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New Type of Riesz Sequence Space of Non-absolute Type

Ganie AH[1]*, Ahmad M[2], Sheikh NA[3] and Jalal T[3]

[1]*Department of Applied Sciences and Humanities, SSM College of Engineering and Technology Kashmir, India*
[2]*Department of Mathematics, Faculty of Sciences Jazan University, Saudi Arabia*
[3]*Department of Mathematics, National Institute of Technology, Srinagar, India*

Abstract

The aim of this paper is to introduce the space $r_\infty^q(u,p)$, we show its completeness property, show that the spaces $r_\infty^q(u,p)$, are linearly isomorphic to the spaces $l_\infty(p)$, respectively and compute their α-, β- and γ- duals.

Keywords: Sequence space of non-absolute type; Paranormed sequence space; α-, β- and γ-duals

AMS Mathematical subject classification: 2010; 46A45; 40C05

Preliminaries, Background and Notation

We denote the set of all sequences with complex terms by ω. It is a routine verification that ω is a linear space with respect to the co-ordinate wise addition and scalar multiplication of sequences which are defined, as usual, by

$$x+y=(x_k)+(y_k)=(x_k+y_k)$$

and

$$\alpha x=\alpha(x_k)=(\alpha x_k),$$

respectively; where $x=(x_k)$, $y=(y_k)$ ∈ ω and α=C. By a sequence space we understand a linear subspace of ω i.e., the sequence space is the set of scalar sequences(real or complex) which is closed under co-ordinate wise addition and scalar multiplication. Throughout the paper N, R and C denotes the set of non-negative integers, the set of real numbers and the set of complex numbers, respectively. Let l_∞, c and c_0, respectively, denotes the space of all bounded sequences , the space of convergent sequences and the sequences converging to zero. Also, by l_1, l(p), cs and bs we denote the spaces of all absolutely, p-absolutely convergent, convergent and bounded series, respectively.

Let X, Y be two sequence spaces and let $A=(a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where n, k ∈ N. Then, the matrix A defines the A-transformation from X into Y, if for every sequence $x=(x_k)$ ∈ X the sequence $Ax=\{(Ax)_n\}$, the A-transform of x exists and is in Y; where $(Ax)_n=\sum_k a_{nk}x_k$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. By A ∈ (X:Y) we mean the characterizations of matrices from X to Y i.e., A:X →Y. A sequence x is said to be A-summable to l if Ax converges to l which is called as the A-limit of x.

For a sequence space X, the matrix domain X_A of an infinite matrix A is defined as

$$X_A=\{x=(x_k):Ax\in X\}. \quad (1)$$

The theory of matrix transformations is a wide field in summability; it deals with the characterizations of classes of matrix mappings between sequence spaces by giving necessary and sufficient conditions on the entries of the infinite matrices.

The classical summability theory deals with a generalization of convergence of sequences and series. One original idea was to assign a limit to divergent sequences or series. Toeplitz [1] was the first to study summability methods as a class of transformations of complex sequences by complex infinite matrices.

Let $A=(a_{nk})$ be any matrix. Then a sequence x is said to be summable to l, written $x_k \to l$, if and only if $A_n x=\sum_k a_{nk}x_k$ exists for each n and $A_n x \to l$ (n → ∞). For example, if I is the unit matrix, then $x_k \to l(I)$ means precisely that $x_k \to l$ (k → ∞), in the ordinary sense of convergence.

We denote by (A) the set of all sequences which are summable A. The set (A) is called summability field of the matrix A. Thus, if $Ax=(a_n(x))$, then (A)={x:Ax ∈ c}, where c is the set of convergent sequences. For example, (I)=c.

A infinite matrix $A=(a_{nk})$ is said to be regular [2] if and only if the following conditions hold:

(i) $\lim_{n\to\infty}\sum_{k=0}^{\infty}a_{nk}=1,$

(ii) $\lim_{n\to\infty}a_{nk}=0,\ (k=0,1,2,\ldots),$

(iii) $\sum_{k=0}^{\infty}|a_{nk}|<M,\ (M>0,n=0,1,2,\ldots).$

Let (q_k) be a sequence of positive numbers and let us write, $Q_n=\sum_{k=0}^{n}q_k$ for all n ∈ ℕ. Then the matrix $R_u^q=(r_{nk}^q)$ of the Riesz mean (R,q_n) is given by

$$r_{nk}^q=\begin{cases}\dfrac{u_k q_k}{Q_n}, & if\ \ 0\le k\le n,\\ 0 & if\ \ k>n\end{cases}$$

The Riesz mean (R,q_n) is regular if and only if $Q_n \to \infty$ as n → ∞ [2].

The approach of constructing a new sequence space by means of matrix domain of a particular limitation method has been studied by

***Corresponding author:** Ganie AH, Department of Applied Sciences and Humanities, SSM College of Engineering and Technology Kashmir, India, E-mail: ashamidg@rediffmail.com

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several authors [3-17].

The Riesz Sequence Space $r_\infty^q(u,p)$ of Non-absolute Type

In the present section, we introduce Riesz sequence space $r_\infty^q(u,p)$, prove that these spaces are complete paranormed linear space and show that the $r_\infty^q(u,p)$ are linearly isomorphic to the space $l_\infty(p)$. We also compute α-, β- and γ- duals of these spaces. Finally, we give basis for the spaces $r_\infty^q(u,p)$, where $u=(u_k)$ is a sequence such that $u_k \neq 0$ for all k ∈ ℕ.

A linear Topological space X over the field of real numbers R is said to be a paranormed space if there is a sub-additive function h: X → R such that h(θ)=0, h(-x)=h(x) and scalar multiplication is continuous, that is, $|\alpha_n - \alpha| \to 0$ and $h(x_n - x) \to 0$ imply $h(a_n x_n - ax) \to 0$ for all α's in R and x's in X, where θ is a zero vector in the linear space X. Assume here and after that (p_k) be a bounded sequence of strictly positive real numbers with $\sup_k p_k = H$ and M=max{1,H}. Then, the linear spaces l(p) and $l_\infty(p)$ were defined by Maddox [8] and [18] as follows :

$$l_\infty(p) = \{x=(x_k) : \sup_k |x_k|^{p_k} < \infty\}$$

which is complete spaces paranormed by

$$h_1(x) = \sup_k |x_k|^{p_k/M}$$

if inf $p_k > 0$.

We shall assume throughout that $p_k^{-1} + \{p_k^{'}\}^{-1}$ provided $1 < inf p_k \le H < \infty$ and we denote the collection of all finite subsets of N by F, where N={0,1,2,…}.

Following Basar and Altay [3], Choudhary and Mishra [4], Edermann [5], Mursaleen et al. [12-14], Neyaz and Hamid [15] we define the spaces $r_\infty^q(u,p)$ as the set of all sequences whose R_u^q -transform is in the spaces $l_\infty(p)$ i.e.,

$$r_\infty^q(u,p) = \left\{ x=(x_k) \in \omega : \sup_k \left| \frac{1}{Q_k} \sum_{j=0}^{k} u_j q_j x_j \right|^{p_k} < \infty \right\},$$

With the notation of (1) that

$$r_\infty^q(u,p) = \{l_\infty(p)\}_{R_u^q}. \tag{2}$$

Define the sequence $y=(y_k)$, which will be used, by the R_u^q -transform of a sequence $x=(x_k)$, i.e.,

$$y_k = \frac{1}{Q_k} \sum_{j=0}^{k} u_j q_j x_j \text{ for } k \in \mathbb{N}. \tag{3}$$

Now, we begin with the following theorem which is essential in the text.

Theorem 1: The spaces $r_\infty^q(u,p)$ are complete linear metric space paranormed by g defined

$$g(x) = \sup_k \left| \frac{1}{Q_k} \sum_{j=0}^{k} u_j q_j x_j \right|^{\frac{p_k}{M}}.$$

Proof: The linearity of $r_\infty^q(u,p)$ with respect to the co-ordinate wise addition and scalar multiplication follows from the inequalities which are satisfied for $z, x \in r_\infty^q(u,p)$ [8].

$$\sup_k \left| \frac{1}{Q_k} \sum_{j=0}^{k} u_j q_j (z_j + x_j) \right|^{\frac{p_k}{M}}$$

$$\le \sup_k \left| \frac{1}{Q_k} \sum_{j=0}^{k} u_j q_j z_j \right|^{\frac{p_k}{M}} + \sup_k \left| \frac{1}{Q_k} \sum_{j=0}^{k} u_j q_j x_j \right|^{\frac{p_k}{M}} \tag{4}$$

and for any α ∈ R [9],

$$|\alpha|^{p_k} \le max\{1, |\alpha|^M\}. \tag{5}$$

It is clear that, g(θ)=0 and g(x)=g(-x) for all $x \in r_\infty^q(u,p)$. Again the inequality (4) and (5), yield the subadditivity of g and

$$g(\alpha x) \le max\{1, |\alpha|\} g(x).$$

Let $\{x^n\}$ be any sequence of points of the space $r_\infty^q(u,p)$ such that $g(x^n - x) \to 0$ and (α_n) is a sequence of scalars such that $\alpha_n \to \alpha$. Then, since the inequality,

$$g(x^n) \le g(x) + g(x^n - x)$$

holds by subadditivity of g, $\{g(x^n)\}$ is bounded and we thus have

$$g(\alpha_n x^n - \alpha x) = \sup_k \left| \frac{1}{Q_k} \sum_{j=0}^{k} u_j q_j (\alpha_n x_j^n - \alpha x_j) \right|^{\frac{p_k}{M}}$$

$$\le |\alpha_n - \alpha| g(x^n) + |\alpha| g(x^n - x)$$

which tends to zero as n → ∞. That is to say that the scalar multiplication is continuous. Hence, g is paranorm on the space $r_\infty^q(u,p)$.

It remains to prove the completeness of the space $r_\infty^q(u,p)$. Let $\{x^i\}$ be any Cauchy sequence in the space $r_\infty^q(u,p)$, where $x^i = \{x_0^i, x_2^i, \ldots\}$. Then, for a given ε > 0 there exists a positive integer $n_0(\varepsilon)$ such that

$$g(x^i - x^j) < \varepsilon \tag{6}$$

for all i,j ≥ $n_0(\varepsilon)$. Using definition of g and for each fixed k ∈ ℕ that

$$\left|(R_u^q x^i)_k - (R_u^q x^j)_k\right| \le \sup_k \left|(R_u^q x^i)_k - (R_u^q x^j)_k\right|^{\frac{p_k}{M}} < \varepsilon$$

for i,j ≥ $n_0(\varepsilon)$, which leads us to the fact that $\{(R_u^q x^0)_k, (R_u^q x^1)_k, \ldots\}$ is a Cauchy sequence of real numbers for every fixed k ∈ ℕ. Since R is complete, it converges,say, $(R_u^q x^i)_k \to ((R_u^q x)_k$ as I → ∞. Using these infinitely many limits $(R_u^q x)_0, (R_u^q x)_1, \ldots$, we define the sequence $\{(R_u^q x)_0, (R_u^q x)_1, \ldots\}$. From (6) for with j → ∞ we have

$$\left|(R_u^q x^i)_k - (R_u^q x)_k\right| \le \varepsilon, \tag{7}$$

for all k, i.e.,

$$g(x^i - x) \le \varepsilon \; (i \ge n_0(\varepsilon)).$$

Finally, taking ε=1 in (7) and letting I ≥ $n_0(1)$. we have by Minkowski's inequality for each m ∈ ℕ that

$$\left|(R_u^q x)_k\right|^{\frac{p_k}{M}} \le g(x^i - x) + g(x^i) \le 1 + g(x^i)$$

which implies that $x \in r_\infty^q(u,p)$. Since $g(x - x^i) \le \varepsilon$ for all i ≥ $n_0(\varepsilon)$, it follows that $x^i \to x$ as i → ∞, hence we have shown that $r_\infty^q(u,p)$ is complete, hence the proof.

Note that one can easily see the absolute property does not hold on the space $r_\infty^q(u,p)$, that is, $g(x) \neq g(|x|)$ for atleast one sequence in the spaces $r_\infty^q(u,p)$ and consequently we see that the space $r_\infty^q(u,p)$ is a sequence space of non-absolute type.

Theorem 2: The sequence spaces $r_\infty^q(u,p)$ of non-absolute type is linearly isomorphic to the spaces $l_\infty(p)$.

Proof : To prove the theorem, we should show the existence of a linear bijection between the spaces $r_\infty^q(u,p)$ and $l_\infty(p)$. With the notation of (3), define the transformation T from $r_\infty^q(u,p)$ to $l_\infty(p)$ by x → y=Tx. The linearity of T is trivial. Further, it is obvious that x=θ whenever Tx=θ and hence T is injective.

Let y ∈ $l_\infty(p)$ and define the sequence $x=(x_k)$ by

$$x_k = \frac{1}{u_k q_k} \{Q_k y_k - Q_{k-1} y_{k-1}\} \; for \; k \in \mathbb{N}.$$

Then,

$$g(x)=\sup_k\left|\frac{1}{Q_k}\sum_{j=0}^{k}u_jq_jx_j\right|^{\frac{p_k}{M}}=\sup_k\left|\sum_{j=0}^{k}\delta_{kj}y_j\right|^{\frac{p_k}{M}}$$

$$=\sup_k|y_k|^{\frac{p_k}{M}}=g_1(y)<\infty.$$

Thus, we have $x\in r_\infty^q(u,p)$. Consequently, T is surjective and is paranorm preserving. Hence, T is a linear bijection and this says us that the spaces $r_\infty^q(u,p)$ and $l_\infty(p)$ are linearly isomorphic, hence the proof.

First we state some lemmas which are needed in proving the theorems.

Lemma 1 [7]: $A\in(l_\infty(p):l_1)$ if and only if there exists an integer B>1 such that

$$\sup_{K\in\mathcal{F}}\sum_n\left|\sum_{k\in K}a_{nk}B^{\frac{1}{p_k}}\right|<\infty. \tag{8}$$

Lemma 2 [7]: Let $0<p_k<\infty$ for every $k\in\mathbb{N}$. Then $A\in(l(p):l_\infty)$ if and only if there exists an integer B > 1 such that

$$\sup_n\sum_k|a_{nk}|B^{\frac{1}{p_k}}<\infty. \tag{9}$$

Lemma 3 [7]: Let $0<p_k<\infty$ for every $k\in\mathbb{N}$. Then $A\in(l_\infty(p):c)$ if and only if (8) and (9) hold and

$$\sum_k|a_{nk}|B^{\frac{1}{p_k}}\ \text{converges uniformly in n for all integers } B>1 \tag{10}$$

and

$$\lim_n a_{nk}=\beta_k\ \text{ for all } k\in\mathbb{N}.$$

Theorem 3: Let $1<p_k\le H<\infty$ for every $k\in\mathbb{N}$. Define the sets $D_1(u,p)$, $D_2(u,p)$ and $D_3(u,p)$, as follows

$$D_1(u,p)=\bigcap_{B>1}\left\{a=(a_k)\in\omega:\sup_{n\in F}\sum_k\left|\sum_{n\in N}(-1)^{n-k}\frac{a_n}{u_nq_n}Q_kB^{-1}\right|^{p_k'}<\infty\right\},$$

$$D_2(u,p)=\bigcap_{B>1}\left\{a=(a_k)\in\omega:\sum_k\left|\Delta\left(\frac{a_k}{u_kq_k}\right)Q_kB^{-1}\right|^{p_k'}<\infty\ and\ \left\{\left(\frac{a_k}{u_kq_k}Q_kB^{-1}\right)^{p_k'}\right\}\in c_0\right\},$$

$$D_3(u,p)=\bigcap_{B>1}\left\{a=(a_k)\in\omega:\sum_k\left|\Delta\left(\frac{a_k}{u_kq_k}\right)Q_kB^{-1}\right|^{p_k'}<\infty\ and\ \left\{\left(\frac{a_k}{u_kq_k}Q_kB^{-1}\right)^{p_k'}\right\}\in l_\infty\right\},$$

Then,

$$\left[r_\infty^q(u,p)\right]^\alpha=D_1(u,p),\left[r_\infty^q(u,p)\right]^\beta=D_2(u,p)\text{ and }\left[r_\infty^q(u,p)\right]^\gamma=D_3(u,p).$$

Proof: Let us take any $a=(a_k)\in\omega$. We can easily derive with (3) that

$$a_nx_n=\sum_{i=n-1}^{n}(-1)^{n-i}\frac{a_n}{u_nq_n}Q_iy_i=(Cy)_n \tag{11}$$

For $n\in\mathbb{N}$ where, $C=(c_{nk})$ is defined as

$$c_{nk}=\begin{cases}(-1)^{n-k}\dfrac{a_n}{u_nq_n}Q_k, & \text{if } n-1\le k\le n,\\ 0, & \text{if } 0\le k<n-1\ or\ k>n,\end{cases}$$

where $n,k\in\mathbb{N}$. Thus we observe by combining (10) with (i) of Lemma 2 that $ax=(a_nx_n)\in l_1$ whenever $x=(x_n)\in r_\infty^q(u,p)$ if and only if $Cy\in l_1$ whenever $y\in l(p)$. This gives the result that $\left[r_\infty^q(u,p)\right]^\alpha=D_1(u,p)$. Consider the equation,

$$\sum_{k=0}^{n}a_kx_k=\sum_{k=0}^{n-1}\Delta\left(\frac{a_k}{u_kq_k}\right)Q_ky_k+\frac{a_n}{u_nq_n}Q_ny_n=(Dy)_n\quad for\ n\in\mathbb{N} \tag{12}$$

where, $D=(d_{nk})$ is defined as

$$d_{nk}=\begin{cases}\Delta\left(\dfrac{a_k}{u_kq_k}\right)Q_k, & \text{if } 0\le k\le n-1,\\ \dfrac{a_n}{u_nq_n}Q_k, & \text{if } k=n,\\ 0, & \text{if } k>n.\end{cases}$$

Where $n,k\in\mathbb{N}$. Thus we deduce from Lemma 3 with (11) that $ax=(a_nx_n)\in cs$ whenever $x=(x_n)\in r_\infty^q(u,p)$ if and only if $Dy\in c$ whenever $y\in l(p)$. Therefore, we derive fom (8) that

$$\sum_k\left|\Delta\left(\frac{a_k}{u_kq_k}\right)Q_kB^{-1}\right|^{p_k'}<\infty\ and\ \sup_{k\in\mathbb{N}}\left|\frac{a_k}{u_kq_k}Q_kB^{-1}\right|^{p_k'}<\infty \tag{13}$$

which shows that that $\left[r_\infty^q(u,p)\right]^\beta=D_2(u,p)$.

As this, from Lemma 2 together with (11) that $ax=(a_kx_k)\in bs$ whenever $x=(x_n)\in r_\infty^q(u,p)$ if and only if $Dy\in l_\infty$ whenever $y=(y_k)\in l(p)$. Therefore, we again obtain the condition (12) which means that $\left[r_\infty^q(u,p)\right]^\gamma=D_2(u,p)$ and the proof of the theorem is complete [19-21].

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