Research Article
Volume 9:1, 2020
DOI: 10.37421/jacm.2020.9.459

Journal of
Applied & Computational Mathematics

ISSN: 2168-9679 Open Access

Modelling and Optimal Control of Toxicants on Fish Population with Harvesting

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Abstract

Toxic in water bodies is a worldwide problem. It kills fish and other aquatic animals in water. Human beings are affected by this indirectly through eating affected fish. In this paper, a model for controlling toxicants in water is formulated and analysed. Boundedness, positivity and analysis of the model are examined where four steady states are determined by using Eigen-value analysis and found to be locally stable under some conditions. The optimal control strategies are established with the help of Pontryagin's maximum principle. The simulations for the model with control show that when control is applied the results reveals that the amount of toxic is reduced and hence there is an increase in fish population for both prey and predator populations. It is recommended that the government has to introduce laws and policies which ensure that the industries treat waste water before they are discharged into water bodies and to develop a system for waste recycling.

Keywords: Toxicants • Fish • Harvesting • Optimal

Introduction

Over several years, the effects of toxicants on ecological communities have become major environmental problem. Due to the growing of human needs, the industries are producing a large amount of waste that contains toxicants which are exposed to environment that cause many species to extinct and several others are at the risk of extinction [1]. Currently, researchers are taking interest in the eco toxicological effects of toxicants released by the marine and industries as well. For example, [2] incorporated the effects of toxic substances in a two species Lotka-Volterra competitive system by considering that some species produce a substance toxic to others only when others are present. The same idea was extended by [3] a two species competing fish effect of a toxicant on the dynamics of a spatial fishery, species which are commercially exploited. [1], in his study of dynamical behaviour of a two-species competitive system affected by toxic substances, modified the deterministic model by incorporating the effect the diffusion and fluctuation of environmental whereas [4] studied a bio-economic harvesting of prey-predator fishery in which all species are infected by toxicants which are released by other species. From reviewed literature, it is seen that most of studies concentrated on the effect of toxic to aquatic organism (fish) and environment as well. In this paper, it is intended to model and apply control strategies to toxic substances in fish population during harvesting.

Model Formulation

The formulation of the model will include ideas of Holling type II- function response which is most typical and applied where the rate of prey consumption by predator rises as prey density increases, but eventually level off at plateau or asymptote at which the rate of consumption remains constant regardless of the increases in prey density. The model to be formulated is based on the findings of [4] and [5]. Motivated by the findings of [5] who did their study on a focus on long-run sustainability of a harvested prey-predator system in the presence of alternative prey introduced the following system of differential equation:

$$\frac{dN}{dT} = rN\left(1-\frac{N}{K}\right) - \frac{\alpha_1 NP}{\alpha + N}, \tag{1}$$

$$\frac{dP}{dT} = \frac{\beta_1 \alpha_1 NP}{\alpha + N} + P\left(1-\frac{N}{K}\right) - y_1 P + d_1, \tag{2}$$

where

$N = N(T)$ is the size of prey at time T,

$P = P(T)$ is the size of predator at T,

K is the environmental carrying capacity of prey,

r is the intrinsic growth rate of prey,

α_1 is the predation coefficient,

α is half saturation constant,

β_1 is the conversion rate factor $(\beta_1 < 1)$,

d_1 is the digestion factor relative to alternative prey,

y_1 is the mortality rate of predator population and

[4]studied the harvesting of prey-predator fishery in the presence of toxicity and developed the following nonlinear system:

$$\frac{dx_1}{dt} = r_1 x_1\left(1-\frac{x}{L}\right) - \alpha x_1 x_2 - c_1 E x_1 - \gamma_1 x_1^3, \tag{3}$$

$$\frac{dx_2}{dt} = -r_2 x_2 + \beta x_1 x_2 - c_2 E x_2 - \gamma_2 x_2^2, \tag{4}$$

where

$x_1 = x_1(t)$ is the size of prey population at time t,

$x_2 = x_2(t)$ is the size of predator population at time t,

r_1 is the maximum specific growth rate of prey population,

r_2 is the relative rate at which the predator dies out in absence of prey,

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Received 18 November 2020; **Accepted** 20 November 2020; **Published** 27 November 2020

L is the environmental carrying capacity of the prey population,

E is the combined harvesting effort,

c_1 is the catchability coefficient of prey,

c_2 is the catchability coefficient of predator,

γ_1 is the coefficient of toxicity to the prey species,

γ_2 is the coefficient of toxicity to predator species.

The model to be formulated lies in the frame work of the systems above i.e. ((1), (2) and (3), and (4)). The model will be formulated by extending the work of [5] who incorporated toxicity terms to both species subject to harvesting efforts to both species which are adopted from a work of [4]. The model has two state variables: $Y(t)$ the population of a prey fish and $Y(t)$, the population of predator fish.

In formulating model, the following assumptions are taken into considerations:

1. Each species is infected by some external toxic substance from external sources,

2. Prey are directly affected by external toxic while predator are indirect affected,

3. The effect of external sources is assumed to be different,

4. The prey reproduction is influenced by predators only,

5. The predator's growth depends on the prey which it catches,

6. In the absence of predation, toxicity and harvesting, the prey grows exponentially,

7. Both prey and predator are subjected to harvesting efforts,

8. There is no any disease that affects prey and predator fish,

9. No natural death of prey population.

Taking into account the above considerations and assumptions; we have the following schematic flow diagram Figure 1:

From the flow diagram (Figure 1) and assumptions, the resulting system is governed by the following equations:

$$\frac{dX}{dt} = rX\left(1-\frac{X}{L}\right)-\frac{\alpha XY}{k+X}-\gamma_1 X^3 - c_1 EX,$$

$$\frac{dY}{dt} = \frac{\beta\alpha XY}{k+X}-\gamma_2 Y^2 - \mu Y - c_2 EY, \tag{5}$$

with initial conditions $X(0)\geq 0$ and $Y(0)\geq 0$,

where:

X is the Prey fish, Y is Predator fish, r is the Prey growth rate, α is the predation rate, β is the conversion rate of prey biomass to predator biomass, k is the amount of prey consume for predator half satisfaction, μ is the natural death rate of predator, E is the harvesting effort, γ_1 is the death rate of prey due to toxic substance, γ_2 is the death rate of predator due to toxic substance, c_1 is the catchability coefficient of prey c_2 is the catchability coefficient of predator, u is the toxicants control variable, c_1EX is the harvesting effort of prey, c_2EY is the harvesting effort of predator, γ_1X^3 is the infection of prey fish by external toxic substance, γ_2Y^2 is the infection of predator fish by external toxic substance μY is the natural death rate of predator, $rX\left(1-\frac{X}{L}\right)$ is the logistic growth of prey fish, $\frac{\alpha XY}{k+X}$ is the predation rate, $\frac{\beta\alpha XY}{k+X}$ is the amount of prey biomass converted to predator biomass.

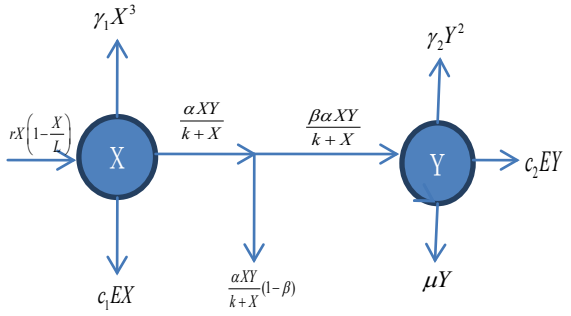

Figure 1. The model flow diagram for prey-predator interaction without control of toxic and harvesting.

Model Analysis

In this section, the formulated model above (5) will be qualitatively analysed to get the dynamical features that help to understand the effects of toxic on a prey-predator system. The boundedness and the positivity of the solution will be determined.

Boundedness

In this section we show that the system (5) is bounded or well behaved by considering the following lemma.

Lemma 1

All the solutions of the system (5) which start in $\mathbb{R}_2^+$ *are uniformly bounded in the region* $C=\left\{(X,Y)\in\mathbb{R}_2^+ : W=\left(\frac{K}{S}\right)+\varepsilon,\ \text{for any } \varepsilon>0\right\}$,[4] where $\mathbb{R}_2^+$ are the positive real numbers.

Proof

We define the following function;

$$W(X,Y)=X+\frac{1}{\beta}Y, \tag{6}$$

whose time derivative is

$$\frac{dW}{dt}=\frac{dX}{dt}+\frac{1}{\beta}\frac{dY}{dt}. \tag{7}$$

Equations (5) are substituted to equation (7) to get

$$\frac{dW}{dt}=rX\left(1-\frac{X}{L}\right)-\frac{\alpha XY}{k+X}-\gamma_1X^3-c_1EX$$

$$+\frac{1}{\beta}\left[\frac{\beta\alpha XY}{k+Y}\right]-\gamma_2Y^2-\mu Y-c_2EY$$

or

$$\frac{dW}{dt}=rX\left(1-\frac{X}{L}\right)-\frac{\mu}{\beta}Y$$

For each $S>0$ we get

$$\frac{dW}{dt}+SW\leq X\left[r\left(1-\frac{X}{L}\right)+S\right]-\left(\frac{\mu-S}{\beta}\right)Y.$$

Choose $S<\min\{0,\mu\}$ such that

$$\frac{dW}{dt}=SW\leq X\left[r\left(1-\frac{X}{L}\right)+S\right]$$

$$\frac{dW}{dt}+SW\leq rX\left(1-\frac{X}{L}\right)+XS \tag{8}$$

The maximum value of (8) is given by $\frac{L(r+S)^2}{2r}$

so that

$$\frac{dW}{dt}+SW\leq\frac{L(r+S)^2}{2r}.$$

Let $\frac{L(r+S)^2}{2r}$ be equal to $K>0$.

Then

$$\frac{dW}{dt}+SW\le K. \tag{9}$$

The inequality (9) is a first order differential inequality. Its solution is obtained by using Integrating factor $IF=e^{st}$.

Multiplying equation (9) by the integrating factor we get

$$e^{st}\frac{dW}{dt}+e^{st}SW\le e^{st}K \tag{10}$$

Equation (10) has a solution

$$W\le\frac{K}{S}+Ce^{-st}\cdot. \tag{11}$$

Apply the initial conditions $W_t=W_0$ and $W_0\le\frac{K}{S}+C$ leading to

$W_0\le\frac{K}{S}+C$, or $C=W_0-\frac{K}{S}$.

This implies that $W=\frac{K}{S}+\left(W_0-\frac{K}{S}\right)e^{-st}$.

As $t\to\infty$, $e^{-st}\to 0$ yielding

$C=W_0-\frac{K}{S}\cdot$

We thus find that all the solutions of the equations 5 that start in $\mathbb{R}_2^+$ are confined in the region C, where $C=\left\{(X,Y)\in\mathbb{R}_2^+ : W=\left(\frac{K}{S}\right)+\varepsilon,\ \text{for any } \varepsilon>0\right\}$ [4].

Positivity of the solution

For the model (5) to be ecologically meaningful and well posed, we need to prove that all solutions with positive initial data will remain positive for all the time $t\ge 0$.

Theorem 1

Let $X_0,\ Y_0>0$. Then the solution of the model (5) is positive for $\forall t\ge 0$.

Proof

To prove the theorem, we use all equations of the system (5)

From the first equation of prey population, we have

$$\frac{dX}{dt}=rX\left(1-\frac{X}{L}\right)-\frac{\alpha XY}{k+X}-\gamma_1X^3-c_1EX$$

or

$$\frac{dX}{dt}\le rX\left(1-\frac{X}{L}\right)\cdot$$

The above equation is a non-linear differential inequality which can be solved which can be solved by using separation of variables. So

$\frac{dX}{XL-X^2}\le\frac{r}{L}dt$,

which imply that

$\frac{dX}{X(L-X)}\le\frac{r}{L}dt,$

which has a solution

$\ln X-\ln(L-X)\le\frac{r}{L}t+c\cdot$

Similarly

$$\ln\left(\frac{X}{L-X}\right)\le\frac{r}{L}t+c$$

or

$$X\le\frac{Lc}{e^{-\frac{r}{L}t}+c}. \tag{12}$$

Applying initial conditions to (12) we get

$$X_0=\frac{cL}{e^{-\frac{r}{L}t}+c}.$$

Then

$c=\frac{X_0}{L-X_0}$,

Substitute the value of c into (12) to get

$$X\le\frac{X_0L}{(L-X_0)e^{-\frac{r}{L}t}+X_0}.$$

As $t\to\infty$ then $e^{-\frac{r}{L}t}\to 0$

Thus

$0\le X\le L\cdot$

Similarly, using the second equation of system (5), positivity of solutions can be established. Hence, both the solutions of the system (5) that are initiated in $\mathbb{R}_2^+$ are confined in the region C, where

$$C=\left\{(X,Y)\in\mathbb{R}_2^+ : W=\left(\frac{K}{S}\right)+\varepsilon,\ \text{for any } \varepsilon>0\right\}\cdot$$

Equilibrium points and stability analysis

Here we study the existence and stability of steady states. The model has four equilibrium points. These are trivial steady state E_0, predator free steady state E_1, prey free steady state E_2 and interior steady state E_3.

Definition

A steady state of a system (5) is a solution $X(t)=X$, $Y(t)=Y$ where *X* and *Y* are solutions of the algebraic equation $f(X,\ Y)=0$.

So we set $\frac{dX}{dt}=\frac{dY}{dt}=0$ and seek for the steady state solution. The system (5) becomes

$$rX^*\left(1-\frac{X^*}{L}\right)-\frac{\alpha X^*Y^*}{k+X^*}-\gamma_1X^{*3}-c_1EX^*=0 \tag{13}$$

$$\frac{\beta\alpha X^*Y^*}{k+X^*}-\gamma_2Y^{*2}-\mu Y^*-c_2EY^*=0 \tag{14}$$

The trivial steady state

The trivial steady state is obtained as follows:

From equation (14), we have

$$\frac{\beta\alpha X^*Y^*}{k+X^*}-\gamma_2Y^{*2}-\mu Y^*-c_2EY^*=0$$

Factor out Y^* to get

$$Y^*\left[\frac{\beta\alpha X^*}{k+X^*}-\gamma_2Y^*-\mu-c_2E\right]=0.$$

Either $Y^*=0$ or $\frac{\beta\alpha X^*}{k+X^*}-\gamma_2Y^*-\mu-c_2E=0$

Substitute $Y^*=0$ into (13) to get

$$rX^*\left(1-\frac{X^*}{L}\right)-\gamma_1X^{*3}-c_1EX^*=0$$

or

$$\gamma_1X^{*3}+\frac{rX^{*2}}{L}-\left(r-c_1E\right)X^*=0$$

Factor out X* and solve for X* to get

$$X^*\left[\gamma_1X^{*2}+\frac{rX^*}{L}-(r-c_1E)\right]=0.$$

Either $X^* = 0$ or

$$\gamma_1 X^{*2} + \frac{rX^*}{L} - (r - c_1 E) = 0 \quad (15)$$

So trivial steady state $E_0\left(X^*, Y^*\right) = (0,0)$

The predator free steady state

The predator free steady state is obtained by solving for X* in (15).

The other $X^* = \frac{\left(\frac{-r}{L}\right) + \sqrt{\left(\frac{r}{L}\right)^2 + 4\gamma_1(r - c_1 E)}}{2\gamma_1}$

Therefore,

the predator free steady state

$$E_1(X^*, Y^*) = \left(\frac{\left(-\frac{r}{L}\right) + \sqrt{\left(\frac{r}{L}\right)^2 + 4\gamma_1(r - c_1 E)}}{2\gamma_1}, 0 \right)$$

The prey free steady state

This steady state E_2 is calculated as follows:

From $\frac{\beta\alpha X^*}{k + X^*} - \gamma_2 Y^* - \mu - c_2 E = 0$ it follows that

$$Y^* = \frac{1}{\gamma_2}\left(\frac{\beta\alpha X^*}{k + X^*} - \mu - c_2 E \right). \quad (16)$$

Substitute (16) into (13) to get

$$rX^*\left(1 - \frac{X^*}{L}\right) - \frac{\alpha X^* Y^*}{k + X^*} - \gamma_1 X^{*3} - c_1 E X^* = 0. \quad (17)$$

Factor out X^* to get

$$X^*\left[r\left(1 - \frac{X^*}{L}\right) - \frac{\alpha Y^*}{k + X^*} - \gamma_1 X^{*2} - c_1 E \right] = 0$$

Either $X^* = 0$ or

$$r\left(1 - \frac{X^*}{L}\right) - \frac{\alpha Y^*}{k + X^*} - \gamma_1 X^{*2} - c_1 E = 0 \quad (18)$$

When $X^* = 0$, then from (16), $Y^* = -\frac{(\mu + c_2 E)}{\gamma_2}$

Thus the prey free steady state is $E_2(X^*, Y^*) = \left(0, -\frac{(\mu + c_2 E)}{\gamma_2}\right)$.

The interior steady state

The interior steady state $E_3(X^*, Y^*)$ is determined as follows:

Substitute (16) into (18) to get

$$\gamma_1\gamma_2 X^{*4} + \left(\frac{\gamma_2 r}{L} + 2k\gamma_1\gamma_2\right)X^{*3}$$
$$+\gamma_1\gamma_2 + \left(2k\gamma_2\frac{r}{L} + k^2\gamma_1\gamma_2 + \gamma_2 c_1 E - \gamma_2 r\right)X^{*2}$$
$$+\left(\gamma_2 k^2\frac{r}{L} + \alpha^2\beta + 2k\gamma_2 c_1 E\right)$$
$$+(\mu + c_2 E) - 2k\gamma_2 r)X - (k(\mu + c_2 E) + \gamma_2 c_1 E k^2 - \gamma_2 k^2 r) = 0 \quad (19)$$

Let

$a = \gamma_1\gamma_2$, $b = \frac{\gamma_2 r}{L} + 2k\gamma_1\gamma_2$,

$c = \gamma_2\left(2k\frac{r}{L} + k^2\gamma_1 + c_1 E - r\right)$,

$d = \gamma_2 k^2\frac{r}{L} + \alpha^2\beta + 2k\gamma_2 c_1 E + (\mu + c_2 E) - 2k\gamma_2 r$,

$e = -\left(k(\mu + c_2 E) + \gamma_2 c_1 E k^2 - \gamma_2 k^2 r\right)$

$$aX^{*4} + bX^{*3} + cX^{*2} + dX^* - e = 0 \quad (20)$$

Equation (20) is very difficult to solve. With help of Mathematica 9.0, the roots of (20) which are the interior steady state $E_3(X^*, Y^*)$ can be estimated by

$rX^*(1 - \frac{X^*}{L}) - \frac{\alpha X^* Y^*}{k + X^*} - \gamma_1 X^{*3} - c_1 E X^* = 0$ and

$\frac{\beta\alpha X^* Y^*}{k + X^*} - \gamma_2 Y^{*2} - \mu Y^* - c_2 E Y^* = 0$

when X^* and Y^* are non-negative and roots of equation (20) are defined.

Stability of the of the equilibrium points: We study the stability of each equilibrium point by first computing the Jacobian matrix corresponding to the model equation calculated at each steady state.

Let

$f = rX(1 - \frac{X}{L}) - \frac{\alpha XY}{k + X} - \gamma_1 X^3 - c_1 E X$ and

$g = \frac{\beta\alpha XY}{k + X} - \gamma_2 Y^2 - \mu Y - c_2 E Y.$

Then the Jacobian of the functions *f* and *g* is given by

$$\mathbf{J}(X, Y) = \begin{pmatrix} \frac{\partial f}{\partial X} & \frac{\partial f}{\partial Y} \\ \frac{\partial g}{\partial X} & \frac{\partial g}{\partial Y} \end{pmatrix}$$

$$\mathbf{J}(X, Y) = \begin{pmatrix} r - \frac{2rX}{L} - \frac{\alpha Yk}{(k + X)^2} - 3\gamma_1 X^2 - c_1 E & -\frac{\alpha X}{k + X} \\ \frac{\beta\alpha Yk}{(k + X)^2} & \frac{\beta\alpha X}{k + X} - 2\gamma_2 Y - \mu - c_2 E \end{pmatrix}. \quad (21)$$

The trivial steady state

$E_0(X^*, Y^*) = (0,0).$

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The Jacobian (21) at trivial steady state becomes

$$\mathbf{J}E_0(0,0) = \begin{pmatrix} r - c_1 E & 0 \\ 0 & -(\mu + c_2 E) \end{pmatrix}$$

Its Eigen values are given by $\lambda_1 = r - c_1 E$ and $\lambda_2 = -(\mu + c_2 E)$

The trivial steady state is stable when $r < c_1 E$

From the result above, it can be seen that the population is at normal state at this point.

The predator free steady state

$$E_1(X^*, Y^*) = \left(\frac{-\frac{r}{L} + \sqrt{\left(\frac{r}{L}\right)^2 + 4\gamma_1(r - c_1 E)}}{2\gamma_1}, 0 \right).$$

The Jacobian (21) at predator free steady state becomes

$$\mathbf{J}E_1\left(\frac{-\frac{r}{L}+\sqrt{\left(\frac{r}{L}\right)^2+4\gamma_1\left(r-c_1E\right)}}{2\gamma_1},\quad 0\right)$$

$$=\begin{pmatrix} r+\frac{rA}{L^2\gamma_1}-3B^2-c_1E & \frac{\alpha A}{-2kL\gamma_1+A} \\ 0 & \frac{\beta\alpha A}{-2kL\gamma_1+A}-\left(\mu+c_2E\right)\end{pmatrix}$$

where

$$A=r-L\sqrt{\frac{r^2}{L^2}+4\gamma_1(r-c_1E)}$$

$$B=\frac{\frac{-r}{L}+\sqrt{\frac{r^2}{L^2}+4\gamma_1(r-c_1E)}}{4\gamma_1}. \tag{22}$$

The Eigen values of the Jacobian are given by

$$\lambda_1=r+\frac{rA}{L^2\gamma_1}-\left(3B^2+c_1E\right)$$

$$\lambda_2=\frac{\beta\alpha A}{-2kL\gamma_1+A}-\left(\mu+c_2E\right).$$

Let $a_1=r+\frac{rA}{L^2\gamma_1}$ and $b_1=\left(3B^2+c_1E\right)$. Then

$$\lambda_1=a_1-b_1$$

or

$$\lambda_1=\left(R_1-1\right)b \tag{23}$$

where $R_1=\frac{a_1}{b_1}$. Thus if $R_1<1$, then λ_1 is negative.

For $\lambda_2=\frac{\beta\alpha A}{-2kL\gamma_1+A}-\left(\mu+c_2E\right)$,

we let $a_2=\frac{\beta\alpha A}{-2kL\gamma_1+A}$ and $b_2=\left(\mu+c_2E\right)$. Thus

$$\lambda_2=a_2-b_2 \text{ or}$$

$$\lambda_2=\left(R_2-1\right)b_2\cdot \tag{24}$$

If $R_2<1$ then the eigenvalue λ_2 is negative. So the predator free steady state is locally asymptotically stable since $R_1<1$ and $R_2<1$

The prey free steady state

$$E_2(X^*,Y^*)=\left(0,-\frac{(\mu+c_2E)}{\gamma_2}\right).$$

Substitute prey free steady state in the Jacobian (21) to get

$$\mathbf{J}\left(0,-\frac{(\mu+c_2E)}{\gamma_2}\right)=\begin{pmatrix} r+\frac{\alpha(\mu+c_2E)}{k\gamma_2}-c_1E & 0 \\ -\frac{\beta\alpha(\mu+c_2E)}{\gamma_2k} & \mu+c_2E\end{pmatrix}$$

The Eigen values are given by

$$\lambda_2=\mu+c_2E \text{ and } \lambda_2=\mu+c_2E$$

For λ_1 we let $a_3=r+\frac{\alpha\left(\mu+c_2E\right)}{k\gamma_2}$ and $b_3=c_1E$.

Then $\lambda_1=a_3-b_3$ or $\lambda_1=\left(R_1-1\right)b_3$ (25)

where $R_1=\frac{a_3}{b_3}$.

Thus when $R_1<1$ the eigenvalue λ_1 is negative. Thus the prey free steady state is locally asymptotically stable when $R_1<1$ and $\mu+c_2E<0$ which means that both eigenvalues are negative.

Model with Control

In the previous section, a model without control was formulated and analysed. The boundedness, positivity, equilibrium and the local stability were examined. In this section, a control variable is incorporated in the model system (5) and then analysed. The main objective is to minimize toxic which affect fish population and optimize cost of implementing control strategies. It is intended to use water hyacinth (*Eichhorniacrassipes* (Martius) Solms-Laubach) ([6]) which is efficient in accumulating heavy metals such as lead, mercury and treat Biological and Chemical waste water [7]. Water hyacinth is a plant which is used to treat polluted water. It deals with biological and chemical wastes which are likely to affect fish population.Water hyacinth, in other sides, can be a problem economically as it negatively affects fisheries, slow or even prevent water traffic, impedes irrigation, obstructs water ways, reduces water supply and slows hydropower generation [8]. It is assumed that the application of water hyacinth to minimize toxic in affected fish population is at a rate of $\mu(t)$. The model equation (5) with control variable at any time *t* is given as

$$\frac{dX}{dt}=rX\left(1-\frac{X}{L}\right)-\frac{\alpha XY}{k+X}-(1-u)\gamma_1X^3-c_1EX$$

$$\frac{dY}{dt}=\frac{\beta\alpha XY}{k+X}-(1-u)\gamma_2Y^2-\mu Y-c_2EY \tag{26}$$

with initial conditions $X(0)\geq 0$ and $Y(0)\geq 0.$

To minimize the cost of applying water hyacinth and its negative effects, we have to formulate an optimal control problem and apply Pontryagin's maximum principle to solve it.

Optimal control problem

Here it is intended to minimize the cost of applying water hyacinth, the amount of toxic in fish population. To address the problem, we first restrict the number of water hyacinth which is administered to water bodies (Ocean, Lake, Ponds *etc.*) by introducing a bound on the control as $t\in\lfloor t_0,t_f\rfloor$ for $t\in\left[t_0,t_f\right]$, and then form the objective function of the optimal control problem as follows:

$$J=\min_u\int_{t_0}^{t_f}\left(AY+\frac{1}{2}Bu^2(t)\right)dt \tag{27}$$

subject to (26) and the initial conditions. Here *A* is the weight associated with the amount of toxicant in fish population and *B* is the weight factor to control variable $u(t)$, t_0 is the initial time and t_f is the final time.

Quadratic control in the objective function is chosen because it is required to minimize toxic in affected fish and cost of applying water hyacinth [9]. For minimizing the toxic in fish population while minimizing control cost, we seek to find the optimal control u^* such that

$$J(u^*) = \min\left(J(u) \in U\right), \tag{28}$$

where $0 \le u \le 1$ such that $0 \le u \le 1$ and $t \in \left[t_0, t_f\right]$ is the control set [10].

Solving the optimal control problem: The Pontryagin maximum principle will be used to suggest the necessary optimal conditions, by forming Hamiltonian function *H* as

$$H = AY + \frac{1}{2}Bu^2(t) + \lambda_1 \frac{dX}{dt} + \lambda_2 \frac{dY}{dt}, \tag{29}$$

Substituting (26) to (29) we get

$$\begin{aligned} H &= AY + \frac{1}{2}Bu^2(t) \\ &+ \lambda_1\left[rX(1-\frac{X}{L}) - \frac{\alpha XY}{k+X} - (1-u)\gamma_1 X^3 - c_1 EX\right] \\ &+ \lambda_2\left[\frac{\beta\alpha XY}{k+X} - (1-u)\gamma_2 Y^2 - \mu Y - c_2 EY\right], \end{aligned} \tag{30}$$

where λ_1 and λ_2 are the adjoint variables or co-state variables. By applying Pontryagin's maximum principle and the existence of results for optimal control, we obtain,

Proposition 1

For the optimal control $u^*(t)$ that minimizes $J(u)$ over *U*, then there exist adjoint variables λ_1 and λ_2 satisfying

$$\begin{aligned} \frac{d\lambda_1}{dt} &= -\frac{\partial H}{\partial X} \\ &= -\lambda_1\left[r - \frac{2rX}{L} - \frac{\alpha kY}{(k+X)^2} - 3(1-u)\gamma_1 X^2 - c_1 E\right] - \lambda_2\left[\frac{\beta\alpha Yk}{(k+X)^2}\right] \\ \frac{d\lambda_2}{dt} &= -\frac{\partial H}{\partial Y} \\ &= -A + \lambda_1\frac{\alpha X}{k+X} - \lambda_2\left[\frac{\beta\alpha X}{k+X} - 2(1-u)\gamma_2 Y - \mu - c_2 E\right] \end{aligned} \tag{31}$$

and with transversality conditions

$$\lambda_1(t_f) = \lambda_2(t_f) = 0 \tag{32}$$

then

$$u^*(t) = \max\{0, \min(1, \bar{u})\} \tag{33}$$

So to find $\bar{u}$, we apply optimality condition (31) to Hamiltonian to get

$$\frac{\partial H}{\partial u} = Bu(t) + \lambda_1\gamma_1 X^3 + \lambda_2\gamma_2 Y^2. \tag{34}$$

Therefore

$$u(t) = -\frac{-(\lambda_1\gamma_1 X^3 + \lambda_2\gamma_2 Y^2)}{B} \tag{35}$$

From (33), then $\bar{u} = u(t)$. Thus the optimality condition is written as

$$u^*(t) = \max\left\{0, \min\left(1, \frac{-(\lambda_1\gamma_1 X^3 + \lambda_2\gamma_2 Y^2)}{B}\right)\right\}. \tag{36}$$

By standard control arguments as suggested by [11] and [12], we have

$$u^*(t) = \begin{cases} 0 & \text{if } \bar{u} \le 0, \\ \bar{u} & \text{if } 0 < \bar{u}_1 < 1, \\ 1 & \text{if } \bar{u} \ge 1. \end{cases} \tag{37}$$

Numerical Simulations for the Model with Control

In order to illustrate some of the analytical results of the study, numerical simulations for the model with control are performed using a set of reasonable values. Table 1 shows the parameter values to be used which are hypothetically chosen following the realistic ecological observations which have been suggested by previous researchers. The initial guess for prey fish and predator fish are set to be 60 and 40 respectively.

Table 1. Table for parameter values.

Parameter Symbol	Parameter value	Source
r	6.5	[4]
L	300	[4]
α	0.006	[4]
k	0.2	[4]
γ_1	0.00005	[4]
c_1	0.03	[4]
E	1	[4]
β	0.7	[4]
γ_2	0.00008	[4]
μ	0.9	[5]
c_2	0.04	[4]

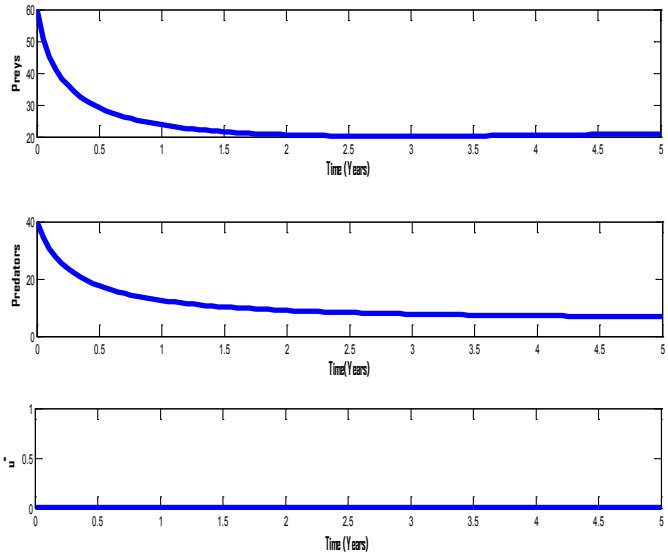

Figure 2. Graph of prey, predator and the control with time.

Figure 2 represents the graphical solutions for the state variables and the control $u(t)$.

Figure 2 shows three graphs: Prey population, Predator population and the control *u*. From the graphs, it is observed that the prey population seems to have sharp decrease. The predator population decreases slowly to zero. This is because there is no toxic control in the whole population so the predator population may vanish. The last graph shows the control profile where $u = 0$. Because of that, the prey and predator populations seem to be decreasing.

Figure 3 shows the behaviour of prey predator populations with intrinsic growth change.

It is observed in figure 3 that, changing the prey growth rate from $r = 0.8$ to $r = 2.5$ tends to decrease both prey population and predator population. With the prey population, the graph shows that the degree of decreasing

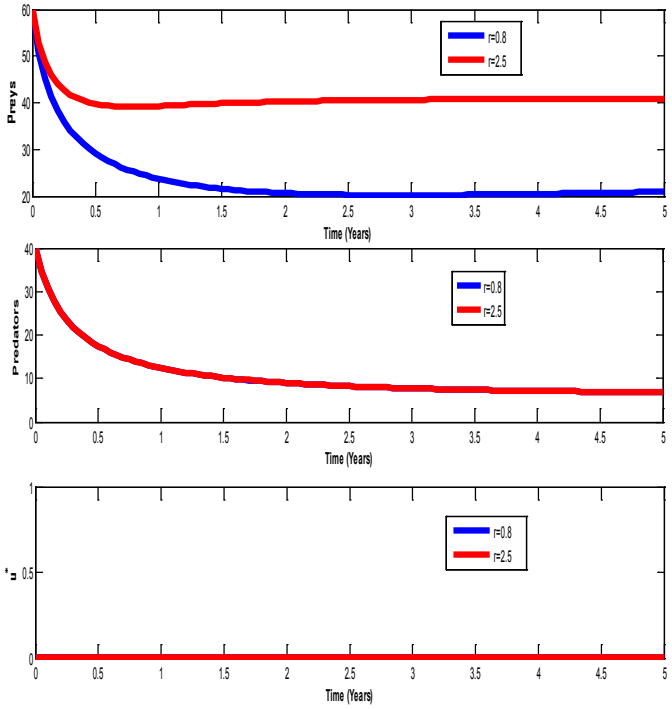

Figure 3. Prey population, predator population and control profile with *r*.

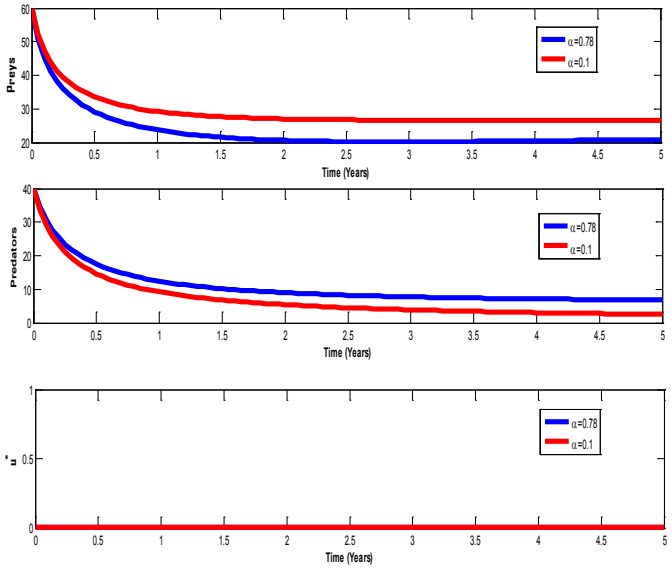

Figure 4. Behavioural changes of prey predator populations when predation growth rate changes.

depends on the growth rate *r*. It can also be seen that the predator population does not vary with the change of growth rate.

Figure 4 shows the behaviour of the populations when predation growth rate changes.

From figure 4, it is observed that changing of predation rate from $\alpha = 0.78$ to $\alpha = 0.1$ affects the predator population. The predator population decreases slowly when $r = 0.1$ as compared to when $r = 0.78$. It is vice versa to prey population, where the lower the predation, the lower the decreasing rate and the higher the predation rate the higher the decreasing rate.

Figure 5 shows the population behaviour when death rate of prey due to toxic varies.

From figure 5, it is observed that, changing prey death rate due to toxic, affects only the prey population. In prey population, there is a sharp decrease when $\gamma_1 = 0.002$ which reflects the population dying in large numbers due

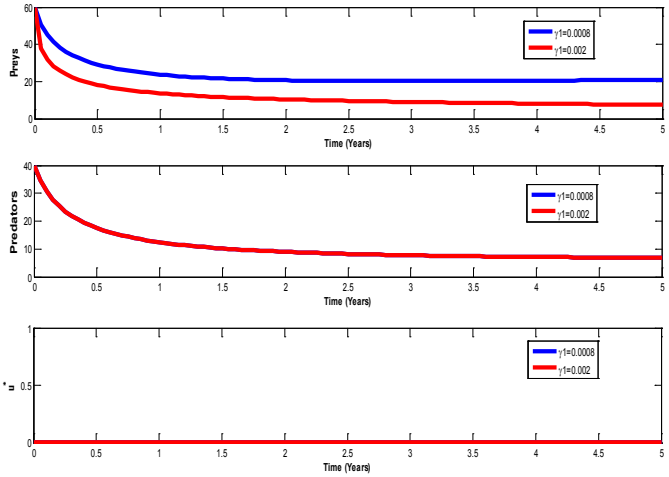

Figure 5. Graph of prey predator population when there is variation of death rate of prey due to toxic.

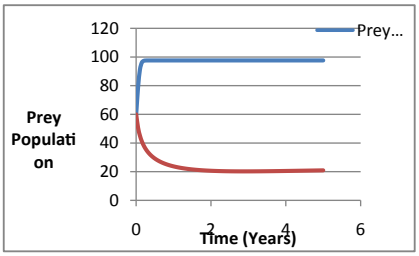

Figure 6. The effect of control on prey population.

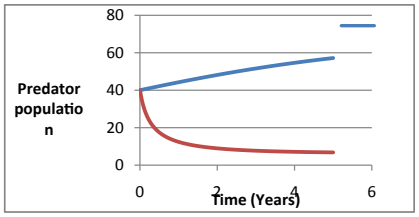

Figure 7. The effect of control on predator population.

to increase of toxic in the population as compared to when the death rate is $\gamma_1 = 0.0008$. The predator population is not affected by changes in γ_1.

Figure 6 below shows the effect of control on prey population.

From Figure 6, it is observed that prey population decreases when there is no control due to toxic in water, harvesting activities and the presence of predator. But the same population increases rapidly and when it approaches the carrying capacity it remain constant.

Figure 7 shows the effect of control on predator population.

From Figure 7, it is observed that the predator population decreases exponentially when there is no the control. This because of the presence of toxic in water, natural death and harvesting activities which cause the population to decrease when there is no control and increasing when the control is applied to a system.

Summary and Conclusion

In this paper, a mathematical model for controlling toxicants in water was formulated and analysed to investigate the dynamical behaviour of the system (26). In formulating the model, harvesting efforts and terms which show the effect of toxicity was introduced to both prey and predator population (fish). Qualitative analysis was performed to the basic model (5). By applying stability theory of ordinary differential equations, four steady states were determined: trivial steady state, prey free steady state, predator

free steady state and interior steady state. Model simulation revealed that application of control strategy (water hyacinth) increases fish population. It is further recommended that waste water with domestic sewage industrial effluents, thermal and radioactive pollutants may be recycled and reused to generate cheaper fuel, gas and electricity.

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How to cite this article: Zachariah Noboth, Estomih S. Massawe, Daniel O. Makinde and Lathika P. Modelling and Optimal Control of Toxicants on Fish Population with Harvesting. *J Appl Computat Math* 9 (2020) doi: 10.37421/jacm.2020.9.459