 1, 2, 3)$ are the unit vectors of the Cartesian coordinate system in R^3.

One can see that the energy density W is simply the module of the complex vector A (half of it). Note that, differently from Maxwell equations, all relations for A-field do not contain the constants of EM medium. In particular, the velocity of electromagnetic waves in this coordinate system is nondimensional and equal to 1.

Here some known statements for A-field are given which are due to Maxwell equations [1].

Theorem 2.1. *For given current and charge, the solution of* (2.1) *satisfies the wave equation*

$$\Box\, A = (\partial_\tau^2 - \Delta) A = i \operatorname{rot} J - \operatorname{grad} \rho - \partial_\tau J, \tag{2.7}$$

and the conservation laws of charge and energy hold:

$$\partial_\tau \rho + \operatorname{div} J = 0, \tag{2.8}$$

$$\partial_\tau W + \operatorname{div} P = -\operatorname{Re}\big(J, \bar{A}\,\big) = c^{-1}\big(j^H H - j^E E\big). \tag{2.9}$$

In Maxwell equations the density of the magnetic charges is $\rho^H = 0$. This means that the magnetic field is a solenoidal one: $\operatorname{div} H = 0$.

But it is known that the classical gravitation is a scalar field. It can be described by a scalar gravitational potential, which depends on masses. Here, these two fields are united in a unique *gravimagnetic field*. It is possible to do so if we introduce a density of gravitational mass in Maxwell equations. In particular, the following hypothesis can be proposed.

Hypothesis. The *density* ρ^H *is equal to the density of gravitational mass.*

Hereinafter we will show that this hypothesis has a theoretical explanation which gives a very plausible effect.

Thus it follows that the potential part of H describes the gravitational field and the solenoidal part describes magnetic field. So H-field is *gravimagnetic field*. Consequently, A-field is *electro-gravimagnetic*. Since its dimensionality is defined by density of energy, it is possible to call it *energetic field*.

We call j^H the *gravimagnetic* current. If $\rho^H = 0$, it is *magnetic* current. If H-field is potential ($\operatorname{rot} H = 0$), then j^H is the *mass* current.

Note that the system of ME is not closed. It allows for given charge and current to define the A-field, or for given A-field to find corresponding charges and currents. If these are unknown, then for its closing usually the equations of mechanics of external medium are used. However, here we will give another presentation by using the biquaternionic form of these equations and Newton's laws.

For going to this form and new equations, we will give the thumbnail sketch on functional space of biquaternions and operation on it.

3 Differential algebra of biquaternions: Bigradients

The functional space of biquaternions is the space of complex quaternions:

$$\mathbf{K}\big(R^{1+3}\big) = \big\{\mathbf{F} = f(\tau, x) + F(\tau, x)\big\},$$

where f is a complex function and F is a three-dimensional vector-function with complex components. We assume further that f and F are locally integrable and differentiable on $\mathbf{M}$.

The space $\mathbf{K}$ is *associative* but noncommutative algebra with addition

$$\mathbf{F} + \mathbf{G} = (f + g) + (F + G),$$

and product

$$\mathbf{F} \circ \mathbf{G} = (f + F) \circ (g + G) = \big(fg - (F, G)\big) + \big(fG + gF + [F, G]\big). \tag{3.1}$$

The biquaternion $\bar{\mathbf{F}} = \bar{f} + \bar{F}$ is called *complex conjugated* and $\mathbf{F}^* = \bar{f} - \bar{F}$ is called *conjugated*. If $\mathbf{F}^* = \mathbf{F}$, it is called *self-conjugated*. An example of self-conjugated biquaternion is $\mathbf{F} = f + iF$, where f and F are real functions.

Definition 3.1. The *scalar product* of $\mathbf{F}_1$, $\mathbf{F}_2$ is defined by

$$\big(\mathbf{F}_1, \mathbf{F}_2\big) = f_1 f_2 + \big(F_1, F_2\big).$$

Definition 3.2. The *norm* of $\mathbf{F}$ is defined by

$$\|\mathbf{F}\| = \sqrt{(\mathbf{F}, \bar{\mathbf{F}})} = \sqrt{f \cdot \bar{f} + (F, \bar{F})} = \sqrt{|f|^2 + \|F\|^2}.$$

Definition 3.3. The *pseudonorm* of $\mathbf{F}$ is defined by

$$\langle \mathbf{F} \rangle = \sqrt{f \cdot \bar{f} - (F, \bar{F})} = \sqrt{|f|^2 - \|F\|^2}.$$

Hereinafter the *mutual complex gradients* are used:

$$\nabla^+ = \partial_\tau + i\nabla, \quad \nabla^- = \partial_\tau - i\nabla,$$

where $\nabla = \mathrm{grad} = (\partial_1, \partial_2, \partial_3)$. The action of these differential operators on $\mathbf{K}$ is determined as in the biquaternion algebra (according to a sign):

$$\begin{aligned}\nabla^\pm \mathbf{F} &= \big(\partial_\tau \pm i\nabla\big) \circ (f + F) = \big(\partial_\tau f \mp i\,(\nabla, F)\big) \pm \partial_\tau F \pm i\nabla f \pm i[\nabla, F] \\ &= \big(\partial_\tau f \mp i \,\mathrm{div}\, F\big) \pm \partial_\tau F \pm i \,\mathrm{grad}\, f \pm i \,\mathrm{rot}\, F.\end{aligned}$$

Further we call these *bigradients*.

4 Biwave equations: Cauchy problem

It is easy to check that the wave operator can be presented in the form

$$\Box = \frac{\partial^2}{\partial \tau^2} - \triangle = \nabla^- \circ \nabla^+ = \nabla^+ \circ \nabla^-.$$

Using this property, it is possible to build the solution of the differential equations of the type

$$\nabla^\pm \mathbf{K} = \mathbf{G}. \tag{4.1}$$

We call such equations the *biwave equations*. From (4.1) it follows that

$$\Box\, \mathbf{K} = \nabla^\mp \mathbf{G}.$$

Its solution is the convolution

$$\mathbf{K} = \nabla^\mp \mathbf{G} * \psi, \tag{4.2}$$

where $\psi(\tau, x)$ is the fundamental solution of the wave equation

$$\Box\, \psi = \delta(\tau)\delta(x).$$

This solution is also the solution of (4.1). Really, by using the property of differentiation of convolution, we have

$$\nabla^\pm \mathbf{K} = \nabla^\pm \nabla^\mp (\mathbf{G} * \psi) = \Box(\mathbf{G} * \psi) = (\mathbf{G} * \Box\, \psi) = \mathbf{G} * \delta(\tau)\delta(x) = \mathbf{G}.$$

The fundamental solutions are defined for constructing solutions of the wave equation. For initial value problems, it is convenient to use as the fundamental solution on the light cone $\tau = \|x\|$ the following solution:

$$\psi = \big(4\pi \|x\|\big)^{-1} \delta\big(\tau - \|x\|\big).$$

In this case, it is easy to show, by writing the convolution in integral form, that the solution (4.2) is equal to zero for $\tau = 0$. We use this for building the solution of the equation (4.1) with the Cauchy data.

Cauchy problem. The initial data are given: $\mathbf{K}(0,x) = \mathbf{K}_0(x)$. It is required to find the solution of (4.1), which satisfies these data.

To solve the Cauchy problem we use here methods of distribution theory [4, 9]. We will consider the regular generalized functions of type $\widehat{\mathbf{G}} = H(\tau)\mathbf{G}(\tau,x)$, where $H(\tau)$ is the Heaviside function. By using differentiation of generalized function we obtain $\nabla^{\pm}\widehat{\mathbf{K}} = \widehat{\mathbf{G}} + \delta(\tau)\mathbf{K}_0(x)$. Hence,

$$\mathbf{H}(\tau)K(\tau,x) = \nabla^{\mp}\{H(\tau)\widehat{\mathbf{G}} * \psi\} + \mathbf{G}(0,x) \underset{x}{*} \psi + \nabla^{\mp}\{\mathbf{K}_0(x) \underset{x}{*} \psi\}, \tag{4.3}$$

where the sign "$\underset{x}{*}$" means that the convolution is given only over x. Its integral form is

$$\begin{aligned} 4\pi\mathbf{K}(\tau,x) = &-\nabla^{\mp}\left\{\int_{r\le\tau} \frac{\mathbf{G}(\tau-r,y)}{r} dV(y) + \tau^{-1}\int_{r=\tau}\mathbf{K}_0(y)dS(y)\right\} \\ &-\tau^{-1}\int_{r=\tau}\mathbf{G}(0,y)dS(y), \end{aligned} \tag{4.4}$$

where $r = \|y - x\|$, $dV(y) = dy_1dy_2dy_3$ and $dS(y)$ is the differential of sphere's area.

This formula is a generalization of the famous Kirchhoff formula for solution of Cauchy problem for wave equation [4].

5 Biquaternions of A-field

Intoduce the following biquaternions: *potential* $\mathbf{\Phi} = i\phi - \Psi$, *tension* $\mathbf{A} = 0 + A$, *charge-current density* $\mathbf{\Theta} = -i\rho - J$, *energy-pulse density* $\mathbf{\Xi} = 0,5\,\mathbf{A}^* \circ \mathbf{A} = W + iP$.

Maxwell equations (2.1)–(2.2) in biquaternions space have the simple form

$$\nabla^{+}\mathbf{A} = \mathbf{\Theta}. \tag{5.1}$$

If the potential satisfies the Lorentz calibration $\partial_\tau\phi - \operatorname{div}\Psi = 0$, then

$$\mathbf{A} = \nabla^{-}\mathbf{\Phi}.$$

If we take the corresponding complex gradient, we get the wave equations

$$\Box\,\mathbf{\Phi} = \mathbf{\Theta}, \tag{5.2}$$

$$\Box\,\mathbf{A} = \nabla^{-}\mathbf{\Theta}. \tag{5.3}$$

Hence, it follows that the the complex gradient from A-field potential defines biquaternions, corresponding to field tension, charge, and current. The scalar part of complex gradient of energy-pulse gives the energy conservation [2] law.

One can see that the *charges and currents are simply physical appearance of the complex gradient of EGM field.*

Cauchy problem for Maxwell equations. From equation (4.3) it follows that for given charge and current and initial data $\mathbf{A}(0,x) = \mathbf{A}_0(x)$, the solution of (5.1) is given by

$$\begin{aligned} 4\pi\mathbf{A} = &-\nabla^{-}\left\{\int_{r\le\tau}\frac{\mathbf{\Theta}(\tau-r,y)}{r}\,dV(y) + \tau^{-1}\int_{r=\tau}\mathbf{A}_0(y)\,dS(y)\right\} \\ &-\tau^{-1}\int_{r=\tau}\mathbf{\Theta}(0,y)dS(y). \end{aligned}$$

Hence it is easy to write the integral representations for vector of the EGM-field tension E, H.

6 Lorentz transformations

Denote

$$\mathbf{Z} = \tau + ix, \quad \bar{\mathbf{Z}} = \tau - ix.$$

It is easy to see that

$$\mathbf{Z} = \mathbf{Z}^*, \quad \bar{\mathbf{Z}} = \bar{\mathbf{Z}}^*, \quad \|\mathbf{Z}\|^2 = \|\bar{\mathbf{Z}}\|^2 = (\mathbf{Z}, \bar{\mathbf{Z}}), \quad \langle \mathbf{Z} \rangle^2 = \langle \bar{\mathbf{Z}} \rangle^2 = \mathbf{Z} \circ \bar{\mathbf{Z}}.$$

Consider the self-conjugated biquaternions by using the hyperbolic sine and cosine:

$$\mathbf{U} = \cosh\theta + ie\sinh\theta, \quad \bar{\mathbf{U}} = \cosh\theta - ie\sinh\theta, \quad \|e\| = 1.$$

Here θ is a real number. It is easy to check that

$$\mathbf{U} \circ \bar{\mathbf{U}} = 1. \tag{6.1}$$

Lemma 6.1. *Classical Lorentz transformation $L : \mathbf{Z} \to \mathbf{Z}'$ has the form*

$$\mathbf{Z}' = \mathbf{U} \circ \mathbf{Z} \circ \mathbf{U}, \quad \mathbf{Z} = \bar{\mathbf{U}} \circ \mathbf{Z}' \circ \bar{\mathbf{U}}.$$

Proof. The direct calculation proves this lemma. The pseudonorm is saved:

$$\langle \mathbf{Z}' \rangle^{\mathbf{2}} = \mathbf{U} \circ \mathbf{Z} \circ \mathbf{U} \circ \bar{\mathbf{U}} \circ \bar{\mathbf{Z}} \circ \bar{\mathbf{U}} = \langle \mathbf{Z} \rangle^{\mathbf{2}}.$$

Here the property of associativity and (6.1) were used.

If we use the notations

$$\cosh 2\theta = \frac{1}{\sqrt{1-v^2}}, \quad \sinh 2\theta = \frac{v}{\sqrt{1-v^2}}, \quad |v| < 1,$$

then the scalar and vector parts of biquaternions can be written in the form of known relativistic formulas:

$$\tau' = \frac{\tau + v(e,x)}{\sqrt{1-v^2}}, \quad x' = \big(x - e(e,x)\big) + e\frac{(e,x) + v\tau}{\sqrt{1-v^2}},$$
$$\tau = \frac{\tau' - v(e,x)}{\sqrt{1-v^2}}, \quad x = \big(x' - e(e,x')\big) + e\frac{(e,x') - v\tau'}{\sqrt{1-v^2}},$$

It corresponds to the motion of coordinate system X in direction of vector e with velocity v. □

Lemma 6.2. *The conjugated quaternions*

$$\mathbf{W} = \cos\varphi + e\sin\varphi, \quad \mathbf{W}^* = \cos\varphi - e\sin\varphi \quad \big(\|e\| = 1\big)$$

define the group of transformation on $\mathbf{M}$*, which are orthogonal on vector part of* $\mathbf{Z}$*:*

$$\mathbf{Z}' = \mathbf{W} \circ \mathbf{Z} \circ \mathbf{W}^*, \quad \mathbf{Z} = \mathbf{W}^* \circ \mathbf{Z}' \circ \mathbf{W}.$$

It is a rotation around the vector e by angle 2φ. As the result of these two lemmas, we have the following.

Lemma 6.3. *The Lorentz transformation on* $\mathbf{M}$ *can be defined by*

$$\begin{aligned} \mathbf{Z}' &= \mathbf{L} \circ \mathbf{Z} \circ \mathbf{L}^*, \quad \mathbf{Z} = \mathbf{L}^* \circ \mathbf{Z}' \circ \mathbf{L}, \\ \mathbf{L} &= \mathbf{W} \circ \mathbf{U} = \cosh(\theta + i\varphi) + ie\sinh(\theta + i\varphi), \\ \mathbf{L}^* &= \mathbf{U}^* \circ \mathbf{W}^* = \cosh(\theta - i\varphi) + ie\sinh(\theta - i\varphi), \end{aligned} \tag{6.2}$$

The pseudonorm is saved for the Lorentz transformation: $\langle \mathbf{Z} \rangle = \langle \mathbf{Z}' \rangle$.

It is easy to see that $\bar{\mathbf{L}} \circ \mathbf{L}^* = \mathbf{L}^* \circ \bar{\mathbf{L}} = 1$ because the pseudonorm $\mathbf{Z}$ is saved.

7 Lorentz transformation of biwave equations

Let us consider how bigradients are transformed under the Lorentz transformation.

Lemma 7.1. *If* $\mathbf{Z}' = \mathbf{L} \circ \mathbf{Z} \circ \mathbf{L}^*$, *then*

$$\mathbf{D}' = \bar{\mathbf{L}}^* \circ \nabla \circ \mathbf{L}, \quad \mathbf{D} = \mathbf{L} \circ \nabla' \circ \bar{\mathbf{L}}^*,$$

where $\mathbf{D} = \nabla^+$ *or* $\mathbf{D} = \nabla^-$.

Based on this lemma, consider how the biwave equation (4.1) is changed by the Lorentz transformation. Using associativity of the product and characteristic of $\mathbf{L}$, we get

$$\nabla'\mathbf{K}' = \big(\bar{\mathbf{L}}^* \circ \nabla \circ \mathbf{L}\big)\big(\bar{\mathbf{L}}^* \circ \mathbf{K} \circ \mathbf{L}\big) = \bar{\mathbf{L}}^* \circ \nabla \circ \mathbf{K} \circ \mathbf{L} = \bar{\mathbf{L}}^* \circ \mathbf{G} \circ \mathbf{L} = \mathbf{G}'.$$

Hence, the form of equation is saved:

$$\left(\frac{\partial}{\partial \tau'} \pm i\nabla'\right)\mathbf{K}' = \mathbf{G}',$$

where $\mathbf{K}' = \bar{\mathbf{L}}^* \circ \mathbf{K} \circ \mathbf{L}$, $\mathbf{G}' = \bar{\mathbf{L}}^* \circ \mathbf{G} \circ \mathbf{L}$. From here we have the following

Theorem 7.2. *A Lorentz transformation of the Maxwell equations can be written as follows:*

$$\mathbf{D}^+\mathbf{A}' = \boldsymbol{\Theta}', \quad \textit{where } \mathbf{A}' = \bar{\mathbf{L}}^* \circ \mathbf{A} \circ \mathbf{L}, \quad \boldsymbol{\Theta}' = \bar{\mathbf{L}}^* \circ \boldsymbol{\Theta} \circ \mathbf{L}.$$

Relativistic formulas for tension, charge, and current (when $\varphi = 0$*) are*

$$A' = \big(A - e(e, A)\big) + e\frac{(e, A)}{\sqrt{1 - v^2}}, \tag{7.1}$$

$$\rho' = \frac{\rho - v(e, J)}{\sqrt{1 - v^2}}, \quad J' = \big(J - e(e, J)\big) + e\frac{(e, J) - v\rho}{\sqrt{1 - v^2}}. \tag{7.2}$$

One can see that the tension of A-field always increases in the direction of vector e. In the absence of current, the charge-mass will increase. At the presence of the current, depending on the directions of their motion, the charge-mass can increase or decrease.

8 The third Newton law: The power and density of acting forces

Let us consider two EGM fields $\mathbf{A}$ and $\mathbf{A}'$. Their generating charges and currents are $\mathbf{\Theta}$ and $\mathbf{\Theta}'$. We call the *power-force density* the biquaternion

$$\mathbf{F} = M - iF = \mathbf{\Theta} \circ \mathbf{A}' = -(i\rho + J) \circ A' = (A', J) - i\rho A' + [A', J], \tag{8.1}$$

which is acting from side of A'-field on the charge and current of A-field. Really, using (2.3) and (2.4), the scalar part is determined as power density of acting forces:

$$M = (A', J) = c^{-1}\big((E', j^E) + (H', j^H)\big) + i\big((B', j^E) - (D', j^H)\big). \tag{8.2}$$

Selecting the real and imaginary parts of vector form of biquaternion, we get expressions for density of acting forces ($F = F^H + i\,F^E$):

$$F^H = \rho^E E' + \rho^H H' + j^E \times B' - j^H \times D', \tag{8.3}$$

$$F^E = c\big(\rho^E B' - \rho^H D'\big) + c^{-1}\big(E' \times j^E + H' \times j^H\big). \tag{8.4}$$

Here $B = \mu H$ is an analog of a vector to magnetic induction and $D = \varepsilon E$ is a vector of the electric offset.

The potential part of H describes the tension of gravitational field. The torsional part of this vector describes magnetic field. The scalar part of $\mathbf{\Theta}$, $\mathbf{\Theta}'$ contains the densities of electric charge and mass, its vector part contains the densities of electric and mass currents.

Coming from these suggestions, in formula (8.3) the known forces are standing, consecutively: the Coulomb force $\rho^E E'$, the gravitational force $\rho^H H'$, the Lorentz force $j^E \times B'$, and a new force $-D' \times j^H$, which we call *gravielectric*. In real part of the power (8.2), we can see the powers of the Coulomb, gravitational, and magnetic forces.

The power of gravielectric force does not enter in real part of (8.2) as it does not work on the mass displacement, because it is perpendicular to its velocity. It is interesting that the Lorentz force also does not enter in real part of (8.2). It proves that this force is perpendicular to mass velocity, though directly from Maxwell equations this does not follow.

Naturally, in analogy, to expect that equations (8.3)-(8.4) describe forces, causing change of electric current, but in imaginary part M the power stands which corresponds to it. On the virtue of the third Newton law about acting and counteracting forces, we can suppose that

$$\mathbf{F}' = -\mathbf{F}.$$

From here we get *the law of field action and reaction*:

$$\mathbf{\Theta} \circ \mathbf{A}' = -\mathbf{\Theta}' \circ \mathbf{A}. \tag{8.5}$$

It is interesting to note that in scalar part it requires the equality of powers corresponding to forces, acting on charges and currents of the other field. That is, it is befitted with what is known in mechanics as the Betti reciprocity identity, which is usually written for the work of forces.

9 The second Newton law: Transformations equation

The charge-current field is changed under influence of the field of other charges and currents. As it is well known, direction of the most intensive change of the scalar field describes its gradient. In analogy we can expect that the most intensive change of the charge-current field occurs toward its complex gradient. Naturally, expect that this change must occur toward power-force, acting on the part of the second field on the first one. So the law of the change of the charge-current field under the action of the others (like second Newton law) is offered in the manner of the following equations.

Equations of the charge-current interaction are

$$\kappa\nabla^{-}\mathbf{\Theta} = \mathbf{F} \equiv \mathbf{\Theta} \circ \mathbf{A}', \quad \kappa\nabla^{-}\mathbf{\Theta}' = \mathbf{\Theta}' \circ \mathbf{A}, \tag{9.1}$$

$$\mathbf{\Theta} \circ \mathbf{A}' = -\mathbf{\Theta}' \circ \mathbf{A}, \tag{9.2}$$

$$\nabla^{+}\mathbf{A} = \mathbf{\Theta}, \quad \nabla^{+}\mathbf{A}' = \mathbf{\Theta}'. \tag{9.3}$$

Here equations (9.1) correspond to the second Newton law which is written for each charge and current of interacting field. Equation (9.2) is the third Newton law. Together with Maxwell equations (9.3) for these fields they give closed system of the nonlinear differential equations for determination $\mathbf{A}$, $\mathbf{A}'$, $\mathbf{\Theta}$, $\mathbf{\Theta}'$. Entering the constant of interaction κ is connected with dimensionality. Revealing the scalar and vector part in (9.1), we have the following.

Equations of charge-current transformations are

$$i\,\kappa\big(\partial_\tau \rho + \operatorname{div} J\big) = M, \tag{9.4}$$

$$i\,\kappa\big(\partial_\tau J - i \operatorname{rot} J + \nabla\rho\big) = F. \tag{9.5}$$

At first let us consider the second equation. By virtue of (2.2), (2.3), (2.4), we obtain analog of *the second Newton law for charge-current field:*

$$\kappa\big(\sqrt{\varepsilon}\,\partial_\tau j^H + \sqrt{\mu}\operatorname{rot} j^E + \mu^{-0,5}\operatorname{grad}\rho^H\big) = \rho^E E' + \rho^H H' + j^E \times B' - j^H \times D', \tag{9.6}$$

$$\kappa\big(\sqrt{\mu}\,\partial_\tau j^E - \sqrt{\varepsilon}\operatorname{rot} j^H + \varepsilon^{-0,5}\operatorname{grad}\rho^E\big) = c\big(\rho^E B' - \rho^H D'\big) + c^{-1}\big(E' \times j^E + H' \times j^H\big). \tag{9.7}$$

In (9.6) the quantity $\kappa\sqrt{\varepsilon} j^H$ is an analog of the linear momentum. Equation (9.7) describes the influence of the external field on electric charge.

If one field is much stronger than the second, then it is possible to neglect the second field change under influence of charge and current from first field. In this case we get the closed system of equations for determination of the charge-current motion of the first field under action of second field:

$$\kappa\nabla^{-}\mathbf{\Theta} = \mathbf{\Theta} \circ \mathbf{A}',$$

where $\mathbf{A}'$ is given. The corresponding A-field is defined by Maxwell equations.

10 First Newton law: Free field

Let us consider A-field, which is generated by $\mathbf{\Theta}$, in the absence of other charges and currents. We call it a *free field*. In this case, $\mathbf{F} = 0$. From (9.1) we get the inertia law, which is analog of *the first Newton law for charge-current field:*

$$\nabla^{-}\mathbf{\Theta} = \mathbf{0}, \tag{10.1}$$

which is equivalent to equations

$$\partial_\tau \rho + \operatorname{div} J = 0, \quad \partial_\tau J - i \operatorname{rot} J + \nabla \rho = 0.$$

For initial notations we have following formulas:

$$\partial_t \rho^E + \operatorname{div} j^E = 0, \quad \partial_\tau j^E = \sqrt{\varepsilon/\mu} \operatorname{rot} j^H - c \operatorname{grad} \rho^E, \tag{10.2}$$

$$\partial_t \rho^H + \operatorname{div} j^H = 0, \quad \partial_\tau j^H = -\sqrt{\mu/\varepsilon} \operatorname{rot} j^E - c \operatorname{grad} \rho^H. \tag{10.3}$$

Consequently, the charge conservation law (2.8) holds in the absence of the external field.

Cauchy problem. For free field, the solution of this problem is given by the following formula:

$$\kappa \mathbf{\Theta}(\tau, x) = \kappa \nabla^- \Big\{ \mathbf{\Theta}_0(x) \underset{x}{*} \psi \Big\} = -\frac{\kappa H(\tau)}{4\pi} \nabla^- \Big\{ \tau^{-1} \int_{r=\tau} \mathbf{\Theta}_0(y) dS(y) \Big\}, \tag{10.4}$$

and tensions of A-field are defined in Section 5.

11 Modified Maxwell equations: Scalar resistance field

Let us consider equation (9.4). Evidently, it is the charge-current conservation law, which contains the power of external acting forces M in the right-hand side. If $M = 0$, then this law has the well-known form (2.8), which we have had for Maxwell equation (see Theorem 2.1).

This means that by EGM field interaction we must enter the scalar part in stress biquaternion:

$$\mathbf{A} = ia(\tau, x) + A(\tau, x).$$

We call $a(\tau, x)$ the A-field *resistance*. From (9.1)–(9.3) it follows that

$$\Box \mathbf{A} = \nabla^- \mathbf{\Theta} = \kappa^{-1} \mathbf{F}.$$

The scalar part of this is

$$\kappa \Box a = iM.$$

Remark 11.1. In system of Maxwell equations (2.1)–(2.2) the first equation defines the current, the second one determines the charge, but the charge conservation law is due to these two equations. It can be obtained if we take divergence in (2.1) with provision for (2.2). However, the biquaternionic approach, as it is shown here, brings to modification of the Maxwell equations, which, in what follows from (9.3), has a following type (*modified Maxwell equations*):

$$J = \operatorname{grad} a - \partial_\tau A - i \operatorname{rot} A, \quad \rho = \operatorname{div} A - \partial_\tau a. \tag{11.1}$$

If ρ and J are known, this system for determining a and A is closed. Only in closed system (in the absence of external field) $a = 0$ and it has the classical type (2.1)–(2.2).

Obviously, by introducing the resistance a, the form of scalar and vector parts of power-force biquaternion (8.1) is changed as follows:

$$\mathbf{F} = \mathbf{\Theta} \circ \mathbf{A}' = \big((A', J) + a' \rho \big) - i(a' J + \rho A') + [A', J]. \tag{11.2}$$

We can see additional summands which appear in the presence of power $(a'\rho)$ and force $(-ia'J)$.

The vector $a'J$ is called a *resistance force* of A'-field. Selecting real and imaginary parts of this vector we get additional summands in expressions for density of electric F^E and gravimagnetic F^H forming F (here $F = F^H + i\,F^E$) in (9.5) with provision for the resistance force of fields, which we must add in right parts of the equations (9.6) and (9.7).

Cauchy problem for equation of transformation. Using formula (4.2), we get

$$\kappa\mathbf{\Theta}(\tau,x) = \nabla^+\Big\{H(\tau)\mathbf{F}(\tau,x) * \psi\Big\} + \mathbf{F}(0,x) \underset{x}{*} \psi + \kappa\nabla^+\Big\{\mathbf{\Theta}(0,x) \underset{x}{*} \psi\Big\}. \tag{11.3}$$

This equation gives the system of integral equations for determining $\mathbf{\Theta}$, as the right-hand side contains $\mathbf{\Theta}$ in $\mathbf{F}$. It can be used for solving the problem if we neglect the second field change. In general case, we write the similar equation for second field $\mathbf{\Theta}'$. These two equations give us the full system of integral equations for determination of charges and currents by their interaction if initial states of fields are known.

The Lorentz transformations of transformation equations (here the sign $'$ means the coordinates in moving coordinate system). According to Theorem 7.2, the Lorentz transformations for $\mathbf{A}$, $\mathbf{\Theta}$, and $\mathbf{F}$ read

$$\mathbf{A}' = \bar{\mathbf{L}}^* \circ \mathbf{A} \circ \mathbf{L}, \quad \mathbf{\Theta}' = \bar{\mathbf{L}}^* \circ \mathbf{\Theta} \circ \mathbf{L}, \quad \mathbf{F}' = \bar{\mathbf{L}}^* \circ \mathbf{F} \circ \mathbf{L}. \tag{11.4}$$

Note that the Lorentz transformation of power-force density at the presence of interaction of two fields of form (8.1) have the same form:

$$\mathbf{F}' = \mathbf{\Theta}_1' \circ \mathbf{A}_2' = \bar{\mathbf{L}}^* \circ \mathbf{\Theta}_1 \circ \mathbf{L} \circ \bar{\mathbf{L}}^* \circ \mathbf{A}_2 \circ \mathbf{L} = \bar{\mathbf{L}}^* \circ \mathbf{\Theta}_1 \circ \mathbf{A}_2 \circ \mathbf{L} = \bar{\mathbf{L}}^* \circ \mathbf{F} \circ \mathbf{L}.$$

For $\varphi = 0$, relations (11.4) are equivalent to equalities (7.1)–(7.2) and

$$\mathbf{F}' = \big(M\cosh 2\theta - (e,F)\sinh 2\theta\big) + i\big\{F + 2e(e,F)\sinh^2\theta - Me\sinh 2\theta\big\}.$$

Relativistic formulas for power-force are

$$M' = \frac{M + v(e,F)}{\sqrt{1-v^2}}, \quad F' = \big(F - e(e,F)\big) + e\frac{(e,F) - vM}{\sqrt{1-v^2}}. \tag{11.5}$$

So, the power also depends on velocity of coordinate system. In initial system it is equal to zero, but in other system it is equal to zero if only external forces are absent ($F = 0$). By this reason the charge conservation law is not postulated in traditional form (2.8) for open systems, which is subjected to external influence.

12 Stress pseudotensor: Equations of EGM-ambiences

The stress pseudotensor may be introduced from formula (9.6):

$$\sigma_{ik}^H = -\kappa\left(\frac{\rho^H}{\sqrt{\mu}}\delta_{ik} + \sqrt{\mu}\, j_l^E e_{ikl}\right), \quad i,k,l = 1,2,3. \tag{12.1}$$

It is analog of the stress tensor of liquid (σ_{ik}). Using this pseudotensor, (9.6) takes form, which looks like equations of hydrodynamics:

$$\frac{\partial \sigma_{ik}^H}{\partial x_k} + F_i^H = \kappa\varepsilon\sqrt{\mu}\,\frac{\partial j_i^H}{\partial t}.$$

Here the second summand on the left-hand side is the density of mass forces:

$$F_i^H = \rho^E E_i' + \rho^H H_i' + j^E \times B_i' - j^H \times D_i' + \mathrm{Re}(a' J).$$

However, there are no traditional index symmetries of the stress tensor: $\sigma_{ik}^H \neq \sigma_{ki}^H$.

Using (9.7), the electric stress pseudotensor may be similarly introduced:

$$\sigma_{ik}^E = -\kappa\left(\frac{\rho^E}{\sqrt{\varepsilon}}\delta_{ik} - \sqrt{\varepsilon}\, j_l^H e_{ikl}\right).$$

By using this equation, (9.7) can be rewritten as

$$\frac{\partial \sigma_{ik}^E}{\partial x_k} + F_i^E = \kappa\mu\sqrt{\varepsilon}\,\frac{\partial j_i^E}{\partial t}. \tag{12.2}$$

Here the second summand on the right-hand side is the density of electric forces:

$$F_i^E = \rho^E B_i' - \rho^H D_i' + c^{-1}\big(E_i' \times j^E + H_i' \times j^H\big) + \mathrm{Im}(a' J).$$

The analog of this formula is unknown to the author.

13 First thermodynamics law

We introduce the energy-impulse density for charge-current field:

$$0,5\boldsymbol{\Theta} \circ \boldsymbol{\Theta}^* = \left(\frac{\|\rho_E\|^2}{\varepsilon} + \frac{\|\rho_H\|^2}{\mu} + Q\right) + i\left(P_J - \sqrt{\frac{\mu}{\varepsilon}}\rho^E j^E - \sqrt{\frac{\varepsilon}{\mu}}\rho^H j^H\right). \tag{13.1}$$

It contains the current of energy density:

$$Q = 0,5\|J\|^2 = 0,5\Big(\mu\big\|j^E\big\|^2 + \varepsilon\big\|j^H\big\|^2\Big),$$

where the first summand includes the Joule heat $\|j^E\|^2$; the second one includes kinetic energy density of mass current $\|j^H\|^2$, it also contains the energy of torsional part of currents (magnetic current). Here vector P_J is analog of the Poynting vector, but for the current,

$$P_J = 0,5i\, J \times \bar{J} = c^{-1}\big[j^H, j^E\big].$$

Only if gravimagnetic and electrical currents are parallel or one of them is equal to zero, then $P_J = 0$. If we take scalar product equation (9.5) with $i\bar{J}$, we get the *charge-current conservation law:*

$$\kappa\big(\partial_\tau Q - \mathrm{div}\, P_J + \mathrm{Re}\,(\nabla\rho, \bar{J})\big) = \mathrm{Im}\,(F, \bar{J}) = c^{-1}\big(\big(F^H, j^H\big) + \big(F^E, j^E\big)\big). \tag{13.2}$$

It is easy to see that this law is like the first thermodynamics law. Here the sum of second and third summands in left part is denoted by $-U$. The function

$$U = \mathrm{div}\, P_J - \sqrt{\mu/\varepsilon}\big(\nabla\rho^E, j^E\big) - \sqrt{\varepsilon/\mu}\big(\nabla\rho^H, j^H\big)$$

characterizes the self-velocity of the change of energy current density of $\boldsymbol{\Theta}$-field. The right-hand side of (13.2), which depends on power of acting external forces, can increase or decrease this velocity.

For the free field, the first thermodynamics law is

$$\partial_\tau Q = U.$$

If we integrate (13.2) on $\{(S^- + S)\times(0,t)\}$ and use the Gauss formula, then the integral representation of this law may be written as

$$\begin{aligned}\int_{S^-}\big(Q(x,t)-Q(x,0)\big)dV(x) &= \int_0^t dt\int_S \big(P_J,n\big)\,dS(x)\\ &- \int_0^t dt\int_{S^-}\big\{\varepsilon^{-1}\big(\nabla\rho^E, j^E\big)+\mu^{-1}\big(\nabla\rho^H, j^H\big)\big\}dV(x)\\ &+ c^{-1}\int_0^t dt\int_{S^-}\big\{\big(F^H, j^H\big)+\big(F^E, j^E\big)\big\}dV(x).\end{aligned}$$

Here $n(x)$ is the unit normal vector to boundary S of the region S^- in space R^3.

14 The total field equations and interaction energy

If there are some (N) interacting fields, generated by different charges and currents, then (9.1) can be written as

$$\kappa\nabla^+\boldsymbol{\Theta}^k + \boldsymbol{\Theta}^k\circ\sum_{m\neq k}\mathbf{A}^m = \mathbf{0},\quad \nabla^+\mathbf{A}^k + \boldsymbol{\Theta}^k = \mathbf{0},\quad k=1,\dots,N, \tag{14.1}$$

$$\nabla^+\mathbf{A}^m\circ\mathbf{A}^k + \nabla^+\mathbf{A}^k\circ\mathbf{A}^m = 0,\quad k\neq m. \tag{14.2}$$

The total field, as is easy to see after summing (14.1) over k, is free because all forces are internal, as also in mechanics of interacting solids.

The interacting fields satisfy the analog of the second Newton law (14.1), (14.2), and for total charge-current there is the equality

$$\nabla^+\boldsymbol{\Theta} = \nabla^+\sum_{m=1}^{M}\boldsymbol{\Theta}^m = \mathbf{0}. \tag{14.3}$$

Let us consider the laws of energy transformation in the case of interaction of different charges and currents. Energy pulse for total charge-current field reads

$$\begin{aligned}\boldsymbol{\Xi}_\Theta &= 0,5\boldsymbol{\Theta}\circ\boldsymbol{\Theta}^* = 0,5\sum_{k=1}^{N}\boldsymbol{\Theta}^k\circ\sum_{l=1}^{N}\boldsymbol{\Theta}^{*l} = 0,5\left(\sum_{k=1}^{N}\boldsymbol{\Theta}^k\circ\boldsymbol{\Theta}^{*k} + \sum_{k\neq l}\boldsymbol{\Theta}^k\circ\boldsymbol{\Theta}^{*l}\right)\\ &= \sum_{k=1}^{N}W_\Theta^{(k)} + i\sum_{k=1}^{N}P_\Theta^{(k)} + \delta\boldsymbol{\Xi}_\Theta.\end{aligned}$$

Here the first summand is an amount of energy pulse of interacting charge and current.

We can introduce the biquaternion of *energy-pulse interaction*. Its real part describes energy-pulse interaction for the same charge and current, but in the imaginary part for different ones,

$$\delta\boldsymbol{\Xi}_\Theta = \delta W_\Theta + i\delta P_\Theta = \sum_{k\neq l}\boldsymbol{\Xi}_\Theta^{kl},\quad \boldsymbol{\Xi}_\Theta^{kl} = 0,5\big(\boldsymbol{\Theta}^k\circ\boldsymbol{\Theta}^{*l} + \boldsymbol{\Theta}^l\circ\boldsymbol{\Theta}^{*k}\big),$$

$$\boldsymbol{\Xi}_\Theta^{kl} = \operatorname{Re}\big(\rho^k\rho^{*l} + \big(J^k, J^{*l}\big)\big) - i\big\{\operatorname{Re}\big(\rho^k J^{*l} + \rho^{*l}J^k\big) + \operatorname{Im}\big[J^k, J^{*l}\big]\big\},$$

or in initial notations,

$$\Xi_{\Theta}^{kl} = \frac{\rho^{E(k)}\rho^{E(l)}}{\sqrt{\varepsilon_k\varepsilon_l}} + \frac{\rho^{(k)H}\rho^{H(l)}}{\sqrt{\mu_k\mu_l}} + \sqrt{\mu_k\mu_l}\big(j^{(k)E}, j^{(l)E}\big) + \sqrt{\varepsilon_k\varepsilon_l}\big(j^{(k)H}, j^{(l)H}\big)$$
$$- i\Bigg\{\sqrt{\frac{\mu_l}{\varepsilon_k}}\rho^{(k)E}j^{(l)E} + \sqrt{\frac{\varepsilon_l}{\mu_k}}\rho^{(k)H}j^{(l)H} + \sqrt{\frac{\mu_k}{\varepsilon_l}}\rho^{(l)E}j^{(k)E} + \sqrt{\frac{\varepsilon_k}{\mu_l}}\rho^{(l)H}j^{(k)H}$$
$$- \sqrt{\varepsilon_k\mu_l}\Big[j^{(l)E}, j^{(k)H}\Big] + \sqrt{\varepsilon_l\mu_k}\Big[j^{(k)E}, j^{(l)H}\Big]\Bigg\}.$$

As a result we get the conditions of energy transformation in the case of charge-current interaction: energy *separation* if $\delta W_\Theta > 0$, energy *absorption* if $\delta W_\Theta < 0$, and energy *conservation* if $\delta \Xi_\Theta = 0$.

15 Conclusion

We considered a model of EGM field (called as *A-field*), which is founded on hypothesis on magnetic charge = mass that has allowed to call such a field *electro-gravimagnetic.*

We used Maxwell equations in biquaternional form and constructed new biquaternional equations for description of charges and currents changing by their interaction. We called these equations an *analog of Newton's laws.*

Investigation of invariance of these equations with respect to Lorentz transformation showed that it is necessary to enter the scalar field of *resistance* $a(\tau, x)$ in scalar part of biquaternion of EGM-field tension. One has to modify the Maxwell equations in the case when charge and current are subjected to influence by the external field. We call these the *modified Maxwell equations for open system.*

When constructing the equation to charge-current transformations aside from the known gravitational and electromagnetic forces, we found the presence of the new forces, which is needed in experimental motivation. Some suggestions on this cause were presented in [3, 4], where this model was offered at first, but with the charge-current conservation law in traditional form. But this is true only for closed system. As it is shown here, for open system we must take into account the power of external forces, which changes the form of this law.

Note also that when considering the model of EGM field it is essential the using of the algebra of biquaternions. Without biquaternions the construction of the differential equations, describing interaction of charge and current in such form, would be practically impossible.

References

[1] L. A. Alexeyeva. Hamiltonian form of the Maxwell equations and its generalized solutions. *Differ. Equ.*, **39** (2003), 807–816.

[2] L. A. Alexeyeva. Quaternions of the Hamiltonian form of Maxwell equations. *Mat. Zh.*, **3** (2003), 20–24.

[3] L. A. Alexeyeva. On a model of an electrogravimagnetic field. Field-interaction equations and conservation laws. *Mat. Zh.*, **4** (2004), 23–34.

[4] L. A. Alexeyeva. Equations of the interaction of A-fields, and Newton's laws. *Izv. Nats. Akad. Nauk Resp. Kaz. Ser. Fiz.-Mat.*, **2004** (2004), 10–19.

[5] P. Rastall. Quaternions in relativity. *Rev. Modern Phys.*, **36** (1964), 820–832.

[6] A. P. Yefremov. Quaternions: algebra, geometry and physical theories. *Hypercomplex Numbers in Geometry and Physics*, **1** (2004), 104–119.

[7] G. Casanova. *Vector Algebra* (in Russian). Mir, Moscow, 1979.

[8] V. V. Kassandrov. Biquaternion electrodynamics and Weyl-Cartan geometry of space-time. *Gravit. Cosmol.*, **1** (1995), 216–222.

[9] V. S. Vladimirov. *Generalized Functions in Mathematical Physics* (in Russian). Nauka, Moscow, 1976.

Received June 25, 2009
Revised July 16, 2009