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New Types of 2D-Integrodifferential Equations and Their Properties

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Abstract

In this paper, we present new type of 2D-volttera integrodifferential equations and study the existence, uniqueness and some other useful properties of solution of these equations. The main tools are based on application of the Banach fixed point theorem.

Keywords: 2D-Volttera nonlinear integrodifferential; ε-Approximate solution

Introduction

Consider the nonlinear integrodifferential equation

$$D_2D_1\, u(x,y) = f(x,y,u(x,y),Gu(x,y),Hu(x,y)) \tag{1.1}$$

with given data

$$u(x,0)=\sigma(x)\ ,\ u(0,y)=\tau(y) \tag{1.2}$$

for $x,y \in \mathbb{R}_+$, where

$$Gu(x,y)=\int_0^x g(x,y,\xi)u(\xi,y)d\xi$$

$$Hu(x,y)=\int_0^x\int_0^y h(x,y,m,n,u(m,n))\,dn\,dm \tag{1.3}$$

and

$f \in C(E\times\mathbb{R}^3,\mathbb{R})$, $g\in C(E\times\mathbb{R}_+\times\mathbb{R},\mathbb{R})$, $h\in C(E^2\times\mathbb{R},\mathbb{R})$, $\sigma\in C(\mathbb{R}_+,\mathbb{R})$,

$\tau\in C(\mathbb{R}_+,\mathbb{R})$ that $E=\mathbb{R}_+\times\mathbb{R}_+$.

Properties of Solution

Let S be the space of functions $z, D_2D_1z \in C(E,\mathbb{R})$ which fulfill the condition [1]

$$|z(x,y)| = o(\exp(\lambda(x+y))) \tag{2.1}$$

For (x,y ∈R), where $\lambda > 0$ is a constant. This space with the norm

$$|z|_s = \sup_{(x,y)\in E}[|z(x,y)|\exp(-\lambda(x+y))] \tag{2.2}$$

is a Banach space.

We note that the condition (2.2) implies that there exists a constant $N \geq 2$ such that

$$|z(x,y)| \leq N\exp(\lambda(x+y)) \tag{2.3}$$

then

$$|z|_s = \sup_{(x,y)\in E}[|z(x,y)|\exp(-\lambda(x+y))]$$

$$\leq \sup_{(x,y)\in E}[N\exp(\lambda(x+y)\exp(-\lambda(x+y))]$$

$$\text{So } z_s \leq N \tag{2.4}$$

Theorem 2.1

Assume that

Functions f, g, h in equation (1.1) satisfy the conditions [2]

$$|f(x,y,u,v,w)-f(x,y,\bar u,\bar v,\bar w)| \leq k(x,y)[|u-\bar u|+|v-\bar v|+|w-\bar w|] \tag{2.5}$$

$$|g(x,y,\xi,u)-g(x,y,\xi,\bar u)| \leq a(x,y,\xi)|u-\bar u| \tag{2.6}$$

$$|h(x,y,m,n,u)-h(x,y,m,n,\bar u)| \leq b(x,y,m,n)|u-\bar u| \tag{2.7}$$

where $k\in C(E,\mathbb{R})$, $a\in C(E\times\mathbb{R}_+,\mathbb{R})$, $b\in C(E^2,\mathbb{R})$.

(i) For λ as in (1.2)

C_1 there exists a nonnegative constant α such that α<1 and

$$\int_0^x\int_0^y k(s,t)[\exp(\lambda(x+y))+\int_0^s a(s,t,\xi)\exp(\lambda(\xi+t))d\xi + \int_0^s\int_0^t b(s,t,m,n)\exp(\lambda(m+n))\,dn\,dm]dt\,ds \leq \alpha\exp(\lambda(x+y)) \tag{2.8}$$

C_2 there exists a nonnegative constant β such that

$$\left|\sigma(x)+\tau(y)-u(0,0)+\int_0^x\int_0^y f(s,t,0,G0,H0)dt\,ds\right| \leq \beta\exp(\lambda(x+y)) \tag{2.9}$$

and $G0=\int_0^x g(x,y,\xi,0)\,d\xi$, $H0=\int_0^x\int_0^y h(x,y,m,n,0)\,dndm$

Then equation (1.1) has a unique [3] solution u(x,y) on E in S.

Proof:

Note that by take integration of (1.1), we have

$$u(x,y)=\sigma(x)+\tau(y)-u(0,0)+\int_0^x\int_0^y f(s,t,u(s,t),Gu(s,t),Hu(s,t))dt\,ds \tag{2.10}$$

Let $u(x,y)\in S$ u(x,y)∈S and define the operator T by

$$(Tu)(x,y)=\sigma(x)+\tau(y)-u(0,0)+\int_0^x\int_0^y f(s,t,u(s,t),Gu(s,t),Hu(s,t))dt\,ds \tag{2.11}$$

Now we shall show that T maps S into itself. Evidently, Tu is continuous on E because σ,τ,u,f are continuous and Tu∈R.

From (2.11), we observe that

$$D_2D_1\,(Tu)(x,y)=f(x,y,u(x,y),Gu(x,y),Hu(x,y))$$

So D_1D_2 Tu is continuous and D_1D_2 Tu∈R. From (2.11) and using the hypotheses and (2.4) we have

$$|Tu(x,y)| \leq \left|\sigma(x)+\tau(y)-u(0,0)+\int_0^x\int_0^y f(s,t,0,G0,H0)dt\,ds\right|$$

$$+\int_0^x\int_0^y|f(s,t,u(s,t),Gu(s,t),Hu(s,t)-f(s,t,0,G0,H0)|dt\,ds$$

$$\leq\beta\exp(\lambda(x+y))+\int_0^x\int_0^y k(s,t)[|u(s,t)|+|Gu(s,t)-G0|+|Hu(s,t)-H0|]dt\,ds$$

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$$\le \beta\exp(\lambda(x+y)) + \int_0^x\int_0^y k(s,t)[|u(s,t)| + \int_0^s |g(s,t,\xi,u(\xi,t)) - g(s,t,\xi,0)| d\xi + \int_0^s\int_0^t |h(s,t,m,n,u(m,n)) - h(s,t,m,n,0)| dn\, dm] dt\, ds$$

$$\le \beta\exp(\lambda(x+y)) + \int_0^x\int_0^y k(s,t)[|u(s,t)| + \int_0^s a(s,t,\xi)|u(s,t)| d\xi + \int_0^s\int_0^t b(s,t,m,n)|u(m,n)| dn\, dm] dt\, ds$$

$$\le \beta\exp(\lambda(x+y)) + |u|_s \int_0^x\int_0^y k(s,t)[\exp(\lambda(s+t)) + \int_0^s a(s,t,\xi)\exp(\lambda(\xi+t)) d\xi + \int_0^s\int_0^t b(s,t,m,n)\exp(\lambda(m+n)) dn\, dm] dt\, ds$$

$$\le \beta\exp(\lambda(x+y)) + |u|_s\, \alpha \exp(\lambda(x+y))$$

$$\le (\beta + N\alpha)\exp(\lambda(x+y)) \qquad (2.12)$$

From (2.12), it follows that Tu∈S. This proves that the operator T maps S into itself. Now, we prove that the operator T is a contraction map.

Let u,v∈S, ,from (2.11) and using the hypothesis, we have

$$|Tu(x,y) - Tv(x,y)| \le \int_0^x\int_0^y |f(s,t,u(s,t),Gu(s,t),Hu(s,t)) - f(s,t,v(s,t),Gv(s,t),Hv(s,t))| dt\, ds$$

$$\le \int_0^x\int_0^y k(s,t)[|u(s,t)-v(s,t)| + |Gu(s,t) - Gv(s,t)| + |Hu(s,t) - Hv(s,t)|] dt\, ds$$

$$\le \int_0^x\int_0^y k(s,t)[|u(s,t)-v(s,t)| + \int_0^s |g(s,t,\xi,u(\xi,t)) - g(s,t,\xi,v(s,\xi))| d\xi + \int_0^s\int_0^t |h(s,t,m,n,u(m,n)) - h(s,t,m,n,v(m,n))| dn\, dm] dt\, ds$$

$$\le \int_0^x\int_0^y k(s,t)[|u(s,t)-v(s,t)| + \int_0^s a(s,t,\xi)|u(s,t)-v(s,t)| d\xi + \int_0^s\int_0^t b(s,t,m,n)|u(m,n)-v(m,n)| dn\, dm] dt\, ds$$

$$\le |u-v|_s \int_0^x\int_0^y k(s,t)[\exp(\lambda(s+t)) + \int_0^s a(s,t,\xi)\exp(\lambda(\xi+t)) d\xi + \int_0^s\int_0^t b(s,t,m,n)\exp(\lambda(m+n)) dn\, dm] dt\, ds$$

$$\le |u-v|_s\, \alpha \exp(\lambda(x+y))$$

So $|Tu - Tv| \le |u-v|_s\, \alpha \exp(\lambda(x+y))$ then $|Tu - Tv|_s \le \alpha\, |u-v|_s$.

Since α<1, it follows from Banach fixed point theorem that T has a unique fixed point in S. The fixed point of T is however a solution of equation (1.1).

The proof is complete.

Some Results

Lemma: Let $u \in C(E,\mathbb{R}_+)$ and c ≥ 0. If

$$u(x,t) \le c + \int_0^x\int_0^s\int_0^t r(\sigma,\tau)u(\sigma,\tau) d\tau\, d\sigma\, ds$$

for $(x,t) \in E$, then

$$u(x,t) \le c\exp(\int_0^x\int_0^s\int_0^t r(\sigma,\tau)\, d\tau\, d\sigma\, ds)$$

for $(x,t) \in E$.

Proof. [1, p. 60]

Theorem 3.1

Suppose that the functions f, g, h in (1.1) satisfy the conditions

$$|f(x,y,u,v,w) - f(x,y,\bar{u},\bar{v},\bar{w})| \le k(x,y)[|u-\bar{u}| + |v-\bar{v}|] + |w-\bar{w}|$$

$$|g(x,y,\xi,u) - g(x,y,\xi,\bar{u})| \le a(\xi,y)|u-\bar{u}|$$

$$|h(x,y,m,n,u) - h(x,y,m,n,\bar{u})| \le b(x,y,m,n)|u-\bar{u}|$$

where $k \in C(E,\mathbb{R}_+)$, $a \in C(E,\mathbb{R}_+)$, $b \in C(E^2,\mathbb{R}_+)$ and

$$c = \sup_{(x,y)\in\mathbb{R}_+} \left|\sigma(x) + \tau(y) - u(0,0) + \int_0^x\int_0^y f(s,t,0,G0,H0) dt\, ds\right|$$

If $u(x,y)$ is any solution of equation (1.1) on E, then

$$u(x,y) \le c\, A(x,y)\exp(\int_0^x\int_0^y A(s,t)[k(s,t) + b(x,y,s,t)] dt\, ds)$$

where $A(x,y) = \exp(\int_0^x\int_0^y\int_0^s a(\xi,t) d\xi\, dt\, ds)$.

Proof:

We know

$$|u(x,y)| \le |\sigma(x) + \tau(y) - u(0,0) + \int_0^x\int_0^y f(s,t,0,G0,H0) dt\, ds| + \int_0^x\int_0^y |f(s,t,u(s,t),Gu(s,t),Hu(s,t)) - f(s,t,0,G0,H0)| dt\, ds$$

$$\le c + \int_0^x\int_0^y (|Gu(s,t) - G0| + k(s,t)[|u(s,t)| + |Hu(s,t) - H0|])\, dt\, ds$$

$$\le c + \int_0^x\int_0^y (\int_0^s a(s,t,\xi)|u(s,t)| d\xi + k(s,t)[|u(s,t)| + \int_0^s\int_0^t b(s,t,m,n)|u(m,n)| dn\, dm]) dt\, ds \qquad (3.1)$$

Let c > 0 and define a function z(x,y) by

$$z(x,y) = c + \int_0^x\int_0^y (k(s,t)[|u(s,t)| + \int_0^s\int_0^t b(s,t,m,n)|u(m,n)| dn\, dm]) dt\, ds$$

Then (3.1) can be restated as

$$|u(x,y)| \le z(x,y) + \int_0^x\int_0^y\int_0^s a(s,t,\xi)|u(s,t)| d\xi\, dt\, ds$$

It is easy to observe that z(x,y) is positive, continuous and non decreasing function and using lemma, we get

$$|u(x,y)| \le z(x,y)\exp(\int_0^x\int_0^y\int_0^s a(s,t,\xi) d\xi\, dt\, ds) = z(x,y)A(x,y) \qquad (3.2)$$

We observe that

$$z(x,y) \le c + \int_0^x\int_0^y k(s,t)[z(s,t)A(s,t) + \int_0^s\int_0^t b(s,t,m,n)z(m,n)A(m,n) dn\, dm]) dt\, ds \qquad (3.3)$$

Define a function v(x,y) by the right hand side of (3.3). Then v(x,y) > 0, v(x,0)= v(0,y)=c and z(x,y) ≤ v(x,y) and

$$D_2D_1v(x,y) = k(x,y)[z(x,y)A(x,y) + \int_0^x\int_0^y b(x,y,m,n)z(m,n)A(m,n) dn\, dm]$$

$$D_2D_1v(x,y) \le k(x,y)[v(x,y)A(x,y) + \int_0^x\int_0^y b(x,y,m,n)v(m,n)A(m,n) dn\, dm]$$

$$\le k(x,y)A(x,y)[v(x,y) + \int_0^x\int_0^y b(x,y,m,n)v(m,n)A(m,n) dn\, dm]$$

Define a function w(x,y) by

$$w(x,y) = v(x,y) + \int_0^x\int_0^y b(x,y,m,n)v(m,n)A(m,n) dn\, dm$$

Then v(x,y) ≤ w(x,y) > 0, we have

$$D_2D_1 w(x,y) = D_2D_1v(x,y) + b(x,y,x,y)v(x,y)A(x,y)$$

$$\le k(x,y)A(x,y)w(x,y) + b(x,y,x,y)w(x,y)A(x,y)$$

$$\le A(x,y)w(x,y)[b(x,y,x,y) + k(x,y)]$$

Because w(x,y) ≥ v(x,y) > 0 $w(x,y) \ge v(x,y) > 0$ then

$$\frac{D_2D_1w(x,y)}{w(x,y)} \le A(x,y)[k(x,y) + b(x,y,x,y)]$$

$$\frac{D_1w(x,y)}{w(x,y)} - \frac{D_1w(x,0)}{w(x,0)} \le \int_0^y A(x,t)[k(x,t) + b(x,y,x,t)] dt$$

$$\ln w(x,y) - \ln w(0,y) \le \int_0^x\int_0^y A(s,t)[k(s,t) + b(x,y,s,t)] dt\, ds$$

$$\ln w(x,y) \le \ln c + \int_0^x\int_0^y A(s,t)\,[k(s,t)+b(x,y,s,t)]\,dt\,ds$$

$$w(x,y) \le c\,\exp\left(\int_0^x\int_0^y A(s,t)\,[k(s,t)+b(x,y,s,t)]\,dt\,ds\right)$$

Using z(x,y) ≤ v(x,y) ≤ w(x,y) and (3.2), we get

$$|u(x,y)| \le c\,A(x,y)\exp\left(\int_0^x\int_0^y A(s,t)\,[k(s,t)+b(x,y,s,t)]\,dt\,ds\right)$$

The proof is complete.▪

We call the function $u\in C(E,\mathbb{R}_+)$ an ε-approximate solution to equation (1.1), if there exists a constant $\varepsilon \ge 0$ such that $D_2D_1u(x,y)$ exists and satisfies the inequality

$$|D_2D_1\,u(x,y) - f(x,y,u(x,y),Gu(x,y),Hu(x,y))| \le \varepsilon \qquad (3.4)$$

The following theorem deals with the estimate on the difference between two approximate solutions of equation (1.1).

Theorem 3.3

Assume that functions f, g, h in equation (1.1) satisfy the conditions

$$|f(x,y,u,v,w) - f(x,y,\bar{u},\bar{v},\bar{w})| \le k(x,y)[|u-\bar{u}| + |v-\bar{v}|] + |w-\bar{w}| \qquad (3.5)$$

$$|g(x,y,\xi,u) - g(x,y,\xi,\bar{u})| \le a(\xi,y)|u-\bar{u}| \qquad (3.6)$$

$$|h(x,y,m,n,u) - h(x,y,m,n,\bar{u})| \le b(x,y,m,n)|u-\bar{u}| \qquad (3.7)$$

that $k,e,q,r \in C(E,\mathbb{R}_+)$.

Let $u_1(x,y), u_2(x,y)$ be respectively ε_1 - and ε_2 - approximate solutions of equation (1.1) on E that

$$u_i(x,0) = \sigma_i(x)\ ,\ u_i(0,y) = \tau_i(y)\quad i=1,2 \qquad (3.8)$$

$$|\sigma_1(x) - \sigma_2(x) + \tau_1(y) - \tau_2(y) + u_1(0,0) - u_2(0,0)| \le \delta \qquad (3.9)$$

That δ ≥ 0 is constant and

$$c = \sup_{(x,y)\in E}\{\varepsilon xy + \delta\} \qquad (3.10)$$

$$p(x,y) = \max\{k(x,y), k(x,y)e(x,y), k(x,y)q(x,y)\} \qquad (3.11)$$

$$\varepsilon = \varepsilon_1 + \varepsilon_2 \qquad (3.12)$$

Then

$$|u_1(x,y) - u_2(x,y)| \le c\,A(x,y)\,\exp\left(\int_0^x\int_0^y A(s,t)[k(s,t)+b(x,y,s,t)]\,dt\,ds\right)$$

where $A(x,y) = \exp\left(\int_0^x\int_0^y\int_0^s a(\xi,t)\,d\xi\,dt\,ds\right)$.

Proof.

Since $u_i(x,y)$ is ε_i - approximate solution to equation (1.1), we have

$$|D_2D_1\,u_i(x,y) - f(x,y,u_i(x,y),Gu_i(x,y),Hu_i(x,y))| \le \varepsilon_i \qquad i=1,2 \quad (3.13)$$

with integration of (3.13) and using the elementary inequalities $|v-z| \le |v| + |z|$, $|v| - |z| \le |v-z|$ we get

$$\varepsilon_i xy \ge \int_0^x\int_0^y |D_2D_1\,u_i(s,t) - f(s,t,u_i(s,t),Gu_i(s,t),Hu_i(s,t))|\,dt\,ds$$

$$\ge \left|\int_0^x\int_0^y (D_2D_1\,u_i(s,t) - f(s,t,u_i(s,t),Gu_i(s,t),Hu_i(s,t)))\,dt\,ds\right|$$

$$= \left|u_i(x,y) - \sigma_i(x) - \tau_i(y) + \sigma_i(0) - \int_0^x\int_0^y f(s,t,u_i(s,t),Gu_i(s,t),Hu_i(s,t))\,dt\,ds\right|$$

i=1,2. We observe that

$$(\varepsilon_1+\varepsilon_2)xy \ge \left|u_1(x,y) - \sigma_1(x) - \tau_1(y) + \sigma_1(0) - \int_0^x\int_0^y f(s,t,u_1(s,t),Gu_1(s,t),Hu_1(s,t))\,dt\,ds\right|$$

$$+\left|u_2(x,y) - \sigma_2(x) - \tau_2(y) + \sigma_2(0) - \int_0^x\int_0^y f(s,t,u_2(s,t),Gu_2(s,t),Hu_2(s,t))\,dt\,ds\right|$$

$$\ge |\{u_1(x,y) - \sigma_1(x) - \tau_1(y) + \sigma_1(0) - \int_0^x\int_0^y f(s,t,u_1(s,t),Gu_1(s,t),Hu_1(s,t))\,dt\,ds\}$$

$$-\{u_2(x,y) - \sigma_2(x) - \tau_2(y) + \sigma_2(0) - \int_0^x\int_0^y f(s,t,u_2(s,t),Gu_2(s,t),Hu_2(s,t))\,dt\,ds\}|$$

$$\ge |u_1(x,y) - u_2(x,y)| - |\sigma_1(x) + \tau_1(y) - \sigma_1(0) - \sigma_2(x) - \tau_2(y) + \sigma_2(0)|$$

$$-\left|\int_0^x\int_0^y \{f(s,t,u_1(s,t),Gu_1(s,t),Hu_1(s,t)) - f(s,t,u_2(s,t),Gu_2(s,t),Hu_2(s,t))\}\,dt\,ds\right|$$

now using (3.9), (3.10), (3.11), (3.12) we have

$$|u_1(x,y) - u_2(x,y)| \le \varepsilon xy + \delta + \int_0^x\int_0^y |f(s,t,u_1(s,t),Gu_1(s,t),Hu_1(s,t)) - f(s,t,u_2(s,t),Gu_2(s,t),Hu_2(s,t))|\,dt\,ds$$

$$\le \varepsilon xy + \delta + \int_0^x\int_0^y k(s,t)[|u_1(s,t)-u_2(s,t)| + |Gu_1(s,t)-Gu_2(s,t)| + |Hu_1(s,t)-Hu_2(s,t)|]\,dt\,ds$$

$$\le c + \int_0^x\int_0^y \Big(k(s,t)[|u_1(s,t)-u_2(s,t)| + \int_0^s\int_0^t |h(s,t,m,n,u_1(m,n)) - h(s,t,m,n,u_2(m,n))|\,dm\,dn] + \int_0^s |g(s,t,\xi,u_1(\xi,t)) - g(s,t,\xi,u_2(\xi,t))|\,d\xi\Big)\,dt\,ds$$

$$\le c + \int_0^x\int_0^y \Big(k(s,t)[|u_1(s,t)-u_2(s,t)| + \int_0^s\int_0^t q(s,t)r(m,n)|u_1(m,n)-u_2(m,n)|\,dm\,dn] + \int_0^s e(\xi,t)|u_1(\xi,t)-u_2(\xi,t)|\,d\xi\Big)\,dt\,ds$$

By theorem 3.1 we have

$$|u_1(x,y) - u_2(x,y)| \le c\,A(x,y)\,\exp\left(\int_0^x\int_0^y A(s,t)[k(s,t)+b(x,y,s,t)]\,dt\,ds\right)$$

that $A(x,y) = \exp\left(\int_0^x\int_0^y\int_0^s e(s,t,\xi)\,d\xi\,dt\,ds\right)$

The proof is complete.

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