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Method for Solving Polynomial Equations

Nahon YJ*

Department of Mathematics, Technion-Israel Institute of Technology, Ashdod, Israel

Abstract

The purpose of my paper is to bring a method for solving polynomial equations using basic algebra and series and also using combinatorics. A series which converges to the solutions of polynomial equations. The contribution of this method is that it leads directly to precise results to find the roots of a polynomial equation of any degree starting from second degree to infinity and also for the solving of radicals since radicals are a particular type of polynomial equations for example to find the square root of 2 sends to solve the equation $x^2=2$. A general formula for the series which converges to the solutions of polynomial equations. For complex solutions we write for a polynomial P(x), P(a+bi)=P(a-bi)=0 and to solve this separately for imaginary part and real part of the solution sends to solve for regular polynomial equations at one variable so we can use the method which is developed to find the solutions.

Keywords: Polynomial Equations; Coefficients; Series; Recurrent Relationships; Finite number

Introduction

Nahon theorem

If a n0 th polynomial equation in powers of X:

X (n0)=A(n0-1)*X(n0-1)+A(n0-2)*X (n0-2)+A(n0-3)*X (n0-3)+…..A(1)*X+A(0) with the condition in the coefficients [1]

A(n0-1)*A(0)≠0

has a solution X in the set of real numbers R then if we define a series E(n) with E(n0)=A(n0-1) and with the recurrent formula [2]:

E(n+1)=A(n0-1)*E(n)+A(n0-2)*E(n-1)+A(n0-3)*E(n-2)+…….A(1)*E(n-n0+2)+A(0)*E(n-n0+1) for (n ≥ n0). Then

$$L = \lim_{(n \to \infty)} \frac{E(n+1)}{E(n)}$$ is a solution of the n0 th polynomial equation [3]. So X=L.

Condition of convergence [4]: for i=1,2,…..(n-1)

$A(i) \neq 0 \rightarrow [A(n0-1)*A(i)+A(i-1)] \neq 0$.

Proof of the Nahon theorem

$X^{(n0)}=A(n0-1)*X^{(n0-1)+}A(n0-2)*X^{(n0-2)}+A(n0-3)*X^{(n0-3)+}….A(1)*X+A(0)$

$X^{(n0+1)}=X^{(n0)}*X$

Multiplication of the two sides by X leads to [5]:

$X^{(n0+1)}=A(n0-1)*X^{(n0)}+A(n0-2)*X^{(n0-1)}+A(n0-3)*X^{(n0-2)}+…A(1)*X^2+A(0)*X$

Substituting $X^{(n0)}$

by its expression in the second expression for $X^{(n0+1)}$ leads to [6]:

$X^{(n0+1)}=A(n0-1) \times [A(n0-1)*X^{(n0-1)}+A(n0-2)*X^{(n0-2)}+A(n0-3)*X^{(n0-3)+}…$

$A(1)*X+A(0)]+A(n0-2)*X^{(n0-1)+}A(n0-3)*X^{(n0-2)+}….$

$A(1)*X^2+A(0)*X=$(Grouping together the coefficients of the same powers)

$X^{(n0+1)}=X^{(n0-1)} \times [A(n0-1)^2+A^{(n0-2)}]+X^{(n0-2)} \times [A(n0-1)*A(n0-2)+A(n0-3)]$

$+X^{(n0-3)} \times [A(n0-1)*A(n0-3)+A(n0-4)]+…X \times [A(n0-1)*A(1)+A(0)]+A(n0-1)*A(0)$

So in this manner by multiplication by X and then substitution of $X^{(n0)}$ by its expression we can express all the subsequent powers [7]

$X^{(n0+2)}, X^{(n0+3)}, X^{(n0+4)}, ……, X^{(n0+k)}$

in terms of a basis composed of the powers $[X^{(n0-1)}, X^{(n0-2)}, X^{(n0-3)}, …X, X^{(0)}]$

So let's denote the coefficients of each power contained in this basis by series E(n) [8],

E(2(n)), E(3(n)),……..E(n0-1(n)),E(n0(n)) (For n ≥ n0)

So we can write

$X^{(n0)}=E(n0)*X^{(n0-1)}+E(2(n0))*X^{(n0-2)+}E(3(n0))*X^{(n0-3)+}…E(n0-1(n0))*X$

$+E(n0(n0))$

Similarly we can write:

$X^{(n0+1)}=E(n0+1)*X^{(n0-1)}+E(2(n0+1))*X^{(n0-2)}+E(3(n0+1))*X^{(n0-3)}+……E(n0-1(n0+1))*X+E(n0(n0+1))$

Since $X^{(n0+1)=}X^{(n0)}*X$

So $X^{(n0+1)=}E(n0)*X^{(n0)+}E(2(n0))*X^{(n0-1)}+E(3(n0))*X^{(n0-2)}+….E(n0-1(n0))*X^2+E(n0(n0))*X$

Substitution of $X^{(n0)}$ by its expression leads to:

$X^{(n0+1)=}E(n0) \times [A(n0-1)*X^{(n0-1)+}A(n0-2)*X^{(n0-2)+}A(n0-3)*X^{(n0-3)+}A(1)*X+A(0)]+E(2(n0))*X^{(n0-1)+}E(3(n0))*X^{(n0-2)}+….. E(4(n0))*X^{(n0-3)+}E(n0-1(n0))*X^2+E(n0(n0))*X=$(Grouping together the coefficients of the same powers)

*Corresponding author: Julien Y, Department of Mathematics, Technion-Israel Institute Of Technology, Ashdod, Israel, Tel: 00972-52-742-1339; E-mail: nahonyehoudajulien@yahoo.fr

$X^{(n0+1)}=X^{(n0-1)}\times[A(n0-1)*E(n0)+E(2(n0))]+X^{(n0-2)}\times[A(n0-2)*E(n0)+E(3(n0))]+X^{(n0-3)}\times[A(n0-3)*E(n0)+E(4(n0))]+\ldots\ldots X^2\times[A(2)*E(n0)+A(1)]+X\times[A(1)*E(n0)+E(n0(n0))]+A(0)*E(n0)$

So we can relate the coeficientes

$E(n0+1)=A(n0-1)*E(n0)+E(2(n0))=A(n0-1)^2+A(n0-2)$

$E(2(n0+1))=A(n0-2)*E(n0)+E(3(n0))=A(n0-1)*A(n0-2)+A(n0-3)$

$E(3(n0+1))=A(n0-3)*E(n0)+E(4(n0))=A(n0-1)*A(n0-3)+A(n0-4)$

$E(n0-2(n0+1))=A(2)*E(n0)+E(n0-1(n0))=A(n0-1)*A(2)+A(1)$

$E(n0-1(n0+1))=A(1)*E(n0)+E(n0(n0))=A(n0-1)*A(1)+A(0)$

$E(n0(n0+1))=A(0)*E(n0)=A(0)*A(n0-1)$

From these recurrent relationships following the pattern of recurrence we can write:

$E(n0+1)=A(n0-1)*E(n0)+E(2(n0))=A(n0-1)*E(n0)+A(n0-2)*E(n0-1)+E(3(n0-1))=$

$A(n0-1)*E(n0)+A(n0-2)*E(n0-1)+A(n0-3)*E(n0-2)+E(4(n0-2))=$

$A(n0-1)*E(n0)+A(n0-2)*E(n0-1)+A(n0-3)*E(n0-2)+A(n0-4)*E(n0-3)+\ldots A(1)*E(2)+A(0)*E$ (1)

To write this the following condition is needed: $E(n0+1)\neq0$ and $E(i(n0+1))\neq0$ for $i=2,3,\ldots n0$. so we can write the condition as follows:

$[A(n0-1)*A(i)+A(i-1)\neq0]$

if this condition applies so it is following from the recurrent relationships $E(i(n))$ that for $n\geq n0$ $E(n)\neq0$ and therefore the condition of convergence of the ratio

$\frac{E(n+1)}{E(n)}$ in infinity is satisfied.

So from this we can write a recurrent relationship for E(n):

$E(n+1)=A(n0-1)*E(n)+A(n0-2)*E(n-1)+A(n0-3)*E(n-2)+\ldots\ldots.A(1)*E(n-n0+2)+A(0)*E(n-n0+1)$

Let's define $L=\lim_{n\to\infty}=\frac{E(n+1)}{E(n)}$

$E(n+1)/E(n)=A(n0-1)+A(n0-2)\times[E(n-1)/E(n)]+A(n0-3)\times[E(n-2)/E(n)]+\ldots$

$A(1)\times[E(n-n0+2)/E(n)]+A(0)\times[E(n-n0+1)/E(n)]$

Let's define G(n+1)=E(n+1)/E(n) so we can write:

$G(n+1)=A(n0-1)+A(n0-2)/Gn+A(n0-3)/[Gn*G(n-1)]+\ldots\ldots.+A(1)/[G(n)*G(n-1)*\ldots.G(n-n0+3)]+A(0)/[Gn*G(n-1)*\ldots G(n-n0+2)]$

$L=\lim_{n\to\infty}=\frac{E(n+1)}{E(n)}=\lim_{n\to\infty}G(n+1)=$

$\lim_{n\to\infty}G(n)=\lim_{n\to\infty}G(n-1)=$

$\lim_{n\to\infty}G(n-n0+2)$

So $L=A(n0-1)+A(n0-2)/L+A(n0-3)/L^2+\ldots\ldots A(1)/L^{(n0-2)+}A(0)/L^{(n0-1)}$

Multiplication of the two sides by $L^{(n0-1)}$ leads to:

$L^{(n0)}=A(n0-1)*L^{(n0-1)}+A(n0-2)*L^{(n0-2)+}A(n0-3)*L^{(n0-3)+}\ldots\ldots A(1)*L+A(0)$

So X=L is a solution of the n0th polynomial equation

$X^{(n0)}=A(n0-1)*X^{(n0-1)}+A(n0-2)*X^{(n0-2)}+A(n0-3)*X^{(n0-3)}\ldots\ldots A(1)*X+A(0)$.

Particular cases:

* When A(n0-1)=0

Before being able to solve the equation, we first need to transform the equation into a classical n0 order equation when at least the coefficient of x^{n0-1} and x^0 will be different from 0. This can be done by setting X=x+a, then we can solve the new equation in x.

Example: Let's get a look at the equation $X^3=3X+4$

Substitution of X by x+a where " a" can be any real leads to:

$(x+a)^3=3(x+a)+4$

so developing the cubic form leads to:

$x^3+a^3+3a^2x+3ax^2=3x+3a+4$ this leads to:

$x^3=-3ax^2+x(3-3a^2)+(3a+4)-a^3$

So X=x+a is the solution of the cubic equation $X^3=3X+4$

* If for a particular $Ai\neq0$ $A(n0-1)*A(i)+A(i-1)=0$

We need to transform the equation for i by setting X=x+a and then we need to recalculate the next terms of the series E(k).

Complex Solutions

For a polynomial P(X)=0,P(a+bi)=P(abi)=0 so we need to solve separately for the real parts and imaginary parts so it sends to solve a regular polynomial equation at one variable to find a and b

Example

Let's try to solve a seventic (n0=7) equation

$X^7=2X^6+4X^5+3X^4+2X^3+X^2+X+6$

Let's express the recurrent relationship for E(n) and G(n):

$E(n+1)=2E(n)+4E(n-1)+3E(n-2)+2E(n-3)+E(n-4)+E(n-5)+6E(n-6)$

$G(n+1)=E(n+1)/E(n)$

$G(n+1)=2+4/G(n)+3/[G(n)*G(n-1)]+2/[G(n)*G(n-1)*G(n-2)]+$

$1/[G(n)*G(n-1)*G(n-2)*G(n-3)]+1/[G(n)*G(n-1)*G(n-2)*G(n-3)*G(n-4)]+$

$6/[G(n)*G(n-1)*G(n-2)*G(n-3)*G(n-4)*G(n-5)]$

Before we can use those formulas we need to calculate the first six terms for E(n) and

G(n):

E(8),E(9),E(10),E(11),E(12),E(13)

G(8), G(9), G(10), G(11), G(12), G(13)

$X^8=X^7X=2[2X^6+4X^5+3X^4+2X^3+X^2+X+6]+$

$4X^6+3X^5+2X^4+X^3+X^2+6X=8X^6+11X^5+8X^4+5X^3+3X^2+8X+12$

So E(8)=8 , from the seventic equation E(7)=2

So G(8)=E(8)/E(7)=8/2=4

$X^9=X^8.X=8X^7+11X^6+8X^5+5X^4+3X^3+8X^2+12X=$

Substitution of X^7 by the equation leads to:

$X^9=8[2X^6+4X^5+3X^4+2X^3+X^2+X+6]+$

$11X^6+8X^5+5X^4+3X^3+8X^2+12X=$

$27X^6+40X^5+29X^4+19X^3+16X^2+20X+48$

G(9)=E(9)/E(8)=27/8=3.375

$X^{10}=X^9.X=27[2X^6+4X^5+3X^4+2X^3+X^2+X+6]+$

$40X^6+29X^5+19X^4+16X^3+20X^2+48X=$

$94X^6+137X^5+100X^4+70X^3+47X^2+75X+162$

G(10)=E(10)/E(9)=94/27=3.481481481

$X^{11}=X^{10}.X=94[2X^6+4X^5+3X^4+2X^3+X^2+X+6]+137X^6+100X^5+70X^4+47X^3+75X^2+$

$162X=325X^6+476X^5+352X^4+235X^3+169X^2+256X+564$

G(11)=E(11)/E(10)=325/94=3.457446809

$X^{12}=X^{11}.X=325[2X^6+4X^5+3X^4+2X^3+X^2+X+6]+$

$476 X^6+352 X^5+235X^4+169X^3+256X^2+564X=$

$1126X^6+1652X^5+1210X^4+819X^3+581X^2+889 X+1950$

G(12)=E(12)/E(11)=1126/325=3.464615385

$X^{13}=X^{12}.X=1126[2X^6+4X^5+3X^4+2X^3+X^2+X+6]+$

$1652X^6+1210X^5+819X^4+581X^3+889X^2+1950X=3904X^6+5714X^5+4197X^4+2833X^3+2015X^2+3076X+6756$

G(13)=E(13)/E(12)=3904/1126=3.46714032

So we are ready now to use the recurrent formula for G(n)

G(14)=2+4/G(13)+3/[G(13)*G(12)]+2/[G(13)*G(12)*G(11)]+

1/[G(13)*G(12)*G(11)*G(10)]+1/[G(13)*G(12)*G(11)* G(10)* G(9)]+

6/[G(13)*G(12)*G(11)*G(10)*G(9)*G(8)]=3.463627049

Applying the formula for the next terms of G(n) for n15 leads to:

G(15)=3.465241828

G(16)=3.46714032

G(17)=3.464868461

G(18)=3.464830021

G(19)=3.464840265

G(20)=3.464839802

G(21)=3.464838453

G(22)=3.464839319

G(23)=3.46483897

G(24)=3.464839088

G(25)=3.46483905

G(26)=3.464839061

G(27)=3.464839059

G(28)=3.464839058

G(29)=3.464839059

G(30)=3.464839059

Since G(29)=G(30)=3.464839059

So we can conclude that all the next terms of the series for n≥30 will be equal to G(30)

So we can conclude that $L=\lim_{n\to\infty}G(n)=3.464839059$ checking this result will show that

X=L=3.464839059 is a solution of the seventic equation.

In this example, we saw that in fact (considering a finite number of digits after the decimal point) a finite number of iterations is needed to calculate the limit L.

So let's formulate a general result for the number of iterations needed to calculate the limit L.

Let it be a no th polynomial equation:

$X^{n0}=A(n0-1)*X^{n0-1}+A(n0-2)*X^{n0-2}+A(n0-3)*X^{n0-3}+….A(1)*X+A(0)$

Let it be the coefficients[A(n0-1), A(n0-2), A(n0-3),A(n0-4),…..A(0)] real integers different from zero [A(i) integers A(i)≠0 and $A(i)\in\Re$]

The number of iterations needed is determined by the minimum common factor Ko.

Ko is the minimum common factor between

$1\times|A(n0\text{-}1)|,2\times|A(n0\text{-}2)|,3\times|A(n0\text{-}3)|,………………(n0\text{-}1)\times|A(1)|,n0\times|A(0)|$.

Ko is determined by simplifying

$1\times|A(n0\text{-}1)|,\quad 2\times|A(n0\text{-}2)|,3\times|A(n0\text{-}3)|,…….(n0\text{-}1)\times|A(1)|,n0\times|A(0)|$

by each pair of prime factors (p,p) contained in the $i\times|A(n0\text{-}i)|$ for $i=1,2,3,…….(n0\text{-}1)$

and then by taking the product between the different prime factors in the $i\times|A(n0\text{-}i)|$,

Then we determine the equivalent of the product in modulo n0 the result is k0[n0],

So ko+n0 iterations will always give at least one digit of the root solution X.

Then for m digits the number of iterations that will always give the m digits is

K=(Ko+n0)×[1+2+…..m]=(Ko+n0)×[m*(m+1)/2].

In the decimal system K+10 iterations will be the maximum number of iterations needed that the m digits are also common to the subsequent terms of the serie G(n).

This says that for k≥ K+10

k iterations correspond to the indice k+n0 for the serie G(n)

So G(k+n0) contains the m digits of the root solution.

So X(with m digits)=

Int[G[(Ko+n0)*m*(m+1)/2+n0+10] ×10$^{(m)}$]/ 10$^{(m)}$

If we call k1=Ko+n0 and Int the integer value of a parameter we can express a general formula for the solution of polynomial equations:

X(with m digits)=Int[G[k1*m*(m+1)/2+10]×10$^{(m)}$]/10$^{(m)}$

For calculating Ko, a rule is that only the $|A(i)| \neq 0$ are taken into account for the product. In any case the minimum condition is

$|A(n0\text{-}1)| \times |A(0)| \neq 0.$

A particular case:

When [A(n0-1), A(n0-2), A(n0-3),……….A(0)] are not integers we express their rational form A(i)=A'(i)/A'(n0)

So we can express the equation in this form

A'(n0)* X$^{(n)}$=A'(n0-1)*X$^{(n0-1)+}$A'(n0-2)*X$^{(n0-2)}$+…..A'(1)*X+A'(0)

[A'(n0), A'(n0-1), A'(n0-2),……….A'(0)] $\neq 0$ and also real integers.

Then we can express the product for Ko

$$Ko = 1 \times A'(n0-1) \times 2\,|\,A'(n0-2)| \times 3\,|\,A'(n0-3) \times \ldots\ldots$$

$$(n0-1) \times A'(1) \times (n0) \times |\,A'(0)| = n0! * \prod_{i=1}^{i=n0} * |\,A'(n0-i)|$$

A rule is that only the $|A'(i)| \neq 0$ are taken into account for the product. In any case the minimum condition is

$|A'(n0\text{-}1)| * |A'(0)| \neq 0.$

Now the question which can be raised up is how to know the number of digits which a rational solution may have.

Determination of the maximum number of digits for a rational solution in the decimal base

Let it be a n0 th polynomial equation

X$^{(n0)}$=A(n0-1)X$^{(n0-1)+}$A(n0-2)X$^{(n0-2)+}$A(n0-3)X$^{(n0-3)}$+……A(1)*X+A(0).

If X has m digits in the decimal base so the number of digits of the left side can be expressed Am+B and in the right side can be expressed Cm+D,So since the left side is equal to the right side so Am+B=Cm+D

(A-C)*m=D-B so m=(D-B)/(A-C) The biggest value for m can be obtained for D maximum, B minimum, A minimum, C maximum C maximum=n0-1A minimum=n0B minimum=-n0+1

$$D = [\sum_{k=0}^{k=n0-1} M(k)] + \sum_{k=1}^{k=n0-1} [k-1]$$

$$D = [\sum_{k=0}^{k=n0-1} M(k)] + [(n0-2)*(n0-1)]$$

So m=[D+(n0-1)]/[n0-(n0-1)]=D+n0-1

So m ≤ D+*n0*-1

So m(Max)=D+n0-1

$$m(Max) = [\sum_{k=0}^{k=n0-1} M(k)] + [(n0-2)*(n0-1)/2] + (n0-1)$$

When M(k) is the number of digits for A(k): example if A(n0-1)=13 so M(n0-1)=2.

What Henrik Niels Abel has demonstrated is that a general formula for the solution of polynomial equations can not be obtained in terms of radicals for degree n ≥ 5 .

However the equation X^2=2 is in itself a mathematical problem to solve because saying that the solution of this equation is $X = \sqrt{2}$ does not give the arithmetic expression of the solution

In order to solve the equation X^2=2 we need first to transform this equation into a classical second order equation. This can be done by transformation with X=x+a

(x+a)^2=2 this leads to:

x^2+a^2+2ax=2⟺ x^2=-2ax+2-a^2.

One example that will lead to the solution can be a=1 this leads to x^2=-2x+2-1=-2x+1

so we need to solve the equation.

x^2=-2x+1 then X=x+1 corresponds to the solution of X^2=2 or in other terms we can express the square root of $2 : \sqrt{2} = x+1$.

Let's solve the equation

x^2=-2x+1

Let's express the recurrent relationship for E(n) and G(n)

E(n+1)=-2E(n)+E(n-1)

G(n+1)=-2+[1/G(n)]

Before we can use those formulas we need first to calculate the first terms for E(n) and

G(n): E(3),G   (3)

x^3=x^{2}*x=-2x^2+x=-2[-2x+1]+x=4x –2+x=5x-2

So E(3)=5, G(3)=E(3)/E(2)=5/(-2)=-2.5

Applying the formula for G(n) leads to:

G(4)=-2+[1/(-5/2)]=-2+(-2/5)=-2-0.4=-2.4

G(4)=-2.4

G(5)=-2.416666667

G(6)=-2.413793103

G(7)=-2.414285714

G(8)=-2.414201183

G(9)=-2.414215686

G(10)=-2.414213198

G(11)=-2.414213625

G(12)=-2.414213552

G(13)=-2.414213564

G(14)=-2.414213562

G(15)=-2.414213562

Since G(15)=G(14)=-2.414213562.

So we can conclude that x1=-2.414213562 we can now find easily the other solution

1 2-1(-2.414213562)

1 (-2.414213562+2) 0

The other solution is $x^2=2.414213562-2=0.414213562$

$X1=x+1=-2.414213562+1=-1.414213562$

$X2=0.414213562+1=1.414213562$

So those are the two square roots of 2

$-\sqrt{2} = -1.414213562$

$\sqrt{2} = 1.414213562$.

So my conclusion is that since we can find the radicals in terms of simple operations like additions multiplications subtractions divisions so we should be able to find general formulas which can be expressed in terms of simple operations to express the solutions of any nth polynomial equation. All what is needed for this is to find explicit formulas for the series E(n) and G(n).

Let's try to develop a formula for the second degree polynomial equations.$X^2=PX+Q$

let's express the recurrent relationship for E(n):

E(n+1)=P*E(n)+Q*E(n-1)

This is the formula for E(k):when k=0 corresponds to n=n0=2 (k=n-n0=n-2)

$E(k)=P^{(k+1)}+k\,[P^{(k-1)}]\times Q+[(k-2)(k-1)/2]\times [P^{(k-3)}]\times Q^2+$

$[\,(k-2)(k-3)(k-4)/6]\times[P^{(k-5)}]\times Q^3+[(k-3)(k-4)(k-5)(k-6)/24]\times[P^{(k-7)}]\times Q^4$

$+[(k-4)(k-5)(k-6)(k-7)(k-8)/120]\times[P^{(k-9)}]\times Q^5+\ldots\ldots\ldots$

The final development of the formula leads to:

$$E(k)=\binom{k+1}{0}*P^{(k+1)}+\binom{k}{1}*P^{(k-1)}*Q+$$

$$\binom{k-1}{2}*p^{(k-3)}*Q^2+\binom{k-2}{3}*P^{(k-5)}*Q^3+$$

$$\binom{k-3}{4}*P^{(k-7)}*Q^4+\binom{k-4}{5}*P^{(k-9)}*Q^5+\ldots$$

$$\binom{L}{L}*Q^L \ (if\ k\ is\ odd)or$$

$$\binom{L+1}{L}*P*Q^L \ (If\ k\ is\ even).$$

The last term of the formula will depend on whether k is odd or k is even

Now we will try to obtain a general formula valid for any n0 th degree polynomial equation with coefficients A(n0-1),A(n0-2), A(n0-3),……..A(1),A(0) :

$X^{(n0)}=A(n0-1)*X^{(n0-1)}+A(n0-2)*X^{(n0-2)}+A(n0-3)*X^{(n0-3)}+\ldots\ldots.A(1)*X+A(0)$

Let's express the recurrent relationship for E(k):

E(k+1)=A(n0-1)*E(k)+A(n0-2)*E(k-1)+A(n0-3)*E(k-2)+………A(1)*E(k-(n0-2))+A(0)*E(k-(n0-1))

Developing a formula for E(k) leads to:

$$E(k)=A(n0-1)^{(k+1)}+$$

$$\binom{k}{1}*A(n0-1)^{(k-1)}*A(n0-2)+$$

$$\binom{k-1}{1}*A(n0-1)^{(k-2)}*A(n0-3)+$$

$$\binom{k-2}{1}*A(n0-1)^{(k-3)}*A(n0-4)+\ldots..$$

$$\binom{k-no+3}{1}*A(n0-1)^{(k-n0+2)}*A(1)+$$

$$\binom{k-n0+2}{2}*A(no-1)^{(k-n0+1)}*A(0)+$$

$$\sum_{S=2}^{S=k}\binom{S+K(n0-1)}{S}*\left[F[\prod_{i=0}^{i=n0-2}A(i)^{K(i)}]\right]*$$

$$[\prod_{i=0}^{i=n0-1}A(i)^{K(i)}] \ \text{where } S \text{ is defined to be}$$

$$S=\sum_{i=0}^{i=n0-2}K(i)$$

K(n0-1) is defined to be

$$K(n0-1)=k-[\sum_{i=0}^{i=n0-2}(K(i)*(n0-i))]+1$$

with one condition: K(n0-1)$\geq$ 0

and F is a function defined as this:

$$F\left(\left[A^K\right]=\frac{K!}{K!}=1 \ \ with \ k\geq 0\right)$$

$$F\left[\prod_{i=0}^{i=n0-2}A(i)^{k(i)}\right]=\frac{\left(\sum_{i=0}^{i=n0-2}K(i))!\right)}{\prod_{i=0}^{i=n0-2}(K(i)!)}=$$

$$\frac{S!}{\prod_{i=0}^{i-n0-2}(K(i)!)} \ with \ S=\sum_{i=0}^{i=n0-2}K(i)$$

In the formula for the serie E(k)

Example

$$F\left(A^2*B^3*C^4\right)=\frac{9!}{2!*3!*4!}=1260$$

We can rewrite the formula for E(k):

$$E(k)=A(n0-1)^{(k+1)}+$$

$$\sum_{i=0}^{i=n0-2}\binom{(k-(n-i-2))}{1}*A(n0-1)^{(k-(n0-i-1))}*A(i)$$

$$+\sum_{S=2}^{S=k}\binom{\sum_{i=0}^{i=n0-2}K(i)+K(n0-1)}{\binom{\sum_{i=0}^{i=n0-2}K(i)}}*$$

$$\frac{\left(\sum_{i=0}^{i=n0-2}K(i)\right)!}{\prod_{i=0}^{i=n0-2}(K(i)!)}*\prod_{i=0}^{i=n0-1}A(i)^{K(i)}$$

Definitions for S and K(n0-1):

$$S = \sum_{i=0}^{i=n0-2} K(i)$$

$$K(n0-1) = k+1-\left[\sum_{i=0}^{i=n0-2} \left(K(i)*(n0-i)\right)\right]$$

Condition K(n0-1)≥ 0

This formula can be rewritten in the following expression:

$$E(k) =$$

$$\sum_{S=0}^{S=k} \left[\frac{\left(\sum_{i=0}^{i=n0-2} K(i) + K(n0-1)\right)}{\left(\sum_{i=0}^{i=n0-2} K(i)\right)} \right] *$$

$$\frac{\left(\sum_{i=0}^{i=n0-2} K(i)\right)!}{\prod_{i=0}^{i=n0-2} \left(K(i)!\right)} * \prod_{i=0}^{i=n0-1} A(i)^{K(i)}$$

Definitions for S and K(n0-1):

$$S = \sum_{i=0}^{i=n0-2} K(i)$$

$$K(n0-1) = k+1-\left[\sum_{i=0}^{i=n0-2} \left(K(i)*(n0-i)\right)\right]$$

condition K(n0-1)≥ 0

Now we will develop the formula E(k) for the quintic equation

$$X^5 = aX^4 + bX^3 + cX^2 + dX + e$$

The expression of E_k developed for the quinitic formula.

$$E_k = a^{k+1} + \binom{k}{1}a^{k-1}b + \binom{k-1}{1}a^{k-2}c + \binom{k-2}{1}a^{k-3}d + \binom{k-3}{1}a^{k-4}e + \binom{k-1}{2}a^{k-3}b^2$$
$$+ 2\binom{k-2}{2}a^{k-4}bc + 2\binom{k-3}{2}a^{k-5}bd + 2\binom{k-4}{2}a^{k-6}be + 2\binom{k-4}{2}a^{k-6}cd +$$
$$2\binom{k-5}{2}a^{k-5}c^2 + \binom{k-5}{2}a^{k-7}d^2 + \binom{k-7}{2}a^{k-9}e^2 + \binom{k-2}{3}a^{k-5}b^3 + 3\binom{k-3}{3}a^{k-6}b^2c +$$
$$3\binom{k-4}{3}a^{k-7}bc^2 + \binom{k-5}{3}a^{k-8}c^3 + 3\binom{k-4}{3}a^{k-7}b^2d$$
$$+ 3\binom{k-6}{3}a^{k-9}bd^2 + \binom{k-8}{3}a^{k-11}d^3 +$$
$$3\binom{k-5}{3}a^{k-8}b^2e + 3\binom{k-8}{3}a^{k-11}bc^2 +$$
$$6\binom{k-5}{3}a^{k-8}bcd + 12\binom{k-7}{4}a^{k-11}b^2ce +$$
$$12\binom{k-8}{4}a^{k-12}bc^2e + 12\binom{k-10}{4}a^{k-14}bce^2 +$$
$$12\binom{k-10}{4}a^{k-14}c^2de + 24\binom{k-11}{4}a^{k-15}cd^2e +$$
$$12\binom{k-12}{4}a^{k-16}cde^2 + 24\binom{k-9}{4}a^{k-13}bcde +$$
$$3\binom{k-6}{3}a^{k-9}c^2d + 3\binom{k-7}{3}a^{k-10}cd^2 + c\binom{k-7}{3}a^{k-10}bde +$$
$$4\binom{k-8}{4}a^{k-12}c^3d + 6\binom{k-9}{4}a^{k-13}c^2d^2 +$$
$$4\binom{k-10}{4}a^{k-14}cd^3 + 4\binom{k-9}{4}a^{k-13}c^2e +$$
$$6\binom{k-11}{4}a^{k-15}c^2e^2 + 4\binom{k-13}{4}a^{k-17}d^2e^2 +$$
$$4\binom{k-14}{4}a^{k-18}de^3 + 6\binom{k-5}{4}a^{k-9}b^2c^2 + 4\binom{k-6}{4}a^{k-10}bc^3 +$$

$$\binom{k-7}{4}a^{k-11}c^4 + 4\binom{k-5}{4}a^{k-9}b^3d + 6\binom{k-7}{4}a^{k-11}b^2d^2 +$$
$$4\binom{k-9}{4}a^{k-13}bd^3 + \binom{k-11}{4}a^{k-15}d^4 + 4\binom{k-6}{4}a^{k-10}b^3c$$
$$+6\binom{k-9}{4}a^{k-13}b^2e^2 + 4\binom{k-12}{4}a^{k-16}bc^3 + \binom{k-15}{4}a^{k-9}e^4 +$$
$$\binom{k-5}{6}a^{k-11}b^6 + 6\binom{k-6}{6}a^{k-12}b^5c + 15\binom{k-7}{6}a^{k-13}b^4c^2$$
$$20\binom{k-8}{6}a^{k-14}b^3c^3 + 15\binom{k-9}{6}a^{k-15}b^2c^4 +$$
$$6\binom{k-10}{6}a^{k-16}bc^5 + \binom{k-11}{6}a^{k-17}c^6 + 6\binom{k-7}{6}a^{k-13}b^5d +$$
$$30\binom{k-8}{6}a^{k-14}b^4cd + 60\binom{k-9}{6}a^{k-15}b^3c^2d +$$
$$60\binom{k-10}{6}a^{k-16}b^2c^3d + 30\binom{k-11}{6}a^{k-17}bc^4d +$$
$$6\binom{k-12}{6}a^{k-18}c^5d + 6\binom{k-8}{6}a^{k-14}b^5c + 30\binom{k-9}{6}a^{k-15}b^4ce +$$
$$60\binom{k-10}{6}a^{k-16}b^2c^3d + 60\binom{k-11}{6}a^{k-17}b^2c^3e + 30\binom{k-12}{6}a^{k-18}bc^4e +$$
$$6\binom{k-13}{6}a^{k-19}c^5e + 30\binom{k-9}{5}a^{k-14}b^2cd^2 + 30\binom{k-10}{5}a^{k-15}bc^2d^2 +$$
$$10\binom{k-11}{5}a^{k-16}c^3d^2 + 20\binom{k-9}{5}a^{k-17}c^3de + 60\binom{k-10}{5}a^{k-15}b^2cde +$$
$$60\binom{k-11}{5}a^{k-16}bc^2de + 20\binom{k-12}{5}a^{k-17}c^3de + 10\binom{k-10}{5}b^3e^2 +$$
$$30\binom{k-11}{5}a^{k-16}b^2ce^2 + 30\binom{k-12}{5}a^{k-17}bc^2e^2 + 15\binom{k-9}{6}a^{k-15}b^4d^2$$

$$+60\binom{k-10}{6}a^{k-16}b^3cd^2 + 90\binom{k-11}{6}a^{k-17}b^2c^2d^2 + 60\binom{k-12}{6}a^{k-18}bc^3d^2 +$$
$$15\binom{k-13}{6}a^{k-19}c^4d^2 + 30\binom{k-10}{6}a^{k-16}b^4de$$
$$120\binom{k-11}{6}a^{k-17}b^3cde + 100\binom{k-12}{6}a^{k-18}b^2c^2de +$$
$$120\binom{k-13}{6}a^{k-19}bc^3de + 30\binom{k-14}{6}a^{k-20}c^4de + 5\binom{k-17}{5}a^{k-22}ce^4 +$$
$$\binom{k-14}{5}a^{k-19}d^5 + 5\binom{k-15}{5}a^{k-20}d^4e + 10\binom{k-16}{5}a^{k-21}d^3e^2 +$$
$$10\binom{k-17}{5}a^{k-22}d^2e^3 + 5\binom{k-18}{5}a^{k-23}de^4 + \binom{k-19}{5}a^{k-24}e^5 +$$
$$20\binom{k-16}{5}a^{k-21}cde^3 + 10\binom{k-10}{5}a^{k-15}b^2d^3 + 6\binom{k-6}{3}a^{k-9}bce +$$
$$3\binom{k-7}{3}a^{k-10}c^2e + 3\binom{k-9}{3}a^{k-12}ce^2 + 6\binom{k-8}{3}a^{k-11}cde +$$
$$3\binom{k-9}{3}a^{k-12}d^2e + 3\binom{k-10}{3}a^{k-13}de^2 + \binom{k-11}{3}a^{k-14}e^3 +$$
$$\binom{k-3}{4}a^{k-7}b^4 + 4\binom{k-4}{4}a^{k-8}b^3c + 10\binom{k-13}{5}a^{k-18}b^2e^3 +$$
$$20\binom{k-14}{5}a^{k-19}bce^3 + 10\binom{k-15}{5}a^{k-20}c^2e^3 + 5\binom{k-12}{5}a^{k-17}bd^4 +$$
$$5\binom{k-13}{5}a^{k-18}b^2e^2 + 20\binom{k-13}{5}a^{k-18}bd^3e + 5\binom{k-14}{5}a^{k-19}cd^3e +$$
$$30\binom{k-14}{5}a^{k-19}bd^2e^2 + 20\binom{k-15}{5}a^{k-20}bde^3 + 5\binom{k-16}{5}a^{k-21}be^4 +$$
$$10\binom{k-13}{5}a^{k-18}c^3e^2 + 20\binom{k-11}{5}a^{k-16}bcd^3 + 10\binom{k-12}{5}a^{k-17}c^2d^3 +$$

$30\binom{k-11}{5}a^{k-16}b^2d^2e + 60\binom{k-12}{5}a^{k-17}bcd^2e + 30\binom{k-13}{5}a^{k-18}c^2d^2e +$

$30\binom{k-12}{5}a^{k-17}b^2de^2 + 60\binom{k-13}{5}a^{k-18}bcde^2 + 30\binom{k-14}{5}a^{k-19}c^2de^2 +$

$30\binom{k-19}{6}a^{k-25}bde^4 + 30\binom{k-20}{6}a^{k-26}cde^4 + 6\binom{k-20}{6}a^{k-26}be^5 +$

$6\binom{k-21}{6}a^{k-27}ce^5 + 6\binom{k-18}{6}a^{k-27}d^2e^4 + 15\binom{k-19}{6}a^{k-25}d^4e^2 +$

$20\binom{k-20}{6}a^{k-26}d^3c^3 + 15\binom{k-21}{6}a^{k-27}d^2e^4 + 6\binom{k-22}{6}a^{k-28}de^5 +$

$\binom{k-23}{6}a^{k-29}e^6 + 60\binom{k-16}{6}a^{k-22}c^3de^2 + 20\binom{k-14}{6}a^{k-20}b^3e^3 +$

$60\binom{k-15}{6}a^{k-24}b^2ce^3 + 60\binom{k-16}{6}a^{k-22}bc^2e^3 + 20\binom{k-17}{6}a^{k-23}c^3e^3 +$

$15\binom{k-13}{6}a^{k-19}b^2d^4 + 30\binom{k-14}{6}a^{k-20}bcd^4 + 15\binom{k-15}{6}a^{k-21}c^2d^4 +$

$60\binom{k-14}{6}a^{k-20}b^2d^3e + 120\binom{k-15}{6}a^{k-21}bcd^3e +$

$60\binom{k-12}{6}a^{k-18}b^3d^2e + 100\binom{k-13}{6}a^{k-19}b^2cd^2e + 100\binom{k-14}{6}a^{k-20}bc^2d^2e +$

$60\binom{k-15}{6}a^{k-21}c^3d^2e + 60\binom{k-13}{6}a^{k-19}b^3de^2 + 100\binom{k-14}{6}a^{k-20}b^2cde^2 +$

$100\binom{k-15}{6}a^{k-21}bc^2de^2 + 15\binom{k-11}{6}a^{k-17}b^4e^2 + 60\binom{k-12}{6}a^{k-18}b^3ce^2 +$

$90\binom{k-13}{6}a^{k-19}b^2c^2e^2 + 60\binom{k-14}{6}a^{k-20}bc^3e^2 + 15\binom{k-15}{6}a^{k-21}c^4e^2 +$

$20\binom{k-11}{6}a^{k-17}b^3d^3 + 60\binom{k-12}{6}a^{k-18}b^2cd^3 + 60\binom{k-13}{6}a^{k-19}bc^2d^3 +$

$20\binom{k-14}{6}a^{k-20}c^3d^3 + 60\binom{k-16}{6}a^{k-22}c^2d^3e + 90\binom{k-15}{6},a^{k-21}b^2d^2e^2 +$

$100\binom{k-16}{6}a^{k-22}bcd^2e^2 + 90\binom{k-17}{6}a^{k-23}c^2d^2e^2 + 60\binom{k-16}{6}a^{k-22}c^2de +$

$120\binom{k-17}{6}a^{k-23}bcde^3 + 60\binom{k-18}{6}a^{k-24}c^2de^3 + 15\binom{k-23}{6},a^{k-23}c^2e^4 +$

$30\binom{k-18}{6}a^{k-24}bce^4 + 15\binom{k-19}{6}a^{k-25}c^2e^4 + \binom{k-17}{6}a^{k-23}d^6 +$

$6\binom{k-15}{6}a^{k-21}bd^5 + 6\binom{k-16}{6}a^{k-22}cd^5 + 30\binom{k-16}{6}a^{k-22}bd^4e +$

$30\binom{k-17}{6}a^{k-23}cd^4e + 60\binom{k-17}{6}a^{k-23}bd^3e^2 + 60\binom{k-18}{6}a^{k-24}cd^3e^2 +$

$60\binom{k-15}{6}a^{k-24}bd^2e^3 + 60\binom{k-19}{6}a^{k-25}cd^2e^3 + 20\binom{k-9}{5}a^{k-14}bc^3d$

$5\binom{k-10}{6}a^{k-15}c^4d + 5\binom{k-7}{5}a^{k-12}b^4e + 20\binom{k-8}{5}a^{k-13}b^3ce +$

$30\binom{k-9}{5}a^{k-14}b^2c^2e + 20\binom{k-10}{5}a^{k-15}bc^3e + 5\binom{k-11}{5}a^{k-16}c^4e +$

$10\binom{k-8}{5}a^{k-13}b^3d^2 + \binom{k-4}{5}a^{k-9}b^5 + 5\binom{k-5}{5}a^{k-10}b^4c +$

$10\binom{k-6}{5}a^{k-11}b^3c^2 + 10\binom{k-7}{5}a^{k-12}b^2c^3 + 5\binom{k-8}{5}a^{k-13}bc^4 +$

$\binom{k-9}{5}a^{k-14}c^5 + 5\binom{k-6}{5}a^{k-11}b^4d + 20\binom{k-7}{4}a^{k-12}b^3cd +$

$12\binom{k-7}{4}a^{k-11}bc^2d + 12\binom{k-8}{4}a^{k-12}b^2c^3 + 5\binom{k-8}{4}a^{k-12}b^2de +$

$12\binom{k-10}{4}a^{k-14}bd^2e + 12\binom{k-11}{4}a^{k-15}bde^2 + \binom{k-6}{7}a^{k-13}b^7 +$

$\binom{k-7}{8}a^{k-15}b^8 + 7\binom{k-7}{7}a^{k-14}b^6c + 21\binom{k-8}{7}a^{k-15}b^5c^2 + 7\binom{k-8}{7}a^{k-15}b^6d$

Let's express the root of a quantic equation by using the expression of E_k developed for the quinitic formula ($n_0=5$)

$X^5=3X^4+4X^3+2X^2+X+1$.

The first thing we need to determine is the minimum number of interaction needed .to calculate the solution of the equation. This number is called $K_1=K0+n0$ following the derivatives given to determine Kc.

We illustrate how to calculate K_0 , K_0 is the minimum common factor between

1×3 , 2×4, 3×2, 4×1 , 5×1.

Where on the right side of each product is represented the coefficient of the quinitic equation and on the left side of each product is represented the indices starting from 1.

So we need to identify the different pair of prime factors contained in the product between the five products.

$[1\times3]\times[2\times4]\times[3\times2]\times[4\times1]\times[5\times1]$.

At next we need to express this product in terms of prime factors

$[1\times3]\times[2\times2\times2]\times[3\times2]\times[2\times2\times1]\times[5\times1]$.

The next thing we need to do is to divide this product by each pair of prime factors Let's do this for each pair of prime factors

$[1\times3]\times[2\times2\times2]\times[3\times2]\times[2\times2\times1]\times[5\times1]/2\times2\times2\times2\times2\times3\times3=5$.

At next we need to determine the equivalent of this result in terms of modulo n0 when n0 is the degree of the equation since the equation is a quintic equation so n0=5.

So 5 modulo 5=5[5]=0

So k0=0 and k1=k0+n0_0+5=5

So K1=5

So K1=5 interacts will give at least one digit after the decimal point of the equation.

For m digits after the decimal point So:

$$K = (Ko + n0)[1+2\ldots\ldots m] = (Ko + n0)*m(m+1)/2$$

This number k is the maximum number of the interactions needed to calculate the solutions until m digits after the decimal point.

So here $k = 5m*(m+1)/2$

For $m=1$ $K = 5*1*(1+1)/2 = 5$

In the decimal system K+10=5+10=15 interactions is the maximum number of the interactions for which m=1 digits after the decimal point is also common to the subsequent terms of the series

$G(K+1) = \dfrac{E(K+1)}{E(K)}$ which means that

L(with m=1 digit)=G(15)=G(16)

Infact checking the result with a calculator will show that

$G(15) = \dfrac{E(15)}{E(14)}$ gives 9 digits after the decimal point of the solution x.

So X (with 9 digits after the decimal point)=[int(G(15)*10^9)]/10^9

Where int is the integers function Now we need to calculate E(15) and E(14) using the expression of E(k) developed for the quantic formula we need to remember that only positive powers are allowed in the expression.

We calculate E(14) So k=14 ,a=3,b=4,c=2,d=1,e=1

$$E_{14} = 3^{14+1} + \binom{14}{1} \times 3^{14-1} \times \binom{14-1}{1} \times 3^{14-2} \times 2 +$$

$$\binom{14-2}{1} \times 3^{14-3} \times 1 + \binom{14-3}{1} \times 3^{14-4} \times 1 + \binom{14-1}{2} \times 3^{14-3} \times 4^2 +$$

$$2\binom{14-2}{2} \times 3^{14-4} \times 4 \times 2 + 2\binom{14-3}{2} \times 3^{14-5} \times 4 \times 1 +$$

$$\binom{14-7}{2} \times 3^{14-9} \times 1^2 + 2\binom{14-4}{2} \times 3^{14-6} \times 4 \times 1 +$$

$$2\binom{14-4}{2} \times 3^{14-6} \times 2 \times 1 + 2\binom{14-5}{2} \times 3^{14-7} \times 2 \times 1 +$$

$$2\binom{14-6}{2} \times 3^{14-8} \times 1 \times 1 + \binom{14-3}{2} \times 3^{14-5} \times 2^2 +$$

$$\binom{14-5}{2} \times 3^{14-7} \times 1^2 + 12\binom{14-6}{4} \times 3^{14-10} \times 4^2 \times 2 \times 1 +$$

$$12 \times \binom{14-7}{4} \times 3^{14-11} \times 4 \times 2^2 \times 1 + 12\binom{14-8}{4} \times 3^{14-12} \times 4 \times 2 \times 1^2 +$$

$$12\binom{14-10}{4} \times 3^{14-14} \times 4 \times 2 \times 1^2 + 12\binom{14-10}{4} \times 3^{14-14} \times 2^2 \times 1 \times 1 +$$

$$24\binom{14-9}{4} \times 3^{14-13} \times 4 \times 2 \times 1 \times 1 + \binom{14-5}{6} \times 3^{14-11} \times 4^6 +$$

$$\binom{14-2}{3} \times 3^{14-5} \times 4^3 \times 2^2 + \binom{14-5}{3} \times 3^{14-8} \times 2^3 +$$

$$3\binom{14-4}{3} \times 3^{14-7} \times 4^2 \times 1 + 3\binom{14-6}{3} \times 3^{14-9} \times 4 \times 1^2 +$$

$$\binom{14-8}{3} \times 3^{14-11} \times 1^3 + 3\binom{14-5}{3} \times 3^{14-8} \times 4^2 \times 1 +$$

$$3\binom{14-8}{3} \times 3^{14-11} \times 4 \times 1^2 + 6\binom{14-5}{3} \times 3^{14-8} \times 4 \times 2 \times 1 +$$

$$6\binom{14-5}{4} \times 3^{14-9} \times 4^2 \times 2^2 + 4\binom{14-6}{4} \times 3^{14-10} \times 4 \times 2^3 +$$

$$\binom{14-7}{4} \times 3^{14-11} \times 4^2 \times 1^2 + 4\binom{14-5}{4} \times 3^{14-9} \times 4^3 \times 1 +$$

$$6\binom{14-7}{4} \times 3^{14-11} \times 4^2 \times 1^2 + 4\binom{14-9}{4} \times 3^{14-13} \times 4 \times 1^3 +$$

$$4\binom{14-6}{4} \times 3^{14-10} \times 4^3 \times 1 + 6\binom{14-9}{4} \times 3^{14-13} \times 4^2 \times 1^2 +$$

$$3\binom{14-6}{3} \times 3^{14-9} \times 2^2 \times 1 + 3\binom{14-7}{3} \times 3^{14-10} \times 2 \times 1^2 +$$

$$6\binom{14-7}{3} \times 3^{14-10} \times 4 \times 1 \times 1 + 4\binom{14-8}{4} \times 3^{14-12} \times 2^3 \times 1 +$$

$$6\binom{14-9}{4} \times 3^{14-13} \times 2^2 \times 1^2 + 4\binom{14-10}{4} \times 3^{14-14} \times 2 \times 1^3$$

$$4\binom{14-9}{4} \times 3^{14-13} \times 2^3 \times 1 + 6\binom{14-6}{3} \times 3^{14-9} \times 4 \times 2 \times 1 +$$

$$3\binom{14-7}{3} \times 3^{14-10} \times 2^2 \times 1 + 3\binom{14-9}{3} \times 3^{14-12} \times 2 \times 1^2 +$$

$$6\binom{14-8}{3} \times 3^{14-11} \times 2 \times 1 \times 1 + 3\binom{14-9}{3} \times 3^{14-14} \times 1^3 +$$

$$\binom{14-3}{4} \times 3^{14-7} \times 4^4 + 4\binom{14-4}{4} \times 3^{14-8} \times 4^3 \times 2 -$$

$$\binom{14-6}{7} \times 3^{14-13} \times 4^7 + 7\binom{14-7}{7} \times 3^{14-14} \times 4^6 \times 2 +$$

$$30\binom{14-9}{5} \times 3^{14-14} \times 4^2 \times 2 \times 1^2 + 20\binom{14-9}{5} \times 3^{14-14} \times 4^3 \times 1 \times 1 +$$

$$6\binom{14-6}{4} \times 3^{14-12} \times 4^5 \times 2^3 + 6\binom{14-7}{6} \times 3^{14-13} \times 4^5 \times 2^2 +$$

$$20\binom{14-8}{6} \times 3^{14-14} \times 4^3 \times 2^3 + 6\binom{14-7}{6} \times 3^{14-13} \times 4^5 \times 1 +$$

$$30\binom{14-8}{6} \times 3^{14-14} \times 4^4 \times 2 \times 1 + 6\binom{14-8}{6} \times 3^{14-14} \times 4^5 \times 1 +$$

$$\binom{14-4}{5} \times 3^{14-9} \times 4^5 + 5\binom{14-5}{6} \times 3^{14-10} \times 4^4 \times 2 +$$

$$10\binom{14-6}{5} \times 3^{14-11} \times 4^3 \times 2^2 + 10\binom{14-7}{5} \times 3^{14-12} \times 4^2 \times 2^3 +$$

$$5\binom{14-8}{5} \times 3^{14-13} \times 4 \times 2^4 + \binom{14-9}{5} \times 3^{14-14} \times 2^5 +$$

$$5\binom{14-6}{5} \times 3^{14-11} \times 4^4 \times 1 + 20\binom{14-7}{5} \times 3^{14-12} \times 4^3 \times 2 \times 1 +$$

$$30\binom{14-8}{5} \times 3^{14-13} \times 4^2 \times 2^2 \times 1 + 20\binom{14-9}{5} \times 3^{14-14} \times 4 \times 2^3 \times 1 +$$

$$30\binom{14-9}{5} \times 3^{14-14} \times 4^2 \times 2^2 \times 1 + 10\binom{14-8}{5} \times 3^{14-13} \times 4^3 \times 1^2$$

E(14)=1231215656.

At next we calculate E_{15}, k=15 a=3,b=4,c=2,d=1,e=1

$$E_{15} = 3^{15+1} + \binom{15}{1} \times 3^{15-1} \times 4 + \binom{15-1}{1} \times 3^{15-2} \times 2 -$$

$$\binom{15-2}{1} \times 3^{15-3} \times 1 + \binom{15-3}{1} \times 3^{15-4} \times 1 + \binom{15-1}{2} \times 3^{15-3} \times 4^2 +$$

$$2\binom{15-2}{2} \times 3^{15-4} \times 4 \times 2 + 2\binom{15-3}{2} \times 3^{15-5} \times 4 \times 1 +$$

$$\binom{15-7}{2} \times 3^{15-9} \times 1^2 + 2\binom{15-4}{2} \times 3^{15-6} \times 4 \times 1 +$$

$$2\binom{15-4}{2} \times 3^{15-6} \times 2 \times 1 + 2\binom{15-5}{2} \times 3^{15-7} \times 2 \times 1 +$$

$$2\binom{15-6}{2} \times 3^{15-8} \times 1 \times 1 + \binom{15-3}{2} \times 3^{15-5} \times 2^2 +$$

$$\binom{15-5}{2} \times 3^{15-7} \times 1^2 + 12\binom{15-6}{4} \times 3^{15-10} \times 4^2 \times 2 \times 1 +$$

$$12\binom{15-7}{4} \times 3^{15-11} \times 4 \times 2^2 \times 1 + 12\binom{15-8}{4} \times 3^{15-12} \times 4 \times 2 \times 1^2 +$$

$$12\binom{15-8}{4} \times 3^{15-12} \times 4^2 \times 1 \times 1 + 12\binom{15-10}{4} \times 3^{15-14} \times 4 \times 1^2 \times 1 +$$

$$12\binom{15-11}{4} \times 3^{15-15} \times 4 \times 1 \times 1^2 + 12\binom{15-7}{4} \times 3^{15-11} \times 4^2 \times 2 \times 1 +$$

$$12\binom{15-8}{4} \times 3^{15-12} \times 4 \times 2^2 \times 1 + 12\binom{15-10}{4} \times 3^{15-14} \times 4 \times 2 \times 1^2 +$$

$$12\binom{15-10}{4} \times 3^{15-14} \times 2^2 \times 1 \times 1 + 12\binom{15-11}{4} \times 3^{15-15} \times 2 \times 1^2 \times 1 +$$

$$24\binom{15-9}{4} \times 3^{15-13} \times 4 \times 2 \times 1 \times 1 + \binom{15-5}{6} \times 3^{15-11} \times 4^6 +$$

$$\binom{15-2}{3} \times 3^{15-15} \times 4 \times 2 \times 1 \times 1 + \binom{15-5}{6} \times 3^{15-11} \times 4^6 +$$

$$\binom{15-2}{3} \times 3^{15-15} \times 4^3 + 3\binom{15-3}{3} \times 3^{15-6} \times 4^2 \times 2 +$$

$$3\binom{15-4}{3} \times 3^{15-7} \times 4 \times 2^2 + \binom{15-5}{3} \times 3^{15-8} \times 2^3 +$$

$$3\binom{15-4}{3} \times 3^{15-7} \times 4^2 \times 1 + 3\binom{15-6}{3} \times 3^{15-9} \times 4 \times 1^2 +$$

$$\binom{15-8}{3} \times 3^{15-11} \times 1^3 + 3\binom{15-5}{3} \times 3^{15-8} \times 4^2 \times 1 +$$

$$3\binom{15-8}{3} \times 3^{15-11} \times 4 \times 1^2 + 6\binom{15-5}{3} \times 3^{15-8} \times 4 \times 2 \times 1 +$$

$$6\binom{15-5}{4} \times 3^{15-9} \times 4^2 \times 2^2 + 4\binom{15-6}{4} \times 3^{15-10} \times 4 \times 2^3 +$$

$$\binom{15-7}{4} \times 3^{15-11} \times 2^4 + 4\binom{15-5}{4} \times 3^{15-9} \times 4^3 \times 1 +$$

$$6\binom{15-7}{4} \times 3^{15-11} \times 4^2 \times 1^2 + 4\binom{15-9}{4} \times 3^{15-13} \times 4 \times 1^3 +$$

$$\binom{15-11}{4} \times 3^{15-15} \times 1^4 + 4\binom{15-6}{4} \times 3^{15-10} \times 4^3 \times 1 +$$

$$6\binom{15-9}{4} \times 3^{15-13} \times 4^2 \times 1^2 + 3\binom{15-6}{3} \times 3^{15-9} \times 2^2 \times 1 +$$

$$3\binom{15-7}{3} \times 3^{15-10} \times 2 \times 1^6 + 6\binom{15-7}{3} \times 3^{15-10} \times 4 \times 1 \times 1 +$$

$$4\binom{15-8}{4} \times 3^{15-12} \times 2^3 \times 1 + 6\binom{15-9}{4} \times 3^{15-13} \times 2^3 \times 1 +$$

$$4\binom{15-10}{4} \times 3^{15-14} \times 2 \times 1^3 + 4\binom{15-9}{4} \times 3^{15-13} \times 2^3 \times 1 +$$

$$6\binom{15-11}{4} \times 3^{15-15} \times 2^2 \times 1^2 + 6\binom{15-6}{3} \times 3^{15-9} \times 4 \times 2 \times 1 +$$

$$3\binom{15-7}{4} \times 3^{15-10} \times 2^2 \times 1 + 3\binom{15-9}{3} \times 3^{15-12} \times 2 \times 1^2 +$$

$$6\binom{15-8}{3} \times 3^{15-11} \times 2 \times 1 \times 1 + 3\binom{15-9}{3} \times 3^{15-12} \times 1^2 \times 1 +$$

$$3\binom{15-10}{3} \times 3^{15-13} \times 1 \times 1^2 + \binom{15-11}{3} \times 3^{15-14} \times 1^3 +$$

$$\binom{15-3}{4} \times 3^{15-7} \times 4^4 + 4\binom{15-4}{4} \times 3^{15-8} \times 4^3 \times 2 +$$

$$10\binom{15-10}{5} \times 3^{15-15} \times 4^2 \times 1^3 + 15\binom{15-9}{6} \times 3^{15-15} \times 4^4 \times 1^2 +$$

$$\binom{15-6}{7} \times 3^{15-13} \times 4^7 + \binom{15-7}{8} \times 3^{15-15} \times 4^8 +$$

$$7\binom{15-7}{7} \times 3^{15-14} \times 4^6 \times 2 + 21\binom{15-8}{7} \times 3^{15-15} \times 4^5 \times 2^2 +$$

$$7\binom{15-8}{7} \times 3^{15-15} \times 4^6 \times 1 + 30\binom{15-9}{5} \times 3^{15-14} \times 4^2 \times 2 \times 1^2 +$$

$$30\binom{15-10}{5} \times 3^{15-15} \times 4 \times 2^2 \times 1^2 + 20\binom{15-9}{5} \times 3^{15-14} \times 4^3 \times 1 \times 1 +$$

$$60\binom{15-10}{5} \times 3^{15-15} \times 4^2 \times 2 \times 1 \times 1 + 10\binom{15-10}{5} \times 3^{15-15} \times 4^3 \times 1^2 +$$

$$6\binom{15-6}{6} \times 3^{15-12} \times 4^5 \times 2 + 15\binom{15-7}{6} \times 3^{15-13} \times 4^4 \times 1^2 +$$

$$20\binom{15-8}{6} \times 3^{15-14} \times 4^3 \times 2^3 + 15\binom{15-9}{6} \times 3^{15-15} \times 4^2 \times 2^4 +$$

$$6\binom{15-7}{6} \times 3^{15-13} \times 4^5 \times 1 + 30\binom{15-8}{6} \times 3^{15-14} \times 4^4 \times 2 \times 1 +$$

$$60\binom{15-9}{6}\times 3^{15-15}\times 4^3\times 2^2\times 1+6\binom{15-8}{6}\times 3^{15-14}\times 4^5\times 1+$$

$$30\binom{15-9}{6}\times 3^{15-15}\times 4^4\times 2\times 1+\binom{15-4}{5}\times 3^{15-9}\times 4^5+$$

$$5\binom{15-5}{5}\times 3^{15-10}\times 4^2\times 2^3+5\binom{15-6}{5}\times 3^{15-11}\times 4^3\times 2^2+$$

$$10\binom{15-7}{5}\times 3^{15-12}\times 4^2\times 2^3+5\binom{15-8}{5}\times 3^{15-13}\times 4\times 2^4+$$

$$\binom{15-9}{5}\times 3^{15-14}\times 2^5+5\binom{15-6}{5}\times 3^{15-11}\times 4^4\times 1+$$

$$20\binom{15-7}{5}\times 3^{15-12}\times 4^3\times 2\times 1+30\binom{15-8}{5}\times 3^{15-13}\times 4^2\times 2^2\times 1+$$

$$20\binom{15-9}{5}\times 3^{15-14}\times 4\times 2^3\times 1+5\binom{15-10}{5}\times 3^{15-15}\times 2^4\times 1+$$

$$5\binom{15-7}{5}\times 3^{15-12}\times 4^4\times 1+20\binom{15-8}{5}\times 3^{15-13}\times 4^3\times 2\times 1+$$

$$30\binom{15-9}{5}\times 3^{15-14}\times 4^2\times 2^2\times 1+20\binom{15-10}{5}\times 3^{15-15}\times 4\times 2^3\times 1+$$

$$10\binom{15-8}{5}\times 3^{15-13}\times 4^3\times 1^2$$

E(15)=5059866125.

So let's calculate the solution of the quantic equation:

$$\underset{\substack{with\,9\,digits\\ after\,the\,decimal\\ po\operatorname{int}}}{X}=\frac{\operatorname{int}\left(G_5\times 10^9\right)}{10^9}=\frac{\operatorname{int}\left(\dfrac{E_{15}}{E_{14}}\times 10^9\right)}{10^9}=\frac{\operatorname{int}\left(\dfrac{5059866125}{1231215656}\times 10^9\right)}{10^9}=4,109650572$$

So X=4.109650572 is a solution of the quantic equation. $X^5=3X^4+4X^3+2X^2+X+1$.

This result can be checked by the Newton method or any other technique.

This How J determined the different power combinations in the E(K) formula for K=15 and K=14.

Let's denote K0,K1,K2,K3 the respective process of b,c,d,e in the qunitic equation.

$X^5=aX^4+bX^3+cX^2+dX+e$

For k0+k1+k2+k3=0 there is only one possibility k0=0, k1=0, k2=0, k3=0 .

For k0+k1+k2+k3=1 this is equivalent to (k0+k1)+(k2+k3)=1 , so the possibilities for this are

(k0+k1)=0, (k2+k3)=1 or (k0+k1)=1, (k2+k3)=0

For (k0+k1)=1 the possibilities are k0=0 , k1=0 .

For (k2+k3)=1 the possibilities are k2=0 , k3=1 or k2=1 , k3=0.

For (k0+k1)=1 the possibilities are k2=0 , k1=1 or k0=1 , k1=0

For (k2+k3)=0 the possibilities are k2=0 , k3=0

So for k0+k1+k2+k3=1 the possibilities are

$$\begin{pmatrix}K_0 & K_1 & K_2 & K_3\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}$$

This gives the followings different power combinations for $b^{k0}\,c^{k1}\,d^{k2}\,e^{k3}$

$b^0c^0d^0e=e$

$b^0c^0de^0=d$

$b^0cd^0e^0=c$

$bc^0d^0e^0=b$

So we need to perform the same things

For $K_0+K_1+K_2+K_3=2$, $K_0+K_1+K_2+K_3=3$

$K_0+K_1+K_2+K_3=4$, $K_0+K_1+K_2+K_3=5$, $K_0+K_1+K_2+K_3=6$, $K_0+K_1+K_2+K_3=7$.

For the qunitic equation $X5=3x^4+4x^3+2x^2+x+1$.

We needed E(15) and E(14)

For k0+k1+k2+k3>7 the powers of a=3 in E(15) will be negative and since the condition in the E(k)formula is K(no-1)≥ 0 , which means that the powers of a is to be positive powers .So we need to step the summation k0+k1+k2+k3 at the sum equal to 7 to calculate E(15).

These are the tables for the sum k0+k1+k2+k3 (Table 1).

K0 +	k1 +	k2 +	k3=2
0	0	0	2
0	0	1	1
0	0	2	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
0	2	0	0
1	1	0	0
2	0	0	0
K0 +	**k1 +**	**k2 +**	**k3=3**
0	0	0	3
0	0	1	2
0	0	2	1
0	0	3	0
0	1	0	2
1	0	0	2
0	1	1	1
1	0	1	1
0	1	2	0
1	0	2	0
0	2	0	1
1	1	0	1
2	0	0	1
0	2	1	0
1	1	1	0
2	0	1	0
3	0	0	0
2	1	0	0
1	2	0	0
0	3	0	0
K0 +	**k1 +**	**k2 +**	**k3=4**
2	0	2	0
0	3	0	1
1	2	0	1
2	1	0	1
3	0	0	1
0	3	1	0
1	2	1	0
2	1	1	0
3	0	1	0
0	4	0	0

K0 +	k1 +	k2 +	k3=5
1	3	0	0
2	2	0	0
3	1	0	0
4	0	0	0
K0 +	k1 +	k2 +	k3=5
0	0	0	5
0	0	1	4
0	0	2	3
0	0	3	2
0	0	4	1
0	0	5	0
0	1	0	4
1	0	0	4
0	1	1	3
1	0	1	3
0	1	2	2
1	0	2	2
0	1	3	1
1	0	3	1
0	1	4	0
1	0	4	0
0	2	0	3
1	1	0	3
2	0	0	3
0	2	1	2
1	1	1	2
K0 +	k1 +	k2 +	k3=5
2	0	1	2
0	2	2	1
1	1	2	1
2	0	2	1
0	2	3	0
1	1	3	0
2	0	3	0
0	3	0	2
1	2	0	2
2	1	0	2
3	0	0	2
0	3	1	1
1	2	1	1
2	1	1	1
3	0	1	1
0	3	2	0
1	2	2	0
2	1	2	0
3	0	2	0
0	4	0	1
1	3	0	1
K0 +	k1 +	k2 +	k3=5
2	2	0	1
3	1	0	1
4	0	0	1
0	4	1	0
1	3	1	0
2	2	1	0
3	1	1	0
4	0	1	0
0	5	0	0
1	4	0	0
2	3	0	0
3	2	0	0

K0 +	k1 +	k2 +	k3=6
4	1	0	0
5	0	0	0
K0 +	k1 +	k2 +	k3=6
0	0	0	6
0	0	1	5
0	0	2	4
0	0	3	3
0	0	4	2
0	0	5	1
0	0	6	0
0	1	0	5
1	0	0	5
0	1	1	4
1	0	1	4
0	1	2	3
1	0	2	3
0	1	3	2
1	0	3	2
0	1	4	1
1	0	4	1
0	1	5	0
1	0	5	0
0	2	0	4
K0 +	k1 +	k2 +	k3=6
1	1	0	4
2	0	0	4
0	2	1	3
1	1	1	3
2	0	1	3
0	2	2	2
1	1	2	2
2	0	2	2
0	2	3	1
1	1	3	1
2	0	3	1
0	2	4	0
1	1	4	0
2	0	4	0
0	3	0	3
1	2	0	3
2	1	0	3
3	0	0	3
0	3	1	2
1	2	1	2
2	1	1	2
K0 +	k1 +	k2 +	k3=6
3	0	1	2
0	3	2	1
1	2	2	1
2	1	2	1
3	0	2	1
0	3	3	0
1	2	3	0
2	1	3	0
3	0	3	0
0	4	0	2
1	3	0	2
2	2	0	2
3	1	0	2
4	0	0	1
0	4	1	1

K0 +	k1 +	k2 +	k3
1	3	1	1
2	2	1	1
3	1	1	1
4	0	1	1
0	4	2	0
1	3	2	0
K0 +	**k1 +**	**k2 +**	**k3=6**
2	2	2	0
3	1	2	0
4	0	2	0
0	5	0	1
1	4	0	1
2	3	0	1
3	2	0	1
4	1	0	1
5	0	0	1
0	5	1	0
1	4	1	0
2	3	1	0
3	2	1	0
4	1	1	0
5	0	1	0
0	6	0	0
1	5	0	0
2	4	0	0
3	3	0	0
4	2	0	0
5	1	0	0
K0 +	**k1 +**	**k2 +**	**k3=6**
6	0	0	0
K0 +	**k1 +**	**k2 +**	**k3=7**
0	0	0	7
0	0	1	6
0	0	2	5
0	0	3	4
0	0	4	3
0	0	5	2
0	0	6	1
0	0	7	0
0	1	0	6
1	0	0	6
0	1	1	5
1	0	1	5
0	1	2	4
1	0	2	4
0	1	3	3
1	0	3	3
0	1	4	2
1	0	4	2
0	1	5	1
K0 +	**k1 +**	**k2 +**	**k3=7**
1	0	5	1
0	1	6	0
1	0	6	0
0	2	0	5
1	1	0	5
2	0	0	5
0	2	1	4
1	1	1	4
2	0	1	4
0	2	2	3

K0 +	k1 +	k2 +	k3
1	1	2	3
2	0	2	3
0	2	3	2
1	1	3	2
2	0	3	2
0	2	4	1
1	1	4	1
2	0	4	1
0	2	5	0
1	1	5	0
2	0	5	0
K0 +	**k1 +**	**k2 +**	**k3=7**
0	3	0	4
1	2	0	4
2	1	0	4
3	0	0	4
0	3	1	3
1	2	1	3
2	1	1	3
3	0	1	3
0	3	2	2
1	2	2	2
2	1	2	2
3	0	2	2
0	3	3	1
1	2	3	1
2	1	3	1
3	0	3	1
0	3	4	0
1	2	4	0
2	1	4	0
3	0	4	0
0	4	0	3
K0 +	**k1 +**	**k2 +**	**k3=7**
1	3	0	3
2	2	0	3
3	1	0	3
4	0	0	3
0	4	1	2
1	3	1	2
2	2	1	2
3	1	1	2
4	0	1	2
0	4	2	1
1	3	2	1
2	2	2	1
3	1	2	1
4	0	2	1
0	4	3	0
1	3	3	0
2	2	3	0
3	1	3	0
4	0	3	0
0	5	0	2
1	4	0	2
K0 +	**k1 +**	**k2 +**	**k3=7**
2	3	0	2
3	2	0	2
4	1	0	2
5	0	0	2
0	5	1	1

K0 +	k1 +	k2 +	k3=7
1	4	1	1
2	3	1	1
3	2	1	1
4	1	1	1
5	0	1	1
0	5	2	0
1	4	2	0
2	3	2	0
3	2	2	0
4	1	2	0
5	0	2	0
0	6	0	1
1	5	0	1
2	4	0	1
3	3	0	1
4	2	0	1
K0 +	**k1 +**	**k2 +**	**k3=7**
5	1	0	1
6	0	0	1
0	6	1	0
1	5	1	0
2	4	1	0
3	3	1	0
4	2	1	0
5	1	1	0
6	0	1	0
0	7	0	0
1	6	0	0
2	5	0	0
3	4	0	0
4	3	0	0
5	2	0	0
6	1	0	0
7	0	0	0

Table 1: The summation k0+k1+k2+k3 at the sum equal to 7.

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