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New Fixed Point Theorems in G-metric Spaces

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Abstract

We prove new theorems for generalized contractions in the setting of G-metric spaces. Our results extend some results of Moradlou and Aggarwal.

Keywords: Fixed point; Generalized contractions; G-metric spaces

Introduction

The concept of G-metric spaces was introduced by Mustafa and Sims [1-9] in order to extend and generalize the notion of metric spaces. Recently, Mustafa studied many fixed point theorems for various contractive conditions on complete G-Metric spaces [2].

Moradlou [7] and Aggarwal [2] proved some fixed point theorems for generalized contractions in the setting of G-Metric spaces, our results extend a result of Edelstein [5] and a result of Suzuki [10-18].

In this paper, we prove fixed point results for generalized contractions in the setting of G-metric spaces, extend the works of Aggarwal [2] and Moradlou [7].

Preliminaries

We recall some basic definitions and results which are important in the sequel. For details on the following notions we refer to [5]. First we give the definition of a G-metric space.

Definition 2.1: *Let X be a non-empty set and* $G: X \times X \times X \to R^+$ *be a function satisfying the following axioms:*

(G1) $G(x,y,z) = 0$ if $x=y=z$;

(G2) $0 < G(x,x,y)$ for all $x,y \in X$, with $x \neq y$;

(G3) $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$, with $z \neq y$,

(G4) $G(x,y,z)=G(x,z,y)=G(y,z,x) = \ldots$(symmetry in all three variables)

(G5) $G(x,y,z) \leq G(x,a,a)+G(a,y,z)$, for all $x,y,z,a \in X$, (rectangle inequality).

Then the function G is called a generalized metric, or, more specifically a G-metric on X, and the pair $(X;G)$ is called a G-metric space.

Example 2.1: Let R be the set of all real numbers. Define $G: R\times R\times R \to R^+$ by $G(x,y,z) = |x- y| + |y- z| + |z-x|$; for all $x,y,z \in X$. Then it is clear that (R,G) is a G-metric space.

Proposition 2.2: *Let* (X,G) *be a G-metric space. Then for any* x,y,z *and* $a \in X$*; it follows that:*

1. If $G(x,y,z)=0$ then $x = y = z$;
2. $G(x,y,z) \leq G(x,x,y)+G(x,x,z)$
3. $G(x,y,y) \leq 2G(y,x,x)$
4. $G(x,y,z) \leq G(x,a,z)+G(a,y,z)$

$$G(x,y,z) \leq \frac{2}{3}\left(G(x,y,a)+G(x,a,x,z)+G(a,y,z)\right),$$

$$G(x,y,z) \leq G(x,a,a)+G(y,a,a)+G(z,a,a).$$

Definition 2.3: *Let* (X,G) *be a G-metric space, and* (x_n) *be a sequence of points of X, we say that* (x_n) *is G-convergent to x if for any* $\epsilon>0$*, there exists* $n_0\, 2\, N$ *such that* $G(x,x_n,x_m) < \epsilon$*, for all* $n,m \geq n_0$.

Proposition 2.4: *Let (X;G) be a G-metric space. Then the following are equivalent:*

1. (x_n) is G-convergent to x,
2. $G(x_n,x_n,x) \to 0$, as $n\to\infty$,
3. $G(x_n,x,x) \to 0$, as $n\to\infty$,
4. $G(x_m,x_n,x) \to 0$, as $m,n \to 1$

Definition 2.5: *Let* (X,G) *be a G-metric space, a sequence* (x_n) *is called G-Cauchy if given* $\epsilon > 0$*; there is* $n_o \in N$ *such that* $G(x_n,x_m,x_l) < \epsilon$ *for all* $n,m,l \geq n_o$*;*

Definition 2.6: *Let* (X,G) *and* (X^*,G^*) *be G-metric spaces and let* $f:(X,G)\to(X^*,G^*)$ *be a function, then f is said to be G continuous at a point* $a\in X$*; if given* $\epsilon > 0$*, there exists* $\delta > 0$ *such that* $x,y\in X$*,* $G(a,x,y)<\delta$ *implies* $G^*(f(a);f(x);f(y)) < \varepsilon$.

A function f is G-continuous on X if, and only if, it is G-continuous at all $a \in X$.

Proposition 2.7: *Let* (X,G) *be a G-metric space. Then the function* $G(x,y,z)$ *is continuous in all variables.*

Definition 2.8: *A G-metric space* (X,G) *is said to be G-complete if every G-Cauchy sequence in* (X,G) *is G-convergent in* (X,G).

Definition 2.9: *A G-metric space* (X,G) *is said to be a compact G-metric space if it is G-complete and G-totally bounded.*

Main Results

Our main results are follows:

Theorem 3.1: *Let* (X,G) *be a complete G-metric space and T be a mapping on X. Assume that there exist* $r \in [0,1), (b+c) \in (0,\frac{1}{2}), a \in [0,1]$

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$$\left((a+b+c)r^2+r\right)\le\frac{1}{2}, if\ r\in[\frac{1}{6},\frac{1}{\sqrt{6}}]$$

and

$$a+(2a+b+c)r\le 1, if\ r\in[\frac{1}{\sqrt{6}},1)$$

such that

$aG(x,Tx,Tx) + bG(y,Tx,Tx) + cG(z,Tx,Tx)\ G(x,y,z)$

implies

$G(Tx,Ty,Tz)\le \gamma M(x,y,z)$ *for all* $x,y,z\in X$

where $M(x,y,z)=\max[G(x,y,z),G(x,Tx,Tx),G(y,Ty,Ty),G(z,Tz,Tz)]$. *Then there exist a unique fixed point w of T, Moreover* $\lim_n T^n x = w$ *for all* $x \in X$ *and T is G-continuous at w.*

***Proof*:** Since $aG(x,Tx,Tx)+ bG(Tx,Tx,Tx)+cG(Tx,Tx,Tx)= aG(x,Tx,Tx)\le G(x,Tx,Tx)$ holds for every $x\in X$; by hypothesis, we get

$G(Tx,T^2x,T^2x) \le \gamma M(x,Tx,Tx)$

where

$M(x,Tx,Tx)=\max[G(x,Tx,Tx),G(Tx,T^2x,T^2x)]$

if $M(x,Tx,Tx)=G(Tx,T^2x,T^2x)$ then $G(Tx,T^2x,T^2x)\le rG(Tx,T^2x,T^2x)$ this due to contradiction, hence $M(x,Tx,Tx)=G(x,Tx,Tx)$ and

$$G(Tx,T^2x,T^2x)\le rG(x,Tx,Tx) \text{ for all } x\in X: \quad (3.1)$$

Define a sequence (u_n) in X by $u_n = Tu_{n-1,\dots,}u_n = T^n u; u\in X$. Then in eqn. (3.1) yields

$G(u_n,u_{n+1},u_{n+1}) = G(T^n u,T^{n+1}u,T^{n+1}u)\in rG(T^{n-1}u,T^n u,T^n u)\le\dots\le r^n G(u,Tu,Tu)$

we have by repeated use of the rectangle inequality and the inequality (3.1) that

$G(u_m,u_m,u_n)\le G(u_{n+1},u_{n+1},u_n)+G(u_{n+2},u_{n+2},u_{n+1})+\dots+G(u_{m-1},u_{m-1},u_{m-2})+G(u_m,u_m,u_{m-1})\le(r^n+r^{n+1}+\dots+r^{m-1})G(u,Tu,Tu)$,

$$G(u_m,u_m,u_n)\le\left(\sum_{i=n}^{m-1}r^i\right)G(u,Tu,Tu)=\frac{r^n}{1-r}G(u,Tu,Tu)$$

Then $\lim G(u_m,u_m,u_n) = 0$; as $n,m\to\infty$: For $n,m,l\in N$ (G5) implies that

$G(u_n;u_m;u_l)\ G(u_n;u_m;u_m) + G(u_m;u_m;u_l)$

taking limit as $n,m,l\to\infty$; we get $G(u_n,u_m,u_l)\to 0$. So (u_n) is a G-Cauchy sequences, since X is a complete G-metric spaces, (u_n) converges to some point w in X: Since $\lim_{n\to\infty}G(u_n,Tu_n,Tu_n)=0$ there exist a positive integer k such that

$$G(u_n,w,w)\le\frac{1}{6}G(x,w,w), G(u_n,u_n,w)\le\frac{1}{12}G(x,w,w) \quad (3.2)$$

$$G(x,u_{n+1},u_{n+1})\le\frac{1}{6}G(x,w,w) \quad (3.3)$$

for all $n \ge k$: Using the rectangle inequality, the proposition (2.2), and the inequalities (3.2),(3.3), we get

$$aG(u_n,u_{n+1},u_{n+1})+(b+c)G(x,u_{n+1},u_{n+1})<G(u_n,u_{n+1},u_{n+1})+\frac{1}{2}G(x,u_{n+1},u_{n+1})$$

$$\le G(u_n,w,w)+G(w,u_{n+1},u_{n+1})+\frac{1}{12}G(x,w,w)\le\frac{2}{6}G(x,w,w)=$$

$$\frac{2}{5}[G(x,w,w)-\frac{1}{6}G(x,w,w)]<\frac{2}{5}G(x,u_n,u_n)<\frac{4}{5}G(u_n,x,x)<G(u_n,x,x)$$

By hypothesis, $G(Tu_n,Tx,Tx) \le rM(u_n,x,x)$. Letting n tend to ∞; we obtain $G(w,Tx,Tx) \le rM(w,x,x)$, where

$M(w,x,x) = \max[G(w,x,x),G(x,Tx,Tx)]$, for all $x\in X$;

if $M(w,x,x)=G(x,Tx,Tx)$, we get $G(w,Tx,Tx) \le rG(x,Tx,Tx)$ and $G(T^{j+1}w,T^{j+1}w,w) \le rG(T^{j+1}w,T^{j+1}w,w)$ this is a contradiction. Hence

$$G(w,Tx,Tx) \le rG(w,x,x) \quad (3.4)$$

$$G(u_{j+1},u_{j+1},w)=G(T^{j+1}w,T^{j+1}w,w) \le rG(T^j w,T^j w,w) \le r^2G(T^{j-1}w,T^{j-1}w,w)\le\dots\le r^j G(Tw,Tw,w) \quad (3.5)$$

Now, we consider the following three cases:

The first case, if $0\le\gamma\le\frac{1}{6}$, then by using the rectangle inequality, the proposition (2.2) and the inequalities (3.1), (3.4), we have

$G(Tw,w,w)\le G(Tw,T^2w,T^2w)+G(T^2w,w,w)\le G(Tw,T^2w,T^2w)+2G(w,T^2w,T^2w)\le 2rG(w,Tw,Tw)$

$+rG(w,Tw,Tw)\le 3rG(w,Tw,Tw)\le 6rG(w,w,Tw)\le G(w,w,Tw)$

a contradiction.

The second case, if $\frac{1}{6}\le\gamma\le\frac{1}{\sqrt{6}}$. If we assume

$aG(T^2w,T^3w,T^3w)+(b+c)G(w,T^3w,T^3w)>G(T^2w,w,w)$,

we have

$G(Tw,w,w) \le G(Tw,T^2w,T^2w)+G(T^2w,w,w)$

$<rG(w,Tw,Tw) + [aG(T^2w,T^3w,T^3w)+(b+c)G(w,T^3w,T^3w)]$

$<rG(w,Tw,Tw) + ar^2G(w,Tw,Tw)+(b+c)r^2G(w,Tw,Tw)$

$< ((a + b + c)r^2 + r)G(w,Tw,Tw)< 2((a + b + c)r^2 + r)G(w,w,Tw)\ G(w,w,Tw)$.

which is a contradiction. Hence,

$aG(T^2w,T^3w,T^3w)+(b+c)G(w,T^3w,T^3w)\le G(T^2w,w,w)$

$G(Tw,w,w)\le G(Tw,T^3w,T^3w)+G(T^3w,w,w)\le rG(w,T^2w,T^2w)+2G(T^3w,T^3w,w) \le r^2G(w,Tw,Tw)+2r^2G(w,Tw,Tw)=3r^2G(w,Tw,Tw) \le 6r^2G(Tw,w,w) \le G(Tw,w,w)$ is also a contradiction.

In the third case, if $\frac{1}{\sqrt{6}}\le\gamma\le 1$ we assume that there exist an integer λ such that

$aG(u_n,u_{n+1},u_{n+1}) + (b + c)G(w,u_{n+1},u_{n+1})>G(u_n,w,w)$ for all $n\ge\lambda$.

Then

$$G(u_n,w,w)<aG(u_n,u_{n+1},u_{n+1})+2(b+c)G(u_{n+1},w,w)$$
$$<aG(u_n,u_{n+1},u_{n+1})+2(b+c)\left[aG(u_{n+1},u_{n+2},u_{n+2})+(b+c)G(w,u_{n+2},u_{n+2})\right]$$
$$<(a+2r(b+c)a)G(u_n,u_{n+1},u_{n+1})+4(b+c)^2G(u_{n+2},w,w)$$
$$<(a+2r(b+c)a)G(u_n,u_{n+1},u_{n+1})+(2(b+c))^2$$
$$\left[aG(u_{n+2},u_{n+3},u_{n+3})+(b+c)G(w,u_{n+3},u_{n+3})\right]$$
$$<\left[a+2(b+c)ar+(2(b+c))^2r^2a+\dots+a(2(b+c))^{p-1}\gamma^{p-1}\right]$$
$$G(u_n,u_{n+1},u_{n+1})+(2(b+c))^pG(w,w,u_{n+p})$$
$$<a\frac{1-(2(b+c)\gamma)^p}{1-2(b+c)\gamma}G(u_n,u_{n+1},u_{n+1})+(2(b+c))^pG(w,w,u_{n+p})$$

for all $n \ge \lambda$, $p \ge 1$. Letting $p\to\infty$, and put $d = 2(b + c)$; we obtain

$$G(u_n,w,w)<\frac{a}{1-dr}G(u_n,u_{n+1},u_{n+1}),\ for\ all\ n\ge\lambda.$$

Thus

$$G(u_{n+1},w,w) < \frac{a}{1-dr}G(u_n,u_{n+2},u_{n+2}) < \frac{ar}{1-dr}G(u_n,u_{n+1},u_{n+1}),\ for\ all\ n \ge \lambda$$

so,

$$G(u_n,u_{n+1},u_{n+1}) \le G(u_n,w,w) + G(w,u_{n+1},u_{n+1})$$
$$< \frac{a}{1-dr}G(u_n,u_{n+1},u_{n+1}) + 2G(w,w,u_{n+1}) < [\frac{a}{1-dr} + \frac{2ar}{1-dr}]G(u_n,u_{n+1},u_{n+1})$$
$$\le G(u_n,u_{n+1},u_{n+1}),\ for\ all\ \ n \ge \lambda$$

a contradiction. Hence there exist a subsequence (u_{nk}) of (u_n) such that

$$aG(u_{n_k},u_{n_{k+1}},u_{n_{k+1}}) + dG(w,u_{n_{k+1}},u_{n_{k+1}}) \le G(u_{n_k},w,w) \text{ for all } k \ge 1:$$

By hypothesis, we get $G(Tu_{nk},Tw,Tw) \le rM(u_{nk},w,w)$, for all $k \ge 1$, where

$M(u_{nk},w,w)=\max[G(u_{nk},w,w),G(u_{nk},Tu_{nk},Tu_{nk}),G(w,Tw,Tw)]$

By taking the limit as $k\to\infty$, we obtain that $G(w,Tw,Tw) \le rG(w,Tw,Tw)$, so $G(w,Tw,Tw) = 0$, that is $Tw = w$: This is a contradiction. Thus there exists an integer j such that $T^jw = w$; by (3.1), we get

$G(w,Tw,Tw) = G(T^jw,T^{j+1}w,T^{j+1}w)\le r^jG(w,Tw,Tw)$,

so $G(w,Tw,Tw) =0$; that is $Tw = w$. Now suppose that y is another fixed point of T, then

$aG(y,Ty,Ty)+(b + c)G(w,Ty,Ty)\ G(w,y,y)$

implies $G(Tw,Ty,Ty) \le rM(w,y,y)=rG(w,y,y)$. Hence $G(w,y,y)=0$, this is a contradiction. To see that T is G-continuous at a fixed point w. Let (y_n) be a sequence such that $\lim_{n\to\infty} y_n = w$. then

$$\alpha G(w,Tw,Tw) + dG(y_n,Tw,Tw) < \frac{1}{2}G(y_n,T_w,T_w) < G(y_n,y_n,w)$$

By hypothesis, we get

$G(Ty_n,Ty_n,Tw) \le rM(y_n,y_n,w)$

where

$M(y_n,y_n,w) = \max[G(y_n,y_n,w),\ G(y_n,Ty_n,Ty_n),G(w,Tw,Tw)] = \max[G(y_n,y_n,w),\ G(y_n,w,w) + G(w,Ty_n,Ty_n)]$

If $M(y_n,y_n,w)=G(y_n,w,w)+G(w,Ty_n,Ty_n)$, we get

$G(Ty_n,Ty_n,Tw) \le r(G(y_n,w,w)+G(w,Ty_n,Ty_n))$

$$G(T_{y_n},Ty_n,w) \le \frac{r}{1-r}G(y_n,w,w) \tag{3.6}$$

If $M(y_n,y_n,w)=G(y_n,y_n,w)$, we get

$$G(Ty_n,Ty_n,w)\le rG(y_n,w,w) \tag{3.7}$$

In each inequality (3.6) and (3.8), take the limit as $n\to\infty$ to see that $G(Ty_n,Ty_n,w)\to 0$ and so by proposition (2.4), we have that the sequence (Ty_n) is G-convergent to $w = Tw$, hence T is G-continuous at w.

Now, we give a fixed point theorem on compact G-metric spaces.

Theorem 3.2: *Let (X,d) be a compact G-metric space and T be a mapping on X. Assume that*

$$aG(x,Tx,Tx)+bG(y,Tx,Tx)+cG(z,Tx,Tx)<G(x,y,z) \tag{3.8}$$

implies

$G(Tx,Ty,Tz) <M(x,y,z)$ *for all* $x,y, z \in X$

where $M(x,y,z) = \max[G(x,y,z),G(x,Tx,Tx),G(y,Ty,Ty),G(z,Tz,Tz)]$ *and* $a>0$, $b>0$, $c>0$, $3a + 2(b + c) < 1$, $2(b + c) < 1$. *Then T has a unique fixed point.*

Proof: If we consider $\beta=\inf\{G(x,Tx,Tx): x\in X\}$; then there exists a sequence (x_n) in X such that $\lim_{n\to\infty} G(x_n,Tx_n,Tx_n) = \beta$ .Since X is compact G- metric space, there exists $v,w\in X$ such that a sequence (x_n) is G- converges to $v\in X$, and (Tx_n) G- converge to $w\in X$. We assume $\beta> 0$. Hence, by the continuity of the function G, we have

$$\beta = \lim_{n\to\infty} G(x_n,Tx_n,Tx_n) = G(v,w,w) = \lim_{n\to\infty} G(x_n,w,w) \tag{3.9}$$

Since

$$\lim_{n\to\infty}[aG(x_n,Tx_n,Tx_n) + (b+c)G(w,Tx_n,Tx_n)] = a\beta < \lim_{n\to\infty} G(x_n,w,w) = \beta \tag{3.10}$$

we can choose a positive integer N such that

$aG(x_n,Tx_n,Tx_n) + (b + c)G(w,Tx_n,Tx_n) <G(x_n,w,w)$, for all $n \ge N$.

By hypothesis, $G(Tx_n,Tw,Tw) <M(x_n,w,w)$, holds for $n \ge N$; where

$M(x_n,w,w) = \max[G(x_n,w,w),G(x_n,Tx_n,Tx_n),G(w,Tw,Tw)]$

this implies

$$G(w,Tw,Tw) = \lim_{n\to\infty} G(Tx_n,Tw,Tw) < \lim_{n\to\infty} M(x_n,w,w) = \max$$
$$[\beta,G(w,Tw,Tw)], \text{If } \max[\beta,G(w,Tw,Tw)] = G(w,Tw,Tw)$$

this is impossible, then $G(w,Tw,Tw) < \beta$. From the definition of β we obtain $\beta=G(w,Tw,Tw)$.

Since $aG(w,Tw,Tw) + (b + c)G(Tw,Tw,Tw) <G(w,Tw,Tw)$

we have $G(Tw,T^2w,T^2w)<M(w,Tw,Tw)=\max[G(w,Tw,Tw),G(Tw,T^2w,T^2w)]=G(w,Tw,Tw)=\beta$, which contradicts the definition of β, therefore $\beta=0$, and

$$\lim_{n\to\infty} G(x_n,w,w) = \lim_{n\to\infty} G(x,Tx_n,Tx_n) = \lim_{n\to\infty} G(v,Tx_n,Tx_n) = G(v,w,w) = 0$$

so $v =w$; thus $\lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n$. We next show that T has a fixed point. We assume that T does not have a fixed point. Since

$aG(x_n,Tx_n,Tx_n) + (b + c)G(Tx_n,Tx_n,Tx_n) <G(x_n,Tx_n,Tx_n)$, for all $n\ge 1$

we get $G(Tx_n,T^2x_n,T^2x_n) <M(x_n,Tx_n,Tx_n) =\max[G(x_n,Tx_n,Tx_n) ,G(Tx_n,T^2x_n,T^2x_n)]=G(x_n,Tx_n,Tx_n)$

By using the rectangle inequality, we have

$G(w,T^2x_n,T^2x_n)\ G(w,Tx_n,Tx_n) + G(Tx_n,T^2x_n,T^2x_n) <G(w,Tx_n,Tx_n) + G(x_n,Tx_n,Tx_n)$

$$\lim_{n\to\infty} G(w,T^2x_n,T^2x_n) < \lim_{n\to\infty}(G(w,Tx_n,Tx_n) + G(x_n,Tx_n,Tx_n)) = 0$$

Thus (T^2x_n) is G-convergent to w. By induction, we obtain that

$G(T^px_n;T^{p+1}x_n;T^{p+1}x_n)\ G(T^{p-1}x_n,T^px_n,T^px_n)\le\ldots\le G(x_n;Tx_n;Tx_n)$

$G(w,T^px_n,T^px_n)\le G(w,T^{p-1}x_n;T^{p-1}x_n) + G(T^{p-1}x_n;T^px_n;T^px_n)$

$$\lim_{n\to\infty} G(w,T^px_n,T^px_n) < \lim_{n\to\infty}(G(w,T^{p-1}x_n,T^{p-1}x_n) + G(x_n,Tx_n,Tx_n))$$

this due to $\lim_{n\to\infty} T^px_n = w$ for all $p \ge 1$. If there exist an integer $p \ge 1$ and a subsequence (x_{nk}) of (x_n) such that

$$aG(T^{p-1}x_{n_k},T^{p-1}x_{n_k}) + (b+c)G(w,T^px_{n_k},T^px_{n_k}) < G(T^{p-1}x_{n_k},w,w), \text{for all } k \ge 1$$

by hypothesis, we have

$G(T^px_{nk},Tw,Tw)<M(T^{p-1}x_{nk},w,w) = \max[G(T^{p-1}x_{nk},w,w),G(T^{p-1}x_{nk},T^px_{nk},T^px_{nk}),G(w,Tw,Tw)]$

Taking the limit as $k \to\infty$, $G(w,Tw,Tw)=0$, $w=Tw$ which is a contradiction. Hence, we can assume that for every $m \ge 1$, there exist an integer $n(m) \ge 1$ such that

$aG(T^{m-1}x_n, T^m x_n, T^m x_n) + (b+c)G(w, T^m x_n, T^m x_n) \geq G(T^{m-1}x_n, w, w)$, *for all* $n \geq n(m)$ (3.11)

We put $\gamma = \max[n(1), n(2), \ldots, n(p)]$. Then by the inequality (3.11) we have

$$\begin{aligned}
&G(x_n,w,w) \le aG(x_n,Tx_n,Tx_n) + (b+c)G(w,Tx_n,Tx_n) \le aG(x_n,Tx_n,Tx_n) + 2(b+c)G(w,w,Tx_n)\\
&\le aG(x_n,Tx_n,Tx_n) + 2(b+c)\left(aG(Tx_n,T^2x_n,T^2x_n) + (b+c)G(w,T^2x_n,T^2x_n)\right)\\
&\le aG(x_n,Tx_n,Tx_n) + 2a(b+c)G(Tx_n,T^2x_n,T^2x_n) + (2(b+c))^2 G(w,w,T^2x_n)\\
&\le aG(x_n,Tx_n,Tx_n) + 2a(b+c)G(Tx_n,T^2x_n,T^2x_n) +\\
&(2(b+c))^2\left(aG(T^2x_n,T^3x_n,T^3x_n) + (b+c)G(w,T^3x_n,T^3x_n)\right) \le \ldots\\
&\le aG(x_n,Tx_n,Tx_n) + 2a(b+c)G(Tx_n,T^2x_n,T^2x_n) + \ldots +\\
&(2(b+c))^2\left(aG(T^2x_n,T^3x_n,T^3x_n) + (b+c)G(w,T^3x_n,T^3x_n)\right) \le \ldots\\
&\le aG(x_n,Tx_n,Tx_n) + 2a(b+c)G(Tx_n,T^2x_n,T^2x_n) + \ldots +\\
&a(2(b+c))^{p-1} G(T^{p-1}x_n,T^P x_n,T^P x_n) + (2(b+c))^p G(w,w,T^2x_n)\\
&\le a\left(1 + 2(b+c) + (2(b+c))^2 + \ldots + (2(b+c))^{p-1} G(x_n,Tx_n,Tx_n) + (2(b+c))^p G(w,w),T^p x_n\right)\\
&G(x_n,w,w) \le \frac{a(1-d^p)}{1-d} G(x_n,Tx_n,Tx_n) + d^p G(w,w,T^p x_n), \textit{where } d = 2(b+c)
\end{aligned} \tag{3.12}$$

Using rectangular inequality and proposition (2.2), we get

$$\begin{aligned}
&G(T^p x_n,w,w) \le G(T^p x_n,x_n,x_n) + G(x_n,w,w) \le 2G(T^p x_n,T^P x_n,x_n) + G(x_n,w,w)\\
&\le 2\left(G(x_n,Tx_n,Tx_n) + G(Tx_n,T^2x_n,T^2x_n) + \ldots + G(T^{p-1}x_n,T^p x_n,T^p x_n)\right) + G(x_n,w,w)\\
&\le 2pG(x_n,Tx_n,Tx_n) + G(x_n,w,w)
\end{aligned} \tag{3.13}$$

substitute (3.13) in (3.12), we get

$$\begin{aligned}
&G(x_n,w,w) \le \frac{a(1-d^p)}{1-d} G(x_n,Tx_n,Tx_n) + d^p\left(2pG(x_n,Tx_n,Tx_n) + G(x_n,w,w)\right)\\
&(1-d^p)G(x_n,w,w) \le \left(\frac{a(1-d^p)}{1-d} + 2pd^p\right)G(x_n,Tx_n,Tx_n)\\
&G(x_n,w,w) \le \left(\frac{a}{1-d} + \frac{2pd^p}{1-d^p}\right)G(x_n,Tx_n,Tx_n), \textit{ for all } n \ge \gamma
\end{aligned} \tag{3.14}$$

Similarly, we obtain

$$\begin{aligned}
&G(Tx_n,w,w) \le \left(\frac{a}{1-d} + \frac{2(p-1)d^{p-1}}{1-d^{p-1}}\right)G(Tx_n,T^2x_n,T^2x_n)\\
&\le \left(\frac{a}{1-d} + \frac{2(p-1)d^{p-1}}{1-d^{p-1}}\right)G(x_n,Tx_n,Tx_n)
\end{aligned} \tag{3.15}$$

Since $G(x_n,Tx_n,Tx_n) \le G(x_n,w,w) + G(w,Tx_n,Tx_n)$ $G(x_n,w,w) + 2G(w,w,Tx_n)$, by use the inequalities (3.14) and (3.15),

$$\begin{aligned}
&G(x_n,Tx_n,Tx_n) \le \left(\frac{a}{1-d} + \frac{2pd^p}{1-d} + \frac{2a}{1-d^p} + \frac{4(p-1)d^{p-1}}{1-d^{p-1}}\right)G(x_n,Tx_n,Tx_n)\\
&\le \left(\frac{3a}{1-d} + \frac{2pd^p}{1-d^p} + \frac{4(p-1)d^{p-1}}{1-d^{p-1}}\right)G(x_n,Tx_n,Tx_n), \textit{ for all } n \ge \gamma
\end{aligned}$$

Since $\lim_{p\to\infty} \frac{2pd^p}{1-d^p} = 0, \textit{and } \frac{3a}{1-d} < 1$ we can choose p satisfying $\frac{3a}{1-d} + \frac{2pd^p}{1-d^p} + \frac{4(p-1)d^{p-1}}{1-d^{p-1}} < 1$. Then

$G(x_n,Tx_n,Tx_n) < G(xn,Tx_n,Tx_n)$

which is a contradiction, therefore there exist $z \in X$ such that $Tz = z$. Fix $y \in X$ with $y \ne x$. Then $aG(x,Tx,Tx) + (b+c)G(y,Tx,Tx) = (b+c)G(y,x,x) \le 2(b+c)G(x,y,y) < G(x,y,y)$, we have $G(Tx,Ty,Ty) < M(x,y,y) = \max[G(x,y,y), G(x,Tx,Tx), G(y,Ty,Ty)] = G(x,y,y)$. Hence y is not a fixed point of T and T has a unique fixed point.

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