New Near Open Set In Topological Space

Sayed MEL* and Mansour FHAL
Department Mathematics, College of Science and Arts, Najran University, Saudi Arabia

Abstract

The aim of this paper is to introduce new class of near open sets namely, \( b^* \)-open set. And study some of their properties, also we study the relation between this class among this classes. Also, we introduce some topological properties and we shall study some of their properties.

Keywords: \( b^* \)-open set, \( b^* \)-interior, \( b^* \)-closure, \( b^* \)-boundary, \( b^* \)-neighbourhood.

Introduction


Definition 1.1: A subset \( A \) of topological space \( (X, \tau) \) is called: \( \text{clint}(\text{int}(A)) \)

(1) open if \( \text{clint}(\text{int}(A)) \) \[6\]
(2) preopen if \( \text{alcl}(\text{cl}(A)) \) \[8\]
(3) semi open if \( \text{ascl}(\text{cl}(A)) \) \[5\]
(4) Regular open if \( A = \text{int}(\text{cl}(A)) \) \[4\]
(5) \( \beta \)-open (or semi pre open) if, \( \text{al}(\text{cl}(\text{int}(A))) \) \[9-15\]
(6) \( \text{b-open} \). \( \text{al}(\text{cl}(A)) \subset \text{cl}(\text{int}(A)) \) \[10\]
(7) A subset \( A \) of a space \( X \) is called \( Bc \)-Open if for each \( x \in A \in bO(X) \), there exists a closed set \( F \) such that \( x \in F \subset A \) \[11\]

Remark 1.1: The complement of a \( \alpha \)-open (resp. preopen, semi open, Regular open, \( \beta \)-open and \( b \)-open) sets is called \( \alpha \)-closed (resp. pre-closed, semi-closed, Regular closed, \( \beta \)-closed and \( b \)-closed) sets. The intersection of all \( \alpha \)-closed (resp. pre-closed, semi-closed, Regular closed, \( \beta \)-closed and \( b \)-closed) sets containing \( A \) is called the \( \alpha \)-closure (resp. pre-closure, semi-closure, Regular closure, \( \beta \)-closure and \( b \)-closure) of \( A \) and is denoted by \( \text{acl}(A) \) (resp. \( \text{pcl}(A) \), \( \text{alcl}(A) \), \( \text{Rcl}(A) \), \( \text{bcl}(A) \) or \( \text{spcl}(A) \), and \( \text{bc}(A) \)).

The union of all \( \alpha \)-open (resp. preopen, semi open, Regular open, \( \beta \)-open, and \( b \)-open) sets contained in \( A \) is called \( \alpha \)-intri (resp. pre-intri, semi-intri, Regular inner, \( \beta \)-intri and \( b \)-intri) of \( A \) and is denoted by \( \text{ain}(A) \) (resp. \( \text{pint}(A) \), \( \text{rint}(A) \), \( \text{rint}(A) \) or \( \text{spint}(A) \), and \( \text{bint}(A) \)). The family of all \( \alpha \)-open (resp. \( \alpha \)-closed, preopen, pre-closed, semi open, semi closed, Regular open, Regular closed, \( \beta \)-open, \( \beta \)-closed and \( b \)-open, \( b \)-closed) sets is denoted by \( \alpha O(X) \) (resp \( \text{cl}(X) \), \( \text{pint}(X) \), \( \text{pc}(X) \), \( \text{SC}(X) \), \( \text{RC}(X) \), \( \text{RO}(A) \), \( \text{C}(A) \), \( \beta O(A) \), \( \beta C(A) \), \( \text{bO}(A) \) and \( \text{bc}(A) \)).

Proposition 1.1: For subset \( A, B \) of space \( (X, r) \), the following statement hold:

(1) \( \text{pcl}(A) = A \cup \text{cl}(\text{int}(A)) \), \( \text{pint}(A) = A \cap \text{cl}(\text{int}(A)) \) \[10\].
(2) \( \text{spcl}(A) = A \cup \text{int}(\text{cl}(A)) \), \( \text{spint}(A) = A \cap \text{cl}(\text{int}(A)) \) \[10\].
(3) \( \text{pcl}(A \cup B) \subseteq \text{pcl}(A) \cup \text{pcl}(B) \), \( \text{spcl}(A \cup B) \subseteq \text{spcl}(A) \cup \text{spcl}(B) \) \[12,13\].
(4) \( \text{pint}(A \cup B) \subseteq \text{pint}(A) \cap \text{pint}(B) \), \( \text{pint}(A \cup B) \subseteq \text{pint}(A) \cap \text{pint}(B) \) \[14\].
(5) \( X \setminus \text{int}(A) = \text{cl}(X \setminus A) \), \( \text{int}(X \setminus A) = X \setminus \text{cl}(A) \).

2. \( b^* \)-Open sets

Definition 2.1: Let \( (X, r) \) be topological space. Then a subset \( A \) of \( X \) said to be

1. \( b^* \)-Open set if \( A \subseteq \text{cl}(\text{int}(A)) \cup \text{int}(\text{cl}(A)) \).
2. A \( b^* \)-closed set if \( A \supseteq \text{int}(\text{cl}(A)) \cap \text{cl}(\text{int}(A)) \).

The family of all \( b^* \)-Open set (resp. \( b^* \)-closed set) subsets of a space \( (X, r) \) will be as always denoted by \( bO(X) \) (resp. \( bC(X) \)).

Example 2.1: Let \( X = \{a, b, c, d\} \) with topology \( \tau = \{x, x, \{a, b\}, \{a, c, d\}\} \). Then the classes of \( b^* \)-open set and \( b^* \)-closed set

\( bO(X) = \{X, \Phi, \{a\}, \{a, b\}, \{a, c\}, \{a, c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\} \), and

\( bC(X) = \{X, \Phi, \{b\}, \{c\}, \{d\}, \{c, d\}, \{b, d\}, \{b, c\}, \{b, c, d\}\} \).

Proposition 2.1: Let \( A \) be a sub set of a space \( (X, r) \). Then (1) Every preopen (resp. \( Bc \)-open) set is \( b^* \)-open

Remark 2.1: The converse of the above proposition is not necessarily true as shown by the following example.

*Corresponding author: Sayed MEL, Professor, Department Mathematics, College of Science and Arts, Najran University, Saudi Arabia, Tel: +966 17 542 8888; E-mail: fa-hamad@hotmail.com

Received November 07, 2016; Accepted November 28, 2016; Published November 30, 2016


Copyright: © 2016 Sayed MEL, et al. This is an open-access article distributed under the terms of the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original author and source are credited.
Example 2.2: Let \( X=\{a,b,c\} \) with topology \( \tau=\{X,\emptyset,\{a\},\{b\},\{a,b\}\} \). Then
\[
(1) \text{ A subset } \{a,c\} \text{ of } X \text{ is } b^*-\text{open but not preopen.}
\]
(2) A subset \( \{a\} \) of \( X \) is \( b^*-\text{open} \) but not \( Bc^*-\text{open}. \)

Remark 2.2: According to Definition (2.1) and Proposition (2.1), the following diagram holds for a subset \( A \) of a space \( X \):

Lemma 2.1: Let \( (X, \tau) \) be topological space. Then the following statements are hold:
(1) The union of \( b^*-\text{Open} \) sets is \( b^*-\text{open} \)
(2) The intersection of \( b^*-\text{Closed} \) sets is \( b^*-\text{closed} \)

Proof: (1) let \( \{A_i, i \in I\} \) be a family of \( b^*-\text{Open sets}. \) Then \( A_i \subseteq \text{cl}(\text{int}(A_i)) \cup \text{int}(A_i) \), hence \( \bigcup_i A_i \subseteq \bigcup_i \text{cl}(\text{int}(A_i)) \cup \bigcup_i \text{int}(A_i) \), for all \( i \). Thus \( \bigcup_i A_i \) is \( b^*-\text{Open} \)

(2) let \( \{A_i, i \in I\} \) be a family of \( b^*-\text{Closed sets}. \) Then \( A_i \supseteq \text{int}(\text{cl}(A_i)) \cap \text{cl}(A_i) \), hence \( \bigcap_i A_i \supseteq \text{int}(\bigcap_i \text{cl}(A_i)) \cap \text{cl}(\bigcap_i A_i) \), for all \( i \). Thus \( \bigcap_i A_i \) is \( b^*-\text{Closed} \)

Remark 2.3: The intersection of any two \( b^*-\text{Open} \) sets is not \( b^*-\text{open}. \) Let \( X=\{a,b,c,d\}, \tau=\{X,\emptyset,\{a\},\{c\},\{a,c\}\} \). Then \( A=\{a,b\} \) and \( B=\{b,c\} \) are \( b^*-\text{open} \) sets, but \( A \cap B=\{b\} \) is not \( b^*-\text{open}. \)

Definition 2.2: Let \( (X, \tau) \) be topological space.
Then:
(1) The union of all \( b^*-\text{Open} \) sets of \( X \) contained in \( A \) is called the \( b^*-\text{interior} \) of \( A \) and is denoted by \( b^*-\text{int}(A). \)
(2) The intersection of all \( b^*-\text{Closed} \) sets of \( X \) contained in \( A \) is called the \( b^*-\text{closure} \) of \( A \) and is denoted by \( b^*-\text{Cl}(A). \)

Example 2.3: Let \( X=\{a,b,c,d\} \) with topology \( \tau=\{X,\emptyset,\{a\},\{c\},\{a,c\}\}. \) and \( A=\{a,b\}, B=\{a,c\} \) are \( b^*-\text{open} \) then
\[
(1) b^*-\text{int}(A) = \{a,b\}, \quad b^*-\text{int}(B) = \{a,c\} \quad \text{and} \quad b^*-\text{Cl}(A) = \{a,b\}, b^*-\text{Cl}(B) = X
\]

Theorem 2.1: Let \( (X, \tau) \) be topological space and \( A \subseteq X \), then the following statement are equivalent:
(1) \( A \) is a \( b^*-\text{Open} \) set,
(2) \( A = \text{spint}(A) \cup \text{pcl}(A) \)

Proof: (1)\( \rightarrow \) (2). Let \( A \) be a \( b^*-\text{Open} \) set. Then \( A \supseteq \text{cl}(\text{int}(A)) \cup \text{int}(A) \), hence by proposition (1.1).
\[
\text{spint}(A) \cup \text{pcl}(A) = (A \cap \text{cl}(\text{int}(A))) \cup (A \cap \text{int}(A)) = A \cap \text{cl}(\text{int}(A)) \cup (A \cap \text{int}(A)) = A
\]

(2)\( \rightarrow \) (1). Suppose that \( A = \text{spint}(A) \cup \text{pcl}(A) \). Then by proposition (1.1)
\[
A = (A \cap \text{cl}(\text{int}(A))) \cup (A \cap \text{int}(A)) \subseteq \text{cl}(\text{int}(A)) \cap \text{int}(A) \text{ . Therefore, } A \text{ is a } b^*-\text{open}.
\]

Theorem 2.2: Let \( (X, \tau) \) be topological space and \( A \subseteq X \), then the following statement are equivalent:
(1) \( A \) is a \( b^*-\text{Closed} \) set,
(2) \( A = \text{scpcl}(A) \cap \text{pocl}(A) \)

Proof: (1)\( \rightarrow \) (2). Let \( A \) be a \( b^*-\text{Closed} \) set. Then \( A \supseteq \text{int}(\text{cl}(A)) \cap \text{cl}(A) \), hence by proposition (1.1).
\[
\text{scpcl}(A) \cap \text{pocl}(A) = (A \cap \text{int}(\text{cl}(A))) \cap (A \cap \text{cl}(\text{int}(A))) = A \cap \text{int}(\text{cl}(A)) \cap \text{cl}(\text{int}(A)) = A
\]

(2)\( \rightarrow \) (1). Suppose that \( A = \text{scpcl}(A) \cap \text{pocl}(A) \). Then by proposition (1.1)
\[
A = (A \cap \text{int}(\text{cl}(A))) \cap (A \cap \text{cl}(\text{int}(A))) \supseteq \text{int}(\text{cl}(A)) \cap \text{cl}(\text{int}(A)) \text{ . Therefore, } A \text{ is a } b^*-\text{closed}.
\]

Theorem 2.3: Let \( (X, \tau) \) be a subset of a space \( (X, \tau) \). Then
(1) \( b^*-\text{Cl}(A) = \text{scpcl}(A) \cap \text{pocl}(A) \)
(2) \( b^*-\text{int}(A) = \text{spint}(A) \cup \text{pcl}(A) \)

Proof: (1) It is easy to see that \( b^*-\text{Cl}(A) \subseteq \text{scpcl}(A) \cap \text{pocl}(A) \). Also \( \text{scpcl}(A) \cap \text{pocl}(A) = (A \cup \text{int}(\text{cl}(A))) \cap (A \cap \text{cl}(\text{int}(A))) \subseteq (A \cap \text{cl}(\text{int}(A))) \cap (A \cup \text{cl}(\text{int}(A))) \). But, \( b^*-\text{Cl}(A) \) is \( b^*-\text{Closed}, \) hence \( X = A \cup \text{int}(\text{cl}(A)) \cap (A \cup \text{cl}(\text{int}(A))) \subseteq A \cup b^*-\text{Cl}(A) = b^*-\text{Cl}(A) \). Thus \( A \cup \text{int}(\text{cl}(A)) \cap (A \cup \text{cl}(\text{int}(A))) \subseteq A \cup b^*-\text{Cl}(A) = b^*-\text{Cl}(A) \) there for, \( \text{scpcl}(A) \cap \text{pocl}(A) \subseteq b^*-\text{Cl}(A) \). So \( b^*-\text{int}(A) = \text{spint}(A) \cup \text{pcl}(A) \).

(2) It is easy to see that \( b^*-\text{int}(A) \subseteq \text{spint}(A) \cup \text{pcl}(A) \). Also \( \text{spint}(A) \cup \text{pcl}(A) = (A \cup \text{cl}(\text{int}(A))) \cap (A \cap \text{int}(\text{cl}(A))) = A \cup (\text{cl}(\text{int}(A))) \cap (\text{int}(\text{cl}(A))) \). But, \( b^*-\text{int}(A) \) is \( b^*-\text{Open}, \) hence \( b^*-\text{int}(A) \subseteq \text{cl}(\text{int}(b^*-\text{int}(A))) \cup \text{int}(\text{cl}(b^*-\text{int}(A))) \subseteq (\text{cl}(\text{int}(A))) \cup \text{int}(\text{cl}(A)) \). Thus \( A \cup (\text{cl}(\text{int}(A))) \cap (\text{int}(\text{cl}(A))) \subseteq b^*-\text{Cl}(A) \) there for \( \text{spint}(A) \cup \text{pcl}(A) \subseteq b^*-\text{int}(A) \).

So \( b^*-\text{int}(A) = \text{spint}(A) \cup \text{pcl}(A) \).
Theorem 2.4: Let A be a subset of a space \((X,\tau)\). Then
1. A is a \(b^*-\)open set if and only if \(A=b^*-\text{int}(A)\)
2. A is a \(b^*-\)closed set if and only if \(A=b^*-\text{cl}(A)\)

Proof: (1) Let A be a \(b^*-\)open set. Then by theorem (2.1), \(A=\text{sp(int}(A))\cup\text{pnt}(A)\) and by theorem (2.3), we have \(A=b^*-\text{int}(A)\). Conversely, let \(A=b^*-\text{int}(A)\). Then by theorem (2.3), \(A=\text{spcl}(A)\cap\text{pct}(A)\) and by theorem (2.1), A is \(b^*-\)closed.

(2) By definition (2.3), \(B\subset A\cup\text{min}(A)\). There for, \(B\subset A\cap\text{min}(A)\). Conversely, suppose that exists a \(b^*-\)open set U and \(x\in U\) such that \(U\cap A\neq\emptyset\).

Theorem 2.5: Let A and B be a subsets of a space \((X,\tau)\). Then the following are hold
1. \(b^*-\text{cl}(X\setminus A)=X\setminus b^*-\text{int}(A)\)
2. \(b^*-\text{int}(X\setminus A)=X\setminus b^*-\text{cl}(A)\)
3. If \(A\subseteq B\), then \(X\setminus b^*-\text{cl}(A)\)
4. \(X\setminus b^*-\text{int}(A)\) and only if there exists a \(b^*-\)open set \(U\) and \(x\in U\) such that \(U\cap A\neq\emptyset\).
5. The inclusion relation in part (6),(7) of the above theorem cannot be replaced by equality as shown by the following example.

Example 2.4. Let \(X=[a,b,c,d]\) with topology \(\tau=[X,\Phi]\{a,\{a\},\{a,b\},\{a,b,c\},\{a,b,c,d\}\}\). Then \(\text{AUB}=[a,b,d]\)
1. If \(A=[a,b], B=[b,d]\) and \(\text{AUB}=[a,b,d]\), then \(b^*-\text{int}(A)=A=b^*-\text{int}(B)=\emptyset\) and \(b^*-\text{int}(AUB)=\emptyset=b^*-\text{int}(B)\).
2. If \(C=[b], B=[d]\) and \(\text{BNC}=[\emptyset]\), then \(b^*-\text{cl}(C)=[\emptyset], b^*-\text{cl}(B)=[B] \neq [b^*-\text{int}(B)=B\neq [b^*-\text{cl}(B)]\) and \(b^*-\text{cl}(BNC)=\emptyset=b^*-\text{cl}(BNC)\).
**Definition 3.1:** Let $(X,r)$ be a space and $A \subset X$. Then the $b'$-boundary of $A$ (briefly, $b'$-$b(A)$) is given by $b'$-$b(A)=b'$-$cl(X/A)$.

**Example 3.1:** From Example (2.1) we have $A=\{a\}$ $B=\{a,b\} C=\{a,b,d\}$ then $b'$-$b(A)=\{b\}$, $b'$-$b(B)=\{c,d\}$ and $b'$-$b(C)=\{c\}$.

**Remark 3.1:** For any subset $A$ of a space $(X,r)$ we have $b'$-$b(A) \subseteq b(A)$ and $b'$-$b(A) \subseteq p-b(A)$.

The inclusion of the above remark can be replaced as shows in the following example.

**Example 3.2:** From Example (2.3) and $A=\{a,b\}$ then $b'$-$b(A) = \phi$, $p-b(A) = \{b\}$ we have $p-b(A) \subseteq b'$-$b(A)$.

**Theorem 3.1:** If $A$ is a sub sets of a space $(X,r)$, then the following statement are hold:

1. $b'$-$b(b(A))=b'(X \setminus A))$.
2. $b'$-$b(A)=b'$-$cl(A) \setminus b'$-$int(A)$.
3. $b'$-$b(A) \cap b'$-$int(A) = \phi$.
4. $b'$-$b(A) \cup b'$-$int(A) = b'$-$cl(A)$.

**Proof:** (1) Since $b'$-$b(b(A))=b'$-$cl(A) \setminus b'$-$int(A)$ then $b'$-$b(b(A))=b'(X \setminus A))=b'$-$cl(X \setminus A)$.

(2) Also, by using (2) $b'$-$b(A) \cap b'$-$int(A)=(b'$-$cl(A) \setminus b'$-$int(A)) \cup b'$-$int(A)=b'$-$cl(A) \setminus b'$-$int(A) \cup b'$-$int(A)$.

(3) By using (3) $b'$-$b(A) \cup b'$-$int(A) \cap b'$-$int(A)=b'$-$cl(A)$.

**Theorem 3.2:** If $A$ is a sub sets of a space $(X,r)$, then the following statement are holds:

1. $A$ is a $b'$-open set if and only if $A \cap b'$-$b(A)=\phi$
2. $A$ is a $b'$-closed set if and only if $b'$-$b(A) \subset A$
3. $A$ is a $b'$-clopen set if and only if $b'$-$b(A) = \phi$.

**Proof:** (1) Let $A$ is a $b'$-open set. Then $A=b'$-$int(A)$ hence by theorem (3.1) $A \cap b'$-$b(A)=\phi$. Conversely, let $A \cap b'$-$b(A)=\phi$ then by theorem (3.1), $\phi=(b'$-$cl(A) \setminus b'$-$int(A)) \cap (A \cap b'$-$b(A))=A \cap b'$-$int(A)$ so, $A=b'$-$int(A)$ and hence $A$ is $b'$-open.

(2) Let $A$ is a $b'$-closed set. Then $A=b'$-$cl(A)$, by theorem (3.1), but $b'$-$b(A)=(b'$-$cl(A) \setminus b'$-$int(A)) \cap A \neq \phi$ then $b'$-$b(A) \subset A$. Conversely, suppose that $b'$-$b(A)=\phi$. Then $b'$-$b(A)=(b'$-$cl(A) \setminus b'$-$int(A)) \neq \phi$ and hence $A$ is a $b'$-clopen set.

**Definition 3.2:** Let $(X,r)$ be a space and $A \subset X$. Then the set $X \setminus \setminus (b'$-$cl(A))$ is called the $b'$-exterior of $A$ and is denoted by $b'$-$ext(A)$. Each point $p \in X$ is called an $b'$-exterior point of $A$, if it is a $b'$-interior point of $X/A$.

**Example 3.3:** Let $X=\{a,b,c,d\}$ with topology $r=\{X, \phi, \{a\}, \{b\}, \{a,b\}, \{b,c,d\}\}$

If $A=\{a\}$ $B=\{a,c\}$ $C=\{b,c,d\}$ then we have $b'$-$ext(A)=\{b,c,d\}$, $b'$-$ext(B)=\{b, d\}$ and $b'$-$ext(C)=\{a\}$.

**Remark 3.2:** For any topology space $(X,r)$ and $A \subset X$, we have $ext(A) \subseteq p-ext(A) \subseteq b'$-$ext(A)$

**Proof:** Since $b'$-$cl(A) \subseteq cl(A)$, then $X \setminus cl(A) \subseteq X \setminus cl(A)$ and $int(X \setminus cl(A) \subseteq b'$-$ext(A)$ i.e $ext(A) \subseteq b'$-$ext(A)$. Since $ext(A) \subseteq p-ext(A)$, then we have $p-ext(A) \subseteq b'$-$ext(A)$. This implies that the relation hold.

**Example 3.4:** Let $X=\{a,b,c,d\}$ with topology $r=\{X, \phi, \{a\}, \{c\}, \{a,c\}, \{b,c\}, \{b,d\}\}$, $A=\{b,d\}$, $B=\{c\}$ we have $b'$-$ext(A)=\{a\}$, $p-ext(A)=\{c\}$, $p-ext(B)=\{b\}$, $ext(B)=\{d\}$

**Theorem 3.3:** If $A$ and $B$ is two sub sets of a space $(X,r)$, then the following statements are hold: $ext(A) \cup b'$

1. $b'$-$ext(A)=b'$-$int(A)$.
2. $b'$-$ext(A)$ is $b'$-open.
3. $b'$-$ext(A) \cup b'$-$int(A)$.
4. $b'$-$ext(A) \cap b'$-$b(A)$.
(5) $b^-\text{ext}(A)\cup b^-\text{int}(A) = b^-\text{cl}(X \setminus A)$.

(6) $\{b^-\text{int}(A), b^-\text{cl}(A)\}$ from partition of $X$.

(7) If $A \subseteq B$, then $b^-\text{ext}(B) \subseteq b^-\text{ext}(A)$.

(8) $b^-\text{ext}(A \cup B) \subseteq b^-\text{ext}(A) \cup b^-\text{ext}(B)$.

(9) $b^-\text{ext}(A \cap B) \supseteq b^-\text{ext}(A) \cap b^-\text{ext}(B)$.

(10) $b^-\text{ext}(X) = \emptyset$ and $b^-\text{ext}(\emptyset) = X$.

**Proof:** (1) By Definition (3.2) $b^-\text{ext}(A) = X \setminus b^-\text{cl}(A) = b^-\text{int}(X \setminus A)$.

(2) From (1) $b^-\text{ext}(A) = b^-\text{int}(X \setminus A)$. Since $b^-\text{int}(A)$ is the union of all $b^-\text{open}$ sets of $X$ contained in $A$ thus $b^-\text{ext}(A)$ is $b^-\text{open}$.

(3) Since $b^-\text{ext}(A) \cap b^-\text{int}(A) = X \setminus b^-\text{cl}(A) \cap b^-\text{int}(A) = b^-\text{int}(X \setminus A) \cap b^-\text{int}(A) = \emptyset$.

(4) By Theorem (3.1), $b^-\text{ext}(A) \cap b^-\text{int}(A) = b^-\text{int}(X \setminus A) \cap b^-\text{int}(A)$.

(5) Also, by Theorem (3.1) $b^-\text{ext}(A) \cap b^-\text{int}(A) = b^-\text{int}(X \setminus A) \cap b^-\text{int}(X \setminus A) = X$.

(6) From (3),(4) we have $b^-\text{ext}(A) = b^-\text{int}(X \setminus A) = \emptyset$ and $b^-\text{ext}(A) = b^-\text{int}(X \setminus A) = \emptyset$. Then by Theorem (3.1) then $b^-\text{ext}(A) = b^-\text{int}(X \setminus A) = \emptyset$.

(7) Let $A \subseteq B$ then $(b^-\text{cl}(B)) \subseteq (b^-\text{cl}(B))$ and hence

$X \setminus (b^-\text{cl}(B)) \subseteq X \setminus (b^-\text{cl}(B))$. So $b^-\text{ext}(B) \subseteq b^-\text{ext}(A)$.

(8) $b^-\text{ext}(A \cup B) = X \setminus (b^-\text{cl}(A \cup B)) \subseteq X \setminus (b^-\text{cl}(A)) \cup b^-\text{ext}(B) = (X \setminus (b^-\text{cl}(A))) \cup (b^-\text{ext}(B)) = b^-\text{ext}(A) \cap b^-\text{ext}(B) \subseteq b^-\text{ext}(A) \cap b^-\text{ext}(B)$.

(9) $b^-\text{ext}(A \cap B) = X \setminus (b^-\text{cl}(A \cap B)) = X \setminus (b^-\text{cl}(A)) \cap b^-\text{ext}(B) = (X \setminus (b^-\text{cl}(A))) \cap (b^-\text{ext}(B)) = b^-\text{ext}(A) \cap b^-\text{ext}(B) \subseteq b^-\text{ext}(A) \cap b^-\text{ext}(B)$.

(10) $b^-\text{ext}(X) = X \setminus (b^-\text{cl}(X)) = X \setminus \emptyset$ and $b^-\text{ext}(\emptyset) = X \setminus (b^-\text{cl}(\emptyset)) = X \setminus \emptyset = X$.

**Remark 3.3:** The inclusion relation in parts (5),(6) of the above theorem cannot be replaced by equality as is show by the following example.

**Example 3.5:** From Example (2.1) we have $A = \{b, c\}$ and $B = \{a, c\}$ then $b^-\text{ext}(A) = \{b, c\}, b^-\text{ext}(B) = \emptyset$ but $b^-\text{ext}(A \cup B) = \emptyset$. Therefore, $b^-\text{ext}(A) \cup b^-\text{ext}(B) \subseteq b^-\text{ext}(A) \cap b^-\text{ext}(B)$. Also, $b^-\text{ext}(A \cap B) = \{a, b, d\}$, hence $b^-\text{ext}(A \cap B) \subseteq b^-\text{ext}(A) \cup b^-\text{ext}(B)$.

**Definition 3.3:** If $A$ is a subset of a space $(X, \tau)$, then a point $pX$ is called a $b^-\text{closed}$ point of a set $A \subseteq X$ if every $b^-\text{open}$ set $G$ containing $p$ contains a point of $A$ other than $p$. The set of all $b^-\text{limit}$ points of $A$ is called an $b^-\text{closed}$ derived set of $A$ and is denoted by $b^-\text{d}(A)$.

**Example 3.6:** Let $X = \{a, b, c, d\}$ with topology $\tau = \{X, \emptyset, \{a, c, d\}, \{a, b, d\}\}$ and if $A = \{a, d\}$ then $b^-\text{d}(A) \subseteq \{a, d\}$.

**Theorem 3.4:** If $A$ and $B$ is two $b^-\text{sub}$ sets of a space $(X, \tau)$, then the following statements are hold:

(1) If $A \subseteq B$, then $b^-\text{d}(A) \subseteq b^-\text{d}(B)$.

(2) $A$ is a $b^-\text{closed}$ set if and only if it contains each of its $b^-\text{limit}$ points.

(3) $b^-\text{d}(A) = A \cup b^-\text{d}(A) = A$.

(4) $b^-\text{d}(A) \subseteq b^-\text{d}(A)$.

(5) $b^-\text{d}(A \cap B) = b^-\text{d}(A) \cap b^-\text{d}(B)$.

**Proof:** (1) By definition (3.3), we have $p \in b^-\text{d}(A)$ if and only if $G \cap (A \setminus \{p\}) = \emptyset$ for every $b^-\text{open}$ set $G$ containing $p$. But $A \subseteq B$, then $G \cap (B \setminus \{p\}) = \emptyset$ for every $b^-\text{open}$ set $G$ containing $p$. Hence, $p \in b^-\text{d}(B)$.

There for $b^-\text{d}(A) \subseteq b^-\text{d}(B)$.

(2) Let $A$ be $b^-\text{closed}$ set and $p \in A$ then $p \in b^-\text{d}(A)$ which is $b^-\text{open}$, hence there exists $b^-\text{open}$ set $X \setminus A$ such that $X \setminus A = \emptyset$.

(3) Since $b^-\text{d}(A) \cup b^-\text{d}(A) \subseteq b^-\text{d}(A) \cup b^-\text{d}(A) \subseteq b^-\text{cl}(A)$.

---

**Citation:** Sayed MEL, Mansour FHAY (2016) New Near Open Set In Topological Space. J Phys Math 7: 204. doi: 10.4172/2090-0902.1000204

**ISSN:** 2090-0902

**Volume:** 7 • Issue **4** • 1000204
Conversely, suppose that $p \notin b^*\text{cl}(A) \cup A$ Then $p \notin b^*\text{cl}(A)$, $p \notin A$ and hence there exists $b^*$-open set $G$ containing $p$ such that $G \cap A = \emptyset$. Thus, $p \notin b^*\text{cl}(A)$ which implies that $b^* - cl(A) \subset b^* - d(A) \cup A$, there for, $b^* - cl(A) = b^* - d(A) \cup A$.

(5) Since $A \supseteq A \cap B$ and $B \supseteq A \cap B$ we have $b^*\text{cl}(A) \supseteq b^*\text{cl}(A \cap B)$ and $b^*\text{cl}(B) \supseteq b^*\text{cl}(A \cap B)$. There for $b^*\text{cl}(A) \cap b^*\text{cl}(B) \supseteq b^*\text{cl}(A \cap B)$.

**Definition 3.4:** Let $(X, b)$ be a space and $AGX$. Then the $b^*$-border of $A$ (briefly, $b^*\text{bd}(A)$) is given by $b^*\text{bd}(A) \equiv A \cap \text{int}(A)$.

**Example 3.7:** Let $X = \{a, b, c, d, e\}$ with topologies $\tau = \{\emptyset, X, [a, b], [c, d], [b, c], [a, d], [a, b, c], [a, d, e]\}$. If $A = \{a, c\}, B = \{c, d\}$ and $C = \{b, e\}$, then $b^*\text{bd}(A) = \{a, c\}$, $b^*\text{bd}(B) = \emptyset$ and $b^*\text{bd}(C) = \{e\}$.

**Theorem 3.5:** For a subset $A$ of a space $X$, the following statements are hold:

1. $A \supseteq \text{int}(A) \cup b^*\text{bd}(A)$.
2. $b^*\text{bd}(A) \subseteq \text{int}(A) \cup b^*\text{bd}(A)$.
3. $b^*\text{bd}(X) \subseteq b^*\text{bd}(A) \cup b^*\text{bd}(\emptyset)$.
4. $b^*\text{bd}(\emptyset) \subseteq b^*\text{bd}(\emptyset) \cup b^*\text{bd}(A)$.

**Proof:** (1) $b^*\text{bd}(A) = \text{int}(A) \cup b^*\text{bd}(A)$. (2) $b^*\text{bd}(X) \subseteq b^*\text{bd}(A)$.

**Theorem 3.6:** For a subset $A$ of a space $X$, the following statements are equivalent:

1. $A = b^*\text{open}$.
2. $A = b^*\text{int}(A)$.
3. $b^*\text{bd}(A) = \emptyset$.

**Example 3.8:** Let $X = \{a, b, c, d\}$ with topology $\tau = \{\emptyset, X, [a, c], [a, b], [a, c, b]\}$, then $b^*\text{Nc} = \{a, c\}$.

**Remark 3.4:** For any topology space $(X, b)$ and for each $x \in X$ we have $N_x \subseteq p \text{-} N_x \supseteq b^*\text{Nc}$. 

**Example 3.9:** From Example (2.2). We have $[a, c] \in b^*\text{Nc}$ but it is not in $p\text{-}N_x$ and not in $N_x$.

**Theorem 3.7:** A subset $G$ of a space $X$ is $b^*$-open if and only if it is $b^*\text{bd}$, for every point $p \in G$.

**Proof:** Necessity. Let $G$ be a $b^*$-open set. Then $G$ is a $b^*\text{bd}$ for each $p \in G$.

Sufficiency. Let $G$ be a $b^*\text{bd}$, for each $p \in G$. Then there exists a $b^*$-open set $W$ containing $p$ such that $p \notin W \subseteq G$, so $G = \bigcup_{p \in W} W$. Therefore, $G$ is $b^*$-open.
Theorem 3.8: For a space \((X,\tau)\). If \(b^-N_p\) is the \(b^-\text{nbd. systems of a point } p \in X\), then the following statements are hold:
(1) if \(b^-N_j\) is not empty and \(p\) belongs to each member of \(b^-N_j\),
(2) Each superset of the members of \(b^-N_j\) belongs to \(b^-N_j\),
(3) Each member \(N \in b^-N_p\) is a superset of the member \(W \in b^-N_p\), where \(W \in b^-\text{nbd of each point } p \in W\).

**Proof:** (1) Since \(X\) is a \(b^-\text{open set containing } p\), then \(X \in b^-\text{Np. So, } b^-\text{Np} \neq \phi\). Also, if \(N \in b^-\text{Np}\), then there exists a \(b^-\text{open set } G\) such that \(p \in G \subseteq N\). Therefore, \(p\) belongs to each member of \(b^-N_p\).

(2) Let \(M\) be a superset of \(N \in b^-\text{Np}\), then there exists a \(b^-\text{open set } G\) such that \(p \in G \subseteq M\) which implies \(p \in G \subseteq M\) and hence, \(M\) is a \(b^-\text{neighbourhood of } p\). Therefore, \(M \in b^-\text{Np}\).

(3) Let \(N\) be a \(b^-\text{neighbourhood of } p \in X\), then there exists a \(b^-\text{open set } W\) such that \(p \in W \subseteq N\). Then by Theorem 2.5.1, \(W\) is a \(b^-\text{neighbourhood of each of its points}.

Definition 3.6: For a space \((X,\tau)\), a subset \(A\) of \(X\) is said to be \(b^-\text{dense}\) in \(X\) if and only if \(b^-\text{-cl}(A) = X\). The family of all \(b^-\text{dense sets in } (X,\tau)\) will be denoted by \(b^-\text{D}(X,\tau)\).

Example 3.10: Let \(X = \{a,b,c\}\) with topology \(\tau = \{X,\emptyset,\{a,b\},\{a,c\}\}\). If \(A = \{b,c,d\}\), then \(b^-\text{-cl}(A) = X\) but \(b^-\text{-Cl}(A) = \{b,c,d\}\). Therefore, \(A\) is dense in \(X\) but not \(b^-\text{dense}\) in \(X\).

Theorem 3.9: For a space \((X,\tau)\) and \(E \subseteq X\), the following statements are equivalent:

(1) \(E\) is \(b^-\text{dense}\) in \(X\).
(2) If \(F\) is an \(b^-\text{closed set in } X\) containing \(E\), then \(F = X\).
(3) \(M \in \text{int}(X/E) = \phi\).

**Proof:** (1)→(2). Let \(E\) be an \(b^-\text{closed set of } X\). Then \(\text{cl}(E) = X\). But \(F\) is an \(b^-\text{closed set contains } E\), then \(\text{cl}(E) \subseteq F\) and therefore \(F = X\).

(2)→(3). Since \(b^-\text{-Cl}(E)\) is an \(b^-\text{closed set contains } E\), By (2) we have \(b^-\text{-Cl}(E) = X\). Hence \(\phi = X \setminus b^-\text{p} \subseteq \text{cl}(E) = b^-\text{-int}(X \setminus E)\).

(3)→(1). Since \(b^-\text{-int}(X/E) = \phi\). Then \(\text{cl}(E) = X\). Hence \(E\) is \(b^-\text{dense}\) in \(X\).

Proposition 3.1: For a space \((X,\tau)\), if \(E \in b^-\text{D}(X,\tau)\), then the following statements are hold:

(1) \(b^-\text{-b}(E) = b^-\text{-cl}(X \setminus E)\).
(2) \(b^-\text{ext}(E) = \phi\).

**Proof:** (1) From Definition (3.1), we have \(b^-\text{-b}(E) = b^-\text{-cl}(E) \cap \text{cl}(X \setminus E)\) and since \(E \in b^-\text{D}(X,\tau)\), then \(b^-\text{-b}(E) = b^-\text{-cl}(X \setminus E)\).

(2) Also, by From Definition (3.2), \(b^-\text{ext}(E) = X \setminus b^-\text{-cl}(E)\) but \(E \in b^-\text{D}(X,\tau)\), then \(b^-\text{ext}(E) = \phi\).

Definition 3.7: For a space \((X,\tau)\), \(AG X\) is called:

(1) \(b^-\text{ - nowhere dense if } \text{int}(A) \subseteq b^-\text{-int } (b^-\text{-cl}(A)) = \phi\)
(2) \(b^-\text{ - residual if } b^-\text{-cl}(X\setminus A) = X\) or \(b^-\text{-int }(A) = \phi\)

\(b^-\text{ - nowhere dense is } b^-\text{ - residual from the fact that } b^-\text{-int}(A) \subseteq b^-\text{-int } (b^-\text{-cl}(A))\) for every \(AG X\).

Example 3.12: Let \(X = \{a,b,c\}\) with topology \(\tau = \{X,\emptyset,\{a,b\}\}\) and \(A = \{b\}\) than \(b^-\text{-int } (b^-\text{-cl}(A)) = \phi\). and \(b^-\text{-int}(A) = \phi\) so \(A\) is \(b^-\text{ - nowhere dense and } b^-\text{-residual}.

Proposition 3.2: A subset \(A\) of a space \((X,\tau)\), \(AG X\) is \(b^-\text{ - nowhere dense of } X\) if \(AG b^-\text{-Cl}(X/b^-\text{-cl}(A))\).

**Proof:** Let \(A\) be \(b^-\text{ - nowhere dense then } b^-\text{-int } (b^-\text{-cl}(A)) = \phi\).

Hence \(X \setminus b^-\text{-int } (b^-\text{-cl}(A)) = b^-\text{-cl}(X \setminus b^-\text{-cl}(A)) = b^-\text{-cl}(X \setminus \text{int}(A)) = X \supseteq A\)

Theorem 3.10: The \(b^-\text{-boundary of each } b^-\text{-open (resp. } b^-\text{-closed) set is } b^-\text{-nowhere dense.}

**Proof:** Let \(AEb^-\text{O}(X)\) then

\(b^-\text{-int}(b^-\text{-cl}(b^-\text{-cl}(A))) = b^-\text{-int}(b^-\text{-cl}(b^-\text{-cl}(A) \cap b^-\text{-cl}(X \setminus A)) = b^-\text{ - cl}(b^-\text{-int}(A) \cap (X \setminus b^-\text{-cl}(A))) \subseteq b^-\text{-int}(b^-\text{-int}(A) \cap (X \setminus b^-\text{-cl}(A)) \subseteq b^-\text{-int}(b^-\text{-int}(A) \cap (X \setminus b^-\text{-cl}(A))) = \phi\)

Also if \(AEb^-\text{Cl}(X)\) Then
Proposition 3.3: For a space \((X, \tau)\), \(A \subseteq X\), then the sets \(A \cap b^-cl(X \setminus A)\) and \(b^-cl(A) \cap (X \setminus A)\) are \(b^-\) residual.

Proof: Since

\[
b^- - int(A \cap b^- - cl(X \setminus A)) \subseteq b^- - int(A \cap \neg cl(X \setminus A)) \subseteq b^- - int(A) \cap b^- - cl(X \setminus A),
\]

\[
cl(X \setminus A) = b^- - int(A) \cap (X \setminus b^- - int(A)) = \phi,
\]

Then \(A \cap b^- - cl(X \setminus A)\) is residual. Similarly

\[
b^- - int(b^- - cl(A) \cap (X \setminus A)) \subseteq b^- - int(b^- - cl(A)) \cap b^- - int(X \setminus A) = b^- - cl(A) \cap (X \setminus b^- - cl(A)) = \phi,
\]

and hence \(b^- - cl(A) \cap (X \setminus A)\) is \(b^-\) residual.

Theorem 3.11: The \(b^-\) boundary of any set contains the union of two \(b^-\) residual sets.

Proof: Let \((X, \tau)\) be a space and \(A \subseteq X\). Then by Proposition (3.3), we have

\[
(A \cap b^- - cl(X \setminus A)) \cup (b^- - cl(A) \cap (X \setminus A)) = ((A \cap b^- - cl(X \setminus A) \cup b^- - cl(A)) \cap (A \cup b^- -

cl(X \setminus A)) \cup (b^- - cl(A) \cap (b^- - cl(X \setminus A)) \cup b^- - cl(A)) \cap (A \cup (X \setminus A) \cap (b^- -

cl(X \setminus A) \cup (X \setminus A)) = b^- - cl(A) \cap (b^- - cl(A) \cup b^- - cl(X \setminus A) \cup (X \setminus A) \subseteq (b^- - cl(A) \cap b^- -

cl(X \setminus A) = b^- - cl(A) \cap b^- - cl(X \setminus A) = b^- - cl(A) \cap b^- - cl(X \setminus A) = b^-.
\]

References