New Fixed Point Theorems in G-metric Spaces

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Abstract

We prove new theorems for generalized contractions in the setting of G-metric spaces. Our results extend some results of Moradlou and Aggarwal.

Keywords: Fixed point; Generalized contractions; G-metric spaces

Introduction

The concept of G-metric spaces was introduced by Mustafa and Sims [1-9] in order to extend and generalize the notion of metric spaces. Recently, Mustafa studied many fixed point theorems for various contractive conditions on complete G-Metric spaces [2]. Moradlou [7] and Aggarwal [2] proved some fixed point theorems for generalized contractions in the setting of G-Metric spaces, our results extend a result of Edelstein [5] and a result of Suzuki [10-18].

In this paper, we prove fixed point results for generalized contractions in the setting of G-metric spaces, extend the works of Aggarwal [2] and Moradlou [7].

Preliminaries

We recall some basic definitions and results which are important in the sequel. For details on the following notions we refer to [5]. First we give the definition of a G-metric space.

Definition 2.1: Let X be a non-empty set and G: X ×X × X → R' be a function satisfying the following axioms:

(G1) G(x, y, z) = 0 if x = y = z;
(G2) 0 < G(x, y, z) for all x, y ∈ X, with x ≠ y;
(G3) G(x, y, z) ≤ G(x, z, z) for all x, y, z ∈ X, with z ≠ y,
(G4) G(x, y, z) = G(x, z, y) = G(y, z, x) = ... (symmetry in all three variables)
(G5) G(x, y, z) ≤ G(x, a, a) + G(a, y, z), for all x, y, z, a ∈ X, (rectangle inequality).

Then the function G is called a generalized metric, or, more specifically a G-metric on X, and the pair (X, G) is called a G-metric space.

Example 2.1: Let R be the set of all real numbers. Define G: R × R × R → R by G(x, y, z) = |x - y| + |y - z| + |z - x|; for all x, y, z ∈ X. Then it is clear that (R, G) is a G-metric space.

Proposition 2.2: Let (X, G) be a G-metric space. Then for any x, y, z and a ∈ X it follows that:

1. If G(x, y, z) = 0 then x = y = z;
2. G(x, y, z) ≤ G(x, y, y) + G(x, x, z)
3. G(x, y, z) ≤ 2G(y, x, x)
4. G(x, y, z) ≤ G(x, a, z) + G(a, y, z)
5. G(x, y, z) ≤ 2(G(x, y, a) + G(x, a, x, z) + G(a, y, z)),
   
   G(x, y, z) ≤ G(x, a, a) + G(y, a, a) + G(z, a, a).

Definition 2.3: Let (X, G) be a G-metric space, and (x_n) be a sequence of points of X, we say that (x_n) is G-convergent to x if for any ε > 0, there exists N such that G(x_n, x) < ε, for all n ≥ N.

Proposition 2.4: Let (X, G) be a G-metric space. Then the following are equivalent:

1. (x_n) is G-convergent to x,
2. G(x_n, x) → 0, as n → ∞,
3. G(x_n, x) → 0, as n → ∞,
4. G(x_n, x) → 0, as m, n → 1

Definition 2.5: Let (X, G) be a G-metric space, a sequence (x_n) is called G-Cauchy if given ε > 0, there is n_0 ∈ N such that G(x_n, x_m) < ε for all n, m ≥ n_0;

Definition 2.6: Let (X, G) and (X', G') be G-metric spaces and let f: (X, G) → (X', G') be a function, then f is said to be G continuous at a point a ∈ X, if given ε > 0, there exists δ > 0 such that x, y ∈ X, G(a, x, y) < δ implies G'(f(a), f(x), f(y)) < ε.

A function f is G-continuous on X if, and only if, it is G-continuous at all a ∈ X.

Proposition 2.7: Let (X, G) be a G-metric space. Then the function G(x, y, z) is continuous in all variables.

Definition 2.8: A G-metric space (X, G) is said to be G-complete if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).

Definition 2.9: A G-metric space (X, G) is said to be a compact G-metric space if it is G-complete and G-totally bounded.

Main Results

Our main results are follows:

Theorem 3.1: Let (X, G) be a complete G-metric space and T be a mapping on X. Assume that there exist r ∈ [0, 1), (b+c) ∈ (0, 1), a ∈ [0, 1]
\[(a+b+c)r^2+r\leq \frac{1}{2} + \frac{1}{6r}\frac{1}{\sqrt{6}}\]
and
\[a+(2a+b+c)r\leq 1, if \ r = \left[\frac{1}{\sqrt{6}}\right]^{-1}\]
such that
\[aG(x,Tx,Tx) + bG(y,Tx,Ty) + cG(z,Ty,Tz)G(x,y,z)\]
implies
\[G(Tx,Ty,Tz)\leq \gamma M(x,y,z)\] for all \(x,y,z \in X\)

where \(M(x,y,z) = \max\{G(x,y,z),G(x,Tx,Ty),G(y,Ty,Tz),G(z,Tz,Tx)\}\).

Then there exist a unique fixed point \(w\) of \(T\). Moreover
\[G(Tw,Tw,Tw) = \max\{G(Tw,w,w)\}\]
by hypothesis, we get
\[\lim n_{\rightarrow \infty} T^n w = w\] for all \(x \in X\) and \(T\) is \(G\)-continuous at \(w\).

Proof: Since \(aG(x,Tx,Tx) + bG(x,Tx,Ty) + cG(Tx,Ty,Tx) = aG(x,Tx,Tx)\) holds for every \(x \in X\), by hypothesis, we get
\[G(Tx,Tx,Tx) \leq \gamma M(x,y,z)\]
where
\[M(x,y,z) = \max\{G(x,y,z),G(x,Tx,Ty),G(y,Ty,Tz),G(z,Tz,Tx)\}\]

if \(M(x,y,z) = \max\{G(x,y,z),G(x,Tx,Ty),G(y,Ty,Tz),G(z,Tz,Tx)\}\) and \(G(x,y,z)\) is a complete \(G\)-metric space, \((3.4)\) holds for every \(x \in X, T\) is \(G\)-continuous at \(w\).

If \(w \rightarrow \infty\), we get
\[G(Tw,Tw,Tw) = \max\{G(Tw,w,w)\}\]
then
\[G(Tw,Tw,Tw) = aG(Tw,w,w) + (b+c)G(Tw,Tw,Tw)\]
which is a contradiction. Hence
\[G(Tw,Tw,Tw) = aG(Tw,w,w) + (b+c)G(Tw,Tw,Tw)\]

In the third case, if \(\frac{1}{6} \leq \gamma \leq \frac{1}{\sqrt{6}}\), we assume
\[aG(Tw,w,w) + (b+c)G(Tw,Tw,Tw)\leq \frac{aG(Tw,w,w)}{1-r}\]
which is a contradiction. Hence
\[G(Tw,Tw,Tw) = aG(Tw,w,w) + (b+c)G(Tw,Tw,Tw)\]

In the third case, if \(\frac{1}{\sqrt{6}} \leq \gamma \leq 1\) we assume that there exist an integer \(k\) such that
\[aG(Tw,w,w) + (b+c)G(Tw,Tw,Tw)\leq \frac{aG(Tw,w,w)}{1-r}\]
for all \(n \geq k\). Using the rectangle inequality, the proposition (2.2), and the inequalities (3.3), we get
\[aG(Tw,w,w) + (b+c)G(Tw,Tw,Tw)\leq \frac{aG(Tw,w,w)}{1-r}\]
for all \(n \geq k\). Using the rectangle inequality, the proposition (2.2), and the inequalities (3.3), we get
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for all \(n \geq k\). Using the rectangle inequality, the proposition (2.2), and the inequalities (3.3), we get
\[aG(Tw,w,w) + (b+c)G(Tw,Tw,Tw)\leq \frac{aG(Tw,w,w)}{1-r}\]
Thus
\[ G(u_n,\ldots,u_n,w) = \frac{n}{1-r} G(u_n,\ldots,u_n,w) + \frac{m}{1-r} G(u_n,\ldots,u_n,w) \]
so,
\[ G(u_n,\ldots,u_n,w) \leq G(u_n,\ldots,u_n,w) + G(u_n,\ldots,u_n,w) \]
\[ \leq \frac{n}{1-r} G(u_n,\ldots,u_n,w) + \frac{m}{1-r} G(u_n,\ldots,u_n,w) \]
\[ \leq G(u_n,\ldots,u_n,w), \text{for all } n \geq \lambda \]

a contradiction. Hence there exist a subsequence \((u_{n_k})\) of \((u_n)\) such that
\[ aG(u_{n_k},u_{n_k},u_{n_k}) + dG(u_{n_k},u_{n_k},u_{n_k}) \leq G(u_{n_k},u_{n_k},w) \]
for all \(k \geq 1\).

By hypothesis, we get \(G(T_{w_n},T_{w_n},T_{w_n}) \leq rM(u_n,\ldots,u_n,w)\), for all \(k \geq 1\), where
\[ M(u_n,\ldots,u_n,w)=\max\{G(u_n,\ldots,u_n,w),G(u_n,\ldots,u_n,T_{w_n}),G(w,T_{w_n},T_{w_n})\} \]

By taking the limit as \(k \to \infty\), we obtain that \(G(w,T_{w_n},T_{w_n}) \leq rG(w,T_{w_n},T_{w_n}), \text{so } G(T_{w_n},T_{w_n}) = 0\), which is a contradiction. Thus there exists an integer \(n_0\) such that \(T_{w_n} \to w\) as \(n \to \infty\).

Theorem 3.2: Let \((X,d)\) be a compact \(G\)-metric space and \(T\) be a mapping on \(X\). Assume that
\[ aG(x,T_{x_n},T_{x_n}) + bG(y,T_{y_n},T_{y_n}) + cG(z,T_{z_n},T_{z_n}) \leq G(x,y,z) \]
implies
\[ G(T_{x_n},T_{y_n},T_{z_n}) < M(x,y,z) \text{ for all } x,y,z \in X \]
where \(M(x,y,z) = \max\{G(x,y,z),G(x,T_{x_n},T_{x_n}),G(y,T_{y_n},T_{y_n}),G(z,T_{z_n},T_{z_n})\}\) and \(a>0, b>0, c>0, 3a+2(b+c) < 1, 2(b+c) < 1\). Then \(T\) has a unique fixed point.

Proof: If we consider \(\beta=\min\{G(x,T_{x_n},T_{x_n}) : x \in X\}\), then there exists a sequence \((x_n)\) in \(X\) such that \(\lim_{n \to \infty} G(x_n,T_{x_n},T_{x_n}) = \beta\). Since \(X\) is compact \(G\)-metric space, there exists \(v \in X\) such that a sequence \((x_n)\) is \(G\)-converges to \(v \in X\), and \((T_{x_n})\) \(G\)-converges to \(v \in X\). We assume \(\beta>0\). Hence, by the continuity of the function \(G\), we have
\[ \beta = \lim_{n \to \infty} G(x_n,T_{x_n},T_{x_n}) = G(v,v,v) = \lim_{n \to \infty} G(x_n,w,w) \]  
(3.9)

Since
\[ \lim_{n \to \infty} aG(x_n,T_{x_n},T_{x_n}) + (b+c)G(v,v,v) = \beta < \lim_{n \to \infty} G(x_n,w,w) = \beta \]  
(3.10)

we can choose a positive integer \(N\) such that
\[ aG(x_n,T_{x_n},T_{x_n}) + (b+c)G(v,v,v) < G(x_n,w,w) \]
for all \(n \geq N\).

By hypothesis, \(G(T_{x_n},T_{x_n},T_{w_n}) < \lambda M(x_n,w,w)\), holds for \(n \geq N\); where
\[ M(x_n,w,w)=\max\{G(x_n,w,w),G(x_n,T_{x_n},T_{x_n}),G(w,T_{w_n},T_{w_n})\} \]

this implies
\[ G(w,T_{w_n},T_{w_n}) = \lim_{n \to \infty} G(T_{x_n},T_{x_n},T_{w_n}) < \lim_{n \to \infty} M(x_n,w,w) = \max_{x \in X} \frac{\beta}{\beta} G(v,v,v) = G(w,T_{w_n},T_{w_n}) \]

is impossible, then \(G(w,T_{w_n},T_{w_n}) < \beta\). From the definition of \(\beta\) we obtain \(\beta=G(w,T_{w_n},T_{w_n})\).

Since \(aG(w,T_{w_n},T_{w_n}) + (b+c)G(T_{w_n},T_{w_n},T_{w_n}) < G(x_n,T_{x_n},T_{x_n})\), for all \(n \geq N\) we get
\[ G(T_{x_n},T_{x_n},T_{w_n}) < \lambda M(x_n,w,w) = \max\{G(x_n,w,w),G(x_n,T_{x_n},T_{x_n}),G(w,T_{w_n},T_{w_n})\} = G(x_n,T_{x_n},T_{x_n}) \]

By using the rectangle inequality, we have
\[ G(w,T_{x_n},T_{x_n}) < G(w,T_{x_n},T_{x_n}) + G(T_{x_n},T_{x_n},T_{x_n}) = G(T_{x_n},T_{x_n},T_{x_n}) \]

for all \(n \geq N\).

Thus \((T_{x_n})\) is \(G\)-convergent to \(w\) as \(n \to \infty\). Hence \(T\) is \(G\)-continuous at \(w\).

Now, we give a fixed point theorem on compact \(G\)-metric spaces.

Theorem 3.3: Let \((X,d)\) be a compact \(G\)-metric space and \(T\) be a mapping on \(X\). Assume that
\[ aG(x,T_{x_n},T_{x_n}) + bG(y,T_{y_n},T_{y_n}) + cG(z,T_{z_n},T_{z_n}) < G(x,y,z) \]
implies
\[ G(T_{x_n},T_{y_n},T_{z_n}) < M(x,y,z) \text{ for all } x,y,z \in X \]
where \(M(x,y,z) = \max\{G(x,y,z),G(x,T_{x_n},T_{x_n}),G(y,T_{y_n},T_{y_n}),G(z,T_{z_n},T_{z_n})\}\) and \(a>0, b>0, c>0, 3a+2(b+c) < 1, 2(b+c) < 1\). Then \(T\) has a unique fixed point.
\[ G(T^{m+1}x_{n},T^{m+1}x_{n}) + \frac{a}{1-d'}G(T^{m}x_{n},T^{m}x_{n}) \leq \frac{a}{1-d'} G(T^{m}x_{n},T^{m}x_{n}) + d' G(T^{m}x_{n},T^{m}x_{n}) \]

(3.11)

We put \( \gamma = \max \{ \eta(1,0), \ldots, \eta(n) \} \). Then by the inequality (3.11) we have

\[ G(x_{n},x_{n}) \leq \frac{a}{1-d'} G(T^{m}x_{n},T^{m}x_{n}) + d' G(T^{m}x_{n},T^{m}x_{n}), \]

(3.12)

which is a contradiction, therefore there exist \( \varepsilon \in \mathbb{X} \) such that \( Tz = \varepsilon \). Fix \( \varepsilon \in \mathbb{X} \) with \( \varepsilon \neq x_{n} \). Then

\[ G(x_{n},Tz) + \frac{(b+c)G(x_{n},x_{n})}{2(b+c)} \leq G(x_{n},x_{n}) \leq G(x_{n},x_{n}) \]

(3.13)

Using rectangular inequality and proposition (2.2), we get

\[ G(x_{n},x_{n}) \leq \frac{a}{1-d'} G(T^{m}x_{n},T^{m}x_{n}) + d' G(T^{m}x_{n},T^{m}x_{n}) \]

(3.14)

Similarly, we obtain

\[ G(Tx_{n},x_{n}) \leq \frac{a}{1-d'} G(Tx_{n},x_{n}) + d' G(Tx_{n},x_{n}) \]

(3.15)

Since \( G(x_{n},Tx_{n}) \leq G(x_{n},w_{n}) + G(w_{n},Tx_{n}) = G(x_{n},w_{n}) + 2G(w_{n},Tx_{n}) \), by using the inequalities (3.14) and (3.15),

\[ G(x_{n},Tx_{n}) \leq \frac{a}{1-d'} \frac{2p}{1-d'} + \frac{a}{1-d'} \frac{2p}{1-d'} G(x_{n},Tx_{n}) \]

(3.16)

Since \( \lim_{p \to +\infty} \frac{2pd'}{1-d'} = 0 \), and \( \frac{3a}{1-d'} \frac{2pd'}{1-d'} < 1 \) we can choose \( p \) satisfying

\[ \frac{3a}{1-d'} + \frac{2pd'}{1-d'} + \frac{4p(1-d')}{1-d'} < 1. \]

Then

\[ G(x_{n},Tx_{n}) < G(x_{n},Tx_{n}) \]

References: