

Research Article

Matrix Transformations of Some Sequence Spaces Over Non-Archmedian Fields

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Abstract

The space $\chi(F)$ of all entire functions $f(z) = \sum_{k} x_k z^k$ of exponential order 1 and type 0 have been defined by Sirajudeen. In the present paper we characterize the matrix class concerning the space $\chi(F)$.

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Preliminaries, Background and Notation

The theory of matrix transformations is a wide field in summability; it deals with the characterizations of classes of matrix mappings between sequence spaces by giving necessary and sufficient conditions on the entries of the infinite matrices.

The classical summability theory deals with a generalization of convergence of sequences and series. One original idea was to assign a limit to divergent sequences or series. Toeplitz [1] was the first to study summability methods as a class of transformations of complex sequences by complex infinite matrices.

Let X, Y be two sequence spaces and let $A=(a_{nk})$ be an infinite matrix of real or complex numbers a_{nk} , where $n, k \in N$. Then, the matrix A defines the A transformation from X into Y, if for every sequence $x=(x_k)E\in X$ the sequence $Ax=\{(Ax)_n\}$, the A-transform of x exists and is in Y; where $(Ax)_n = \sum_{k=1}^{k} a_{nk}X_k$ For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞ . By $A \in X : Y$ we mean the characterizations of matrices from X to Y i.e., $A: X \to Y$. A sequence x is said to be A-summable to l if Ax converges to l which is called as the A-limit of x [1-5].

For a sequence space X, the matrix domain $\mathbf{X}_{_{\!\!A}}$ of an infinite matrix A is defined as

$$X_A = \{x = (x_k) : x = (x_k) \in \omega\}$$

Let F be a non-trivial, non-archimedian field which is complete under the metric of valuation. If $\mathbf{x}=(\mathbf{x}_k)=(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{X}_k,\ldots)$, $X_k \in F$ is a sequence defined over F, this assumption ensures not only the completion of the sequence spaces we consider but also the absolute convergence of a series in F implies convergence in F. In what follows $\sum x_k$ denote $\sum_{k=1}^{\infty} x_k$ and the notion of convergence and boundedness will be in k=1 relation to the metric of valuation of the field.

Assume here and after that $p=(P_k)$ is bounded sequence of positive reals, so that $0 < P_k \le \sup P_k = H < \infty$ and $M = \max\{1, H\}$. We shall assume throughout the text that $p_k^{-1} + q_k^{-1} = 1$ provided $P_k > 1$ for all $k \in N$.

The spaces $l_{_{\rm \infty}}(p)$, c(p) and $c_{_0}(p)$ were defined by Maddox [6-10] as follows:

$$l_{\infty}(P,F):\left\{x = (x_k) : \sup_{k} |x_k|^{p_k} < \infty\right\}.$$

$$c(P,F):\left\{x = (x_{k}) : \lim_{k} |x_{k} - l|^{p_{k}} = 0, l \in C\right\}.$$

$$c_{0}(P,F):\left\{x = (x_{k}) : \lim_{k} |x_{k}|^{p_{k}} \to 0\right\}.$$

$$l(P,F):\left\{x = (x_{k}) : \lim_{k} |x_{k}|^{p_{k}} < \infty\right\}.$$

$$\chi(F):\left\{x = (x_{k}) : (k!|x_{k}|^{\frac{1}{k}} \to 0 \text{ as } k \to \infty\right\}.$$

We denote by $\chi(F)$ the collect ion of all entire functions $f(z) = \sum z_k x^k$ of exponential order 1 and type 0[3,8].

 $c_0(F)$, c(f) and l_{∞} are non-Archimedian Banach spaces with Non-Archimedian norm, $||x|| = \sup_k |x_k|$. If $x=(x_k)$ is an element of $\chi(F)$ then $||x|| = \sup\{(k!|x_k|^{\frac{1}{k}}, k \ge 1\}$ satisfies the following conditions:

(i) |x| > 0, |x|=0 if and only if x =(0, 0, ...) where 0 is the zero element of the field F.

(ii) $|x + Y| \le max\{|x|, |y|\}$.

(iii) $|tx| \le G(t) |x|, t \in F, G(t) = max \{1, |t|\}.$

Hence $\chi(F)$ is a metric space defined over F with a metric d(x, y) = |x - Y|.

Main Results

We begin with the following lemma [7] which is essential in the text.

Lemma 4.1: Let $T_n(x)$ be a sequence of continuous linear functionals defined on a complete linear metric space E over F. Let $\lim |T_n(x)| < \infty$ for each $x \in E$. Then there exists a fixed number M and a closed sphere $S \subset E$ sue that Tn(x)| < M for all x E S and for all n>1.

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Since $||x|| = (\sum |x_k|^{n_k})^{\frac{1}{n_k}}$ is evidently a non-Archmedian norm in the sense that, it satisfies the stronger form of triangular inequality $||x + Y|| \le max\{||x||, ||y||\}$. With this as norm as in the Archimedian case, we can establish the following theorem.

Theorem 4.2:

(i) l(p, F) is non-Archimedian Banach space.

(ii) If $P_k > 1$ for all $k \in N$ so that $p_k^{-1} + q_k^{-1} = 1$ and $\sum a_k x_k$ converges for every $x \in l(p, F)$, then $\sum |a_k|^{q_k}$ is convergent.

Theorem 4.3:

 $A \in \left(l(p, F) : \chi(F)\right) \text{ if and only if}$ $\sup_{k} (n! |a_{nk}| q_{k})^{\frac{1}{n}} \to 0 \text{ as } n \to \infty, \qquad (1)$ where $P_{k} > 1$ for all $k \in N$ and $p_{k}^{-1} + q_{k}^{-1} = 1.$

Proof: Sufficiency: Let $x = (x_k) \in l(p, F)$ and (1) holds so that $\sum |x_k|^{p_k}$ converges, converging to L(say). Then by Holders inequality, we have

$$(n!|y_{n}|)^{\frac{1}{n}} = \left(n!\sum_{k}|a_{nk}x_{k}|\right)^{\frac{1}{n}}$$

$$\leq \left(n!\sum_{k}|a_{nk}|^{q_{k}}\right)^{\frac{1}{nq_{k}}} \left(n!\sum_{k}|x_{k}|^{p_{k}}\right)^{\frac{1}{np_{k}}}$$

$$\leq \sup_{k} \left(n!\sum_{k}|a_{nk}|^{q_{k}}\right)^{\frac{1}{nq_{k}}} (n!L)^{\frac{1}{np_{k}}}$$

$$\leq \sup_{k} \left(n!\sum_{k}|a_{nk}|^{q_{k}}\right)^{\frac{1}{n}} (n!L)^{\frac{1}{n}}$$

Hence, by using (1) we get $(n!|y_{nk}|^{q_1})^{\frac{1}{n}} \to 0$ as $n \to \infty$ so that $y = (Y_n) \in \chi(F)$.

Necessity: We now suppose that $A \in (l(p, F) : \chi(F))$. If condition (1) does not hold, then for some E > 0, there exists subsequences of n, such that

$$\sup_{k} \left(n! \sum \left| a_{nk} \right|^{q_{k}} \right)^{\frac{1}{n}} > \varepsilon$$
⁽²⁾

for sufficiently large n.

Since $y_n = \sum a_{nk} x_k$ is defined for all $x = (x_k) \in l(p, F)$, from Theorem 2.2 (ii) (above), $\sum |a_{nk}|^{q_k}$ is convergent, so that we have $|a_{nk}|^{q_k} \to 0$ as $k \to \infty$ for every fixed n.

Hence we have

$$\lim_{k} (n! |a_{nk}|^{q_k})^{\frac{1}{n}} = 0 \quad \text{for ever y fixed n.}$$
(3)

Taking
$$x = (e_k) \in l(p, F)$$
 in $Y_n = \sum a_{nk} x_k$ to get $Y_n = (a_{nk}) \in \chi(F)$,

This gives

$$(n!|a_{nk}|^{q_k})^{\frac{1}{n}} < \frac{\varepsilon}{2} \text{ for } n > n_k \text{ and every fixed } k.$$
(4)

Now, we shall construct a sequence $x \in l(p, F)$ and prove that the corresponding $y = (y_n) \notin \chi(F)$ using (2), (3) and (4). Then that will suffice to prove the necessity of the condition (1).

By (2), first choose n_1 for n such that

$$\sup_{k} \left(n! \left| a_{n_{1}k} \right|^{q_{k}} \right)^{\frac{1}{n_{1}}} > \varepsilon$$
(5)

Having fixed an n_1 , by (3) we can choose ak_{n_1} for k such that

$$\sup_{k_{m}+1 \le k < \infty} \left(n_{1} ! | a_{n_{1}k} |^{q_{k}} \right)^{\frac{1}{n_{1}}} < \frac{\varepsilon}{2}$$
(6)

Hence, from (5) and (6) we get

$$\sup_{1 \le k < k_{n_{1}}} \left(n! |a_{n_{1}k}|^{q_{k}} \right)^{\frac{1}{n_{1}}} > \varepsilon$$

Therefore, there is a k_1 , $1 \le k_1 \le k_{n_1}$ such that

$$\left(n! \left| a_{n_{k_{1}}} \right|^{q_{k}} \right)^{\frac{1}{n_{1}}} > \varepsilon$$

$$\tag{7}$$

Next by (2) and (4) choose $n_2 > n_1$ such that

$$\sup_{1 \le k < \infty} \left(n_2 ! \left| a_{n_2 k} \right|^{q_k} \right)^{\frac{1}{n_2}} > \varepsilon$$
(8)

And

$$\sup_{1 \le k < k_m} \left(n_2 ! |a_{n_2 k}|^{q_k} \right)^{\frac{1}{n_2}} > \frac{\varepsilon}{2} \,. \tag{9}$$

This is possible if n_2 is large enough that $n_2 > max(n_k)$ such that

$$\sup_{k_{n_2}+1\le k< k_n} \left(n_2 \, | \, | \, a_{n_2 k} \, |^{q_k} \, \right)^{\frac{1}{n_2}} < \frac{\varepsilon}{2} \tag{10}$$

Now from (8) and (10) we get

$$\sup_{1 \le k < k_{n_2}} \left(n_2 ! |a_{n_2 k}|^{q_k} \right)^{\frac{1}{n_2}} > \varepsilon$$

Therefore, there exists a $k_2>k_1$ in $1\leq k\leq k_{n_2}$, that is in $k_{n_1+1}\leq k\leq k_{n_2}$ such that

$$n_{2}! |a_{n_{2}k_{2}}|^{q_{k}} |^{\frac{1}{n_{2}}} > \varepsilon$$
(11)

Proceeding like this, by (2), (3) and (4) we can find $n_m > n_{m-1}$ and km $> k_{m-1}$ in $1 \le k \le k_{n_m}$ such that

$$\sup_{1 \le k < k_{n_{m-1}}} \left(n_m ! | a_{n_m k} |^{q_k} \right)^{\frac{1}{n_m}} < \frac{\varepsilon}{2}.$$
 (12)

$$\sup_{k_{n_m+1} \le k < \infty} \left(n_m ! | a_{n_m k} |^{q_k} \right)^{\frac{1}{n_m}} < \frac{\varepsilon}{2}.$$
(13)

$$\left(n_{m}! \left| a_{n_{m}k_{m}} \right|^{q_{k}} \right)^{\frac{1}{n_{m}}} > \varepsilon.$$

$$(14)$$

$$\mathbf{X}_{k} = \begin{cases} |a_{nk}|^{q_{k}-1}, & \text{if } k = k_{1}, k_{2}, \dots \\ 0, & \text{if } k \neq k_{1}, k_{2}, \end{cases}$$
(15)

so that $x \in l(p, F)$, then

$$|n_{1}!y_{n_{1}}| = \left|n_{1}!\sum_{1}^{k_{n_{1}}}a_{n_{1}k}x_{k} + n_{1}!\sum_{k_{n_{1}}+1}^{\infty}a_{n_{1}k}x_{k}\right|$$

Gives

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$$\left| n_{1}! \sum_{1}^{k_{n_{1}}} a_{n_{1}k} x_{k} \right| = \left| n_{1}! y_{n_{1}} - n_{1}! \sum_{k_{n_{1}}+1}^{\infty} a_{n_{1}k} x_{k} \right|$$

$$\leq max \left\{ \left| n_{1}! \right| y_{n_{1}} \right|, \left| n_{1}! \sum_{k_{n_{1}}+1}^{\infty} a_{n_{1}k} x_{k} \right| \right\}.$$
(16)

Now, by using (15) and (7) we have

$$\left| n_{1}! \sum_{1}^{k_{n_{1}}} a_{n_{k}} x_{k} \right| = n_{1}! \left| a_{n_{k_{1}}} \right| \left| x_{k_{1}} \right| = n_{1}! \left| a_{n_{k_{1}}} \right|^{q_{k}} < \varepsilon^{n_{1}}$$

$$(17)$$

$$\begin{vmatrix}
n_{1}! \sum_{k_{n_{1}}+1}^{\infty} a_{n_{l}k} x_{k} \\
\leq \sup_{k_{n_{1}}+1 \leq k < \infty} \left(n_{1}! |a_{n_{l}k}|^{a_{k}} \right) \\
\leq \left(\sum_{k_{n_{1}}+1 \leq k < \infty} \left(n_{1}! |a_{n_{l}k}|^{a_{k}} \right), \text{ by (15)} \\
< \left(\frac{\varepsilon}{2} \right)^{n_{1}}, by (6).
\end{cases}$$
(18)

Using (17) and (18) in (16), we have

$$\varepsilon^{n_1} < \max\left\{n_1 \mid |y_{n_1}|, \left(\frac{\varepsilon}{2}\right)^{n_1}\right\}.$$

Then

$$y_{n_2} = \sum_{1}^{k_n} a_{n_2k} x_k + \sum_{k_n+1}^{k_{n_2}} a_{n_2k} x_k + \sum_{k_{n_2}+1}^{\infty} a_{n_2k} x_k$$
Gives

$$\left| n_{2}! \sum_{k_{n_{1}}+1}^{k_{n_{2}}} a_{n_{2}k} x_{k} \right| \leq \max \left\{ n_{2}! |y_{n_{2}}|, \left| n_{2}! \sum_{k_{n_{1}}+1}^{\infty} a_{n_{2}k} x_{k} \right|, \left| n_{2}! \sum_{k_{n_{2}}+1}^{\infty} a_{n_{2}k} x_{k} \right| \right\}.$$
(19)

Now, by using (15) and (11) we have

$$\left| n_{2}! \sum_{1}^{k_{n_{2}}} a_{n_{2}k} x_{k} \right| = n_{2}! \left| a_{n_{2}k_{2}} \right| \left| x_{k_{2}} \right| = n_{2}! \left| a_{n_{2}k_{2}} \right|^{q_{k}} < \varepsilon^{n_{2}}.$$
(20)

Thus,

$$\begin{vmatrix} n_{2} ! \sum_{k=1}^{k_{n_{2}}} a_{n_{2}k} x_{k} \end{vmatrix} = n_{2} ! |a_{n_{2}k_{2}}| |x_{k}|, \text{ by (15)} = n_{2} ! |a_{n_{2}k_{2}}|^{q_{k}} < \left(\frac{\varepsilon}{2}\right)^{n_{2}}, \text{ by (11)} \left| n_{2} ! \sum_{k_{n_{2}}+1}^{\infty} a_{n_{2}k} x_{k} \right| \le \sup_{k_{2}+1 \le k < \infty} \left(n_{2} ! |a_{n_{2}k}|^{q_{k}} \right), \text{ by (15)} < \left(\frac{\varepsilon}{2}\right)^{n_{2}}, \text{ by (10)}$$
(22)

Now, by using (20), (21) and (22) in (19) we get

$$\varepsilon^{n_2} < \max\left\{n_2! \mid y_{n_2} \mid \left(\frac{\varepsilon}{2}\right)^{n_2}, \left(\frac{\varepsilon}{2}\right)^{\frac{n}{2}}\right\}.$$

Hence, $n_2! |y_{n_2}| > \varepsilon^{n_2}$ so that $(n_2! |y_{n_2}|)^{\frac{1}{n_2}} > \varepsilon$.

Continuing in this way using (15) and the inequalities (12), (13) and (14), we can show that

 $(n_m ! | y_{n_m} |)^{\frac{1}{n_m}} > \varepsilon.$ So that $(n_m ! | y_{n_m} |)^{\frac{1}{n_m}}$ does not tend to zero as $n_m \to \infty$.

Hence, $(y_n) \notin \chi(F)$, which gives a contradiction so that (1) is

necessary. Hence the proof is complete.

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