Journal of Physical Mathematics
ISSN: 2090-0902

Soltani, J Phys Math 2014, 5:1
DOI: 10.4172/2090-0902.1000127

Research Article

Open Access

L^P Donoho-Stark Uncertainty Principles for the Dunkl Transform on $\mathbb{R}^d$

Fethi Soltani*

Department of Mathematics, Faculty of Science, Jazan University, Saudi Arabia

Abstract

In the Dunkl setting, we establish three continuous uncertainty principles of concentration type, where the sets of concentration are not intervals. The first and the second uncertainty principles are L^p versions and depend on the sets of concentration T and W, and on the time function f. The time-limiting operators and the Dunkl integral operators play an important role to prove the main results presented in this paper. However, the third uncertainty principle is also L^p version depends on the sets of concentration and he is independent on the band limited function f. These uncertainty principles generalize the results obtained for the Fourier transform and the Dunkl transform in the case p=2.

Keywords: Dunkl transform; Dunkl integral operators; Concentration uncertainty principles

Introduction

According to the classical uncertainty principle a function *f(t)* is essentially zero outside an interval of length Δt and its Fourier transform $\hat{f}(w)$ is essentially zero outside an interval of length Δw, then $\Delta t\ \Delta w \geq 1$; a function and its Fourier transform cannot both be highly concentrated [1,2]. The uncertainty principle is widely known for its"philosophical" applications: in quantum mechanics, it shows that a particle's position and momentum cannot be determined simultaneously [3]; in signal processing, it establishes limits on the extent to which the "instantaneous frequency" of a signal can be measured [4]. However, it has also technical applications, such as in the theory of partial differential equations [5,6]. In this paper, we consider $\mathbb{R}^d$ with the Euclidean inner product $\langle .,. \rangle$ and norm $|y| := \sqrt{\langle y,y\rangle}$. For $\alpha \in \mathbb{R}^d \setminus \{0\}$, let σ_α be the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$ orthogonal to α:

$$\sigma_\alpha y := y - \frac{2\langle \alpha, y\rangle}{|\alpha|^2}\alpha$$

A finite set $\Re \subset \mathbb{R}^d \setminus \{0\}$ is called a root system, if $\Re \cap \mathbb{R}.\alpha = \{-\alpha, \alpha\}$ and $\sigma_\alpha \Re = \Re$, for all $\alpha \in \Re$. We assume that it is normalized by $|\alpha|^2 = 2$, for all $\alpha \in \Re$. For a root system $\Re$, the reflections $\sigma_\alpha, \alpha \in \Re$, generate a finite group $G \subset O(d)$, the reflection group associated with $\Re$. All reflections in G correspond to suitable pairs of roots. For a given $\beta \in \mathbb{R}^d \setminus \bigcup_{\alpha\in\Re} H_\alpha$, we fix the positive subsystem $\Re_+ := \{\alpha \in \Re : \langle \alpha, \beta\rangle > 0\}$. Then for each $\alpha \in \Re$ either $\alpha \in \Re_+ or -\alpha \in \Re_+$. Let k: $\Re \to C$ be a multiplicity function on $\Re$ (a function which is constant on the orbits under the action of G). As an abbreviation, we introduce the index

$$\gamma = \gamma_k := \sum_{\alpha\in\Re_+} k(\alpha)$$

Throughout this paper, we will assume that the multiplicity is nonnegative, that is, $k(\alpha) \geq 0$, for all $\alpha \in \Re$. Moreover, let w_k denote the weight function

$$w_k(y) := \prod_{\alpha\in\Re_+} |\langle \alpha, y\rangle|^{2k(\alpha)}, y \in \mathbb{R}^d$$

which is G-invariant and homogeneous of degree 2γ. Let c_k be the Mehta-type constant given by

$$c_k := \left(\int_{\mathbb{R}^d} e^{-|y|^2/2} w_k(y)\,dy\right)^{-1}$$

We denote by μ_k the measure on $\mathbb{R}^d$ given by $d\mu_k(y) := c_k w_k(y)\,dy$; and by $L_k^p, 1 \leq p \leq \infty$, the space of measurable functions f on $\mathbb{R}^d$, such that

$$\|f\|_{L_k^p} := \left(\int_{\mathbb{R}^d} |f(y)|^p\, d\mu_k(y)\right)^{1/p} < \infty,\ \ 1 \leq p < \infty$$

$$\|f\|_{L_k^\infty} := ess \sup_{y\in\mathbb{R}^d} |f(y) < \infty|$$

For $f \in L_k^1$ the Dunkl transform is defined [6] by

$$F_k(f)(x) := \int_{\mathbb{R}^d} E_k(-ix, y) f(y)\, d\mu_k(y), x \in \mathbb{R}^d,$$

where $E_k(-ix, y)$ denotes the Dunkl kernel. (For more details see the next section). Many uncertainty principles have already been proved for the Dunkl transform, namely by Rösler [7] and Shimeno [8] who established (by two different methods) the Heisenberg-Pauli-Weyl inequality. Kawazoe and Mejjaoli gave some related versions of the uncertainty principle (Cowling-Price's theorem, Miyachi's theorem, Beurling's theorem and Donoho-Stark's theorem). Recently, the author [9,10] proved a general forms of the Heisenberg-Pauli-Weyl inequality and he also established a logarithmic uncertainty principle [11].

Let T and W be a measurable subsets of $\mathbb{R}^d$. We say that a function $f \in L_k^p, 1 \leq p \leq 2$, is ε -concentrated to T in L_k^p, is concentrated to T in L_k^p -norm, if there is a measurable function $g(t)$ vanishing outside T such that $\|f - g\|_{L_k^p} \leq \varepsilon \|f\|_{L_k^p}$. Similarly, we say that $F_k(f)$ is ε -concentrated to W in L_k^p -norm, $q = p/(p-1)$, if there is a function $h(w)$ vanishing outside W with $\|F_k(f) - h\|_{L_k^q} \leq \varepsilon \|F(f)\|_{L_k^q}$.

Based on the ideas of Donoho and Stark , we show a continuous-time uncertainty principle of concentration type for the L_k^p theory: If f is ε_T -concentrated to T in L_k^p norm, $1 < p \leq 2$, and $F_k(f)$ is ε_W-concentrated to W in L_k^q norm, $q = p/(p-1)$, then

$$\|F_k(f)\|_{L_k^q} \leq \frac{(\mu_k(T))^{1/q}(\mu_k(W))^{1/q} + \varepsilon_T}{1 - \varepsilon_w}\|f\|_{L_k^p}$$

***Corresponding author:** Fethi Soltani, Department of Mathematics, Faculty of Science, Jazan University, P.O. Box 114, Jazan, Kingdom of Saudi Arabia, Tel: 073230029; E-mail: fethisoltani10@yahoo.com

Received January 03, 2014; **Accepted** November 20, 2014; **Published** November 24, 2014

Citation: Soltani F (2014) L^P Donoho-Stark Uncertainty Principles for the Dunkl Transform on $\mathbb{R}^d$. J Phys Math 5: 127 doi: 10.41722090-0902.1000127

Citation: Soltani F (2014) L^p Donoho-Stark Uncertainty Principles for the Dunkl Transform on $\mathbb{R}^d$. J Phys Math 5: 127 doi: 10.41722090-0902.1000127

Next, we prove another version of continuous-time uncertainty principle of concentration type for the $L_k^1 \cap L_k^p$ theory: If $f \in L_k^1 \cap L_k^p, 1 < p \le 2$, is ε_T -concentrated to T in L_k^1 -norm and $F_k(f)$ is ε_W -concentrated to W *in* L_k^q -norm, *q=p/(p-1)*, then

$$\|F_k(f)\|_{L_k^q} \le \frac{(\mu_k(T))^{1/p}(\mu_k(W))^{1/q}}{(1-\varepsilon_T)(1-\varepsilon_w)}\|f\|_{L_k^p}$$

Let $B_k^p(W)$, 1 ≤ p ≤ 2, be the set of functions $g \in L_k^p$ that are bandlimited to W (i.e. $g \in B_k^p(W)$ implies $S_W g=g$). Here S_W is the Dunkl integral operator given by

$$F_k(S_W f) = F_k(f)1_W$$

where 1_W is the indicator function of the set W. We say that f is ε-bandlimited to W in L_k^p norm if there is a $g \in B_k^p(W)$ with

$$\|f-g\|_{L_k^p} \le \varepsilon\|f\|_{L_k^p}$$

The space $B_k^p(W)$ leads to establish the following version of continuous-bandlimited uncertainty principle for L_k^p theory: If f is ε_T -concentrated to T and ε_W -bandlimited to W in L_k^p norm, 1 ≤ p ≤ 2, then

$$\frac{1-\varepsilon_T-\varepsilon_W}{1+\varepsilon_W} \le (\mu_k(T))^{1/p}(\mu_k(W))^{1/p}$$

This paper is organized as follows. The Section 2 is devoted to recalling some basic properties of the Dunkl transform F_k: Plancherel theorem, inversion formula and Hausdorff-Young inequality, which are tools to prove the main results presented in this paper. In Section 3, we introduce some properties of the time-limiting operators and the Dunkl integral operators. These operators play an important role to establish the concentration uncertainty principles in the next sections. In Section 4, we present two continuous-time uncertainty principles of concentration type. These principles depend on the sets of concentration T and W, and on the time function f. In the last section, we establish continuous-bandlimited uncertainty principle of concentration. This principle depends also on the sets of concentration T and W, but he is independent on the bandlimited function f.

The Dunkl transform on $\mathbb{R}^d$

The Dunkl operators D_j; j=1,..., d, on $\mathbb{R}^d$ associated with the finite reflection group G and multiplicity function k are given, for a function f of class C^1 on $\mathbb{R}^d$, by

$$D_j f(y) := \frac{\partial}{\partial y_j} f(y) + \sum_{\alpha\in\Re_+} k(\alpha)\alpha_j \frac{f(y)-f(\sigma_\alpha y)}{\langle \alpha, y\rangle}$$

For $y \in \mathbb{R}^d$, the initial problem $D_j u(.,y)(x) = y_j u(x,y), j=1,...,d$, with μ(0, y) = 1 admits a unique analytic solution on $\mathbb{R}^d$, which will be denoted by $E_h(x, y)$ and called Dunkl kernel [12,13]. This kernel has a unique analytic extension to $\mathbb{C}^d \times \mathbb{C}^d$.

The Dunkl kernel has the Laplace-type representation [14]

$$E_k(x,y) = \int_{\mathbb{R}^d} e^{\langle y,z\rangle} d\Gamma_x(z), x \in \mathbb{R}^d, y \in \mathbb{C}^d$$

where $\langle y,z\rangle := \sum_{i=1}^d y_i z_i$ *and* Γ_x is a probability measure on $\mathbb{R}^d$ such that

$$supp(\Gamma_x) \subset \{z \in \mathbb{R}^d : |z| \le |x|\}.$$

In our case,

$$|E_k(ix,y)| \le 1, x, y \in \mathbb{R}^d. \tag{2.1}$$

The Dunkl kernel gives rise to an integral transform, which is called Dunkl transform on $\mathbb{R}^d$, and was introduced by Dunkl in, where already many basic properties are established. Dunkl's results have been completed and extended later by De Jeu. The Dunkl transform of a function f in L_k^1, is defined by

$$F_k(f)(x) := \int_{\mathbb{R}^d} E_k(-ix,y) f(y) d\mu_k(y)$$

We notice that F_0 agrees with the Fourier transform F, that is given by

$$F_k(f)(x) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i\langle x,y\rangle} f(y) dy, x \in \mathbb{R}^d$$

Some of the properties of Dunkl transform F_k are collected bellow.

(a) L L^∞ ***-boundedness*:** For all $f \in L_k^1, F_k(f) \in L_k^\infty$ and

$$\|F_k(f)\|_{L_k^\infty} \le \|f\|_{L_k^1} \tag{2.2}$$

(b) ***Inversion theorem*:** Let $f \in L_k^1$, such that $F_k(f) \in L_k^1$. Then

$$f(x) = F_k(F_k(f))(-x), a.e. x \in \mathbb{R}^d \tag{2.3}$$

(c) ***Plancherel theorem*:** The Dunkl transform F_k extends uniquely to an isometric isomorphism of L_k^2 onto itself. In particular,

$$\|f\|_{L_k^2} = \|F_k(f)\|_{L_k^2} \tag{2.4}$$

(d) ***Hausdorff-Young inequality*:** Using relations (2.2) and (2.4) with Marcinkiewicz's interpolation theorem [15,16], we deduce that for every 1 ≤ p ≤ 2, and for every $f \in L_k^p$ the function $F_k(f)$ belongs to the space L_k^q *,q=p/(p-1), and* $\|F_k(f)\|_{L_k^q} \le \|f\|_{L_k^p}$ (2.5)

The Dunkl integral operators

Let T and W be a measurable subsets of $\mathbb{R}$. We introduce the time-limiting operator P_T [1] by

$$P_t f := f 1_T \tag{3.1}$$

And, we introduce the Dunkl integral operator S_W by

$$F_k(S_W f) = F_k(f)1_W \tag{3.2}$$

In the case k=0, the operator S_W is the frequency-limiting operator given in [1].

Theorem 3.1: If $\mu_k(W) < \infty$ *and* $f \in L_k^p, 1 \le p \le 2$,

$$S_W f(x) = \int_W E_k(ix,y) F_k(f)(y) d\mu_k(y)$$

Proof. Let f $\in L_k^p$, 1 ≤ p ≤ 2 and let q=p/(p−1). Then by (2.1), H¨older's inequality and (2.5),

$$\|F_k(f)1_W\|_{L_k^1} = \int_W |F_k(f)(w)| d\mu_k(w)$$

$$\le (\mu_k(W))^{1/p}\|F_k(f)\|_{L_k^q}$$

$$\le (\mu_k(W))^{1/p}\|f\|_{L_k^q}$$

And

$$\|F_k(f)1_W\|_{L_k^2} = \left(\int_W |F_k(f)(w)|^2 d\mu_k(w)\right)^{1/2}$$

$$\le (\mu_k(W))^{\frac{q-2}{2q}}\|F_k(f)\|_{L_k^q} \le (\mu_k(W))^{\frac{q-2}{2q}}\|f\|_{L_k^p}$$

Thus $F_k(f)1_W \in L_k^1 \cap L_k^2$ and by (3.2)

$$S_W f = F_k^{-1}(F_k(f)1_W)$$

This combined with (2.3) gives the result.

Lemma 3.2: If 1 ≤ p ≤ 2, q=p/(p−1) and f ∈ L_k^p, then

$$\left\|F_k\left(S_W f\right)\right\|_{L_k^q} \le \|f\|_{L_k^p}$$

Proof: Let f∈ L_k^p, 1 ≤ p ≤ 2 and let q=p/(p−1). From (2.5) and (3.2),

$$\left\|F_k\left(S_W f\right)\right\|_{L_k^q} = \left(\int_W \left|F_k(f)(w)\right|^q d\mu_k(w)\right)^{1/q} \le \left\|F_k(f)\right\|_{L_k^q} \le \|f\|_{L_k^p}$$

This yields the desired result.

Lemma 3.3: Let T and W be measurable subsets of $\mathbb{R}^d$. If 1 < p ≤ 2, q = p/(p−1) and f∈ L_k^p, then

$$\left\|F_k\left(S_W P_T f\right)\right\| \le \left(\mu_k(T)\right)^{1/q}\left(\mu_k(W)\right)^{1/q}\|f\| .$$

Proof: Assume that $\mu_k(T) < \infty$ and $\mu_k(W) < \infty$.

Let f∈ L_k^p, 1 < p ≤ 2 and let q=p/(p−1). From (3.2),

$$\left\|F_k\left(S_W P_T f\right)\right\| = 1_W F_k\left(P_T f\right)$$

Thus

$$\left\|F_k\left(S_W P_T f\right)\right\|_{L_k^q} = \left(\int_W \left|F_k\left(P_T f\right)(w)\right|^q d\mu_k(w)\right)^{1/q} \tag{3.3}$$

So

$$F_k\left(P_T f\right)(w) = \int_T E_k(-iw,t) f(t) d\mu_k(t)$$

and by Holder's inequality and (2.1),

$$\left|F_k\left(P_T f\right)(w)\right| \le \left(\int_T \left|E_k(-iw,t)\right|^q d\mu_k(t)\right)^{1/q}\left(\int_T |f(t)|^P d\mu_k(t)^{1/p}\right)$$

$$\le \left(\mu_k(T)\right)^{1/q}\|f\|_{L_k^p}$$

Then by (3.3),

$$\left\|F_k\left(S_W P_T f\right)\right\|_{L_k^q} \le \left(\mu_k(T)\right)^{1/q}\left(\mu_k(W)\right)^{1/q}\|f\|_{L_k^p}$$

Thus, the proof is complete.

Concentration uncertainty principle

Let T and W be measurable subsets of $\mathbb{R}^d$. We say that a function f $\in L_k^p$, 1 ≤ p ≤ 2, is ε -concentrated to T in L_k^p-norm, if there is a measurable function g(t) vanishing outside T such that $\|f-g\|_{L_k^p} \le \varepsilon\|f\|_{L_k^p}$. Similarly, we say that $F_k(f)$ is ε-concentrated to W in L_k^q-norm, q=p/(p−1), if there is a function h(w) vanishing outside W with $\left\|F_k(f)-h\right\|_{L_k^q} \le \varepsilon\left\|F_k(f)\right\|_{L_k^q}$.

If f is ε_T -concentrated to T in L_k^p-norm (g being the vanishing function) then by (3.1),

$$\left\|f-P_T f\right\|_{L_k^p} = \left(\int_{\mathbb{R}^d \setminus T} |f(t)|^p d\mu_k(t)\right)^{1/p} \le \|f-g\|_{L_k^p} \le \varepsilon_T\|f\|_{L_k^p} \tag{4.1}$$

and therefore f is ε_T -concentrated to T in L_k^p-norm if and only if $\left\|f-P_T f\right\|_{L_k^p} \le \varepsilon_T\|f\|_{L_k^p}$.

From (3.2) it follows as for P_T that $F_k(f)$ is ε_W-concentrated to *W* in L_k^q-norm, q=p/(p − 1), if and only if

$$\left\|F_k(f)-F_k\left(S_W f\right)\right\|_{L_k^q} \le \varepsilon_W\left\|F_k(f)\right\|_{L_k^q} \tag{4.2}$$

The following theorem, states the first continuous-time uncertainty principle of concentration type for the theory.

Theorem 4.1: Let T and W be a measurable subsets of $\mathbb{R}^d$ and f∈ L_k^p, 1 < p ≤ 2. If f is ε_T -concentrated to T in L_k^p-norm and $F_k(f)$ is ε_W-concentrated to W in L_k^q-norm, q=p/(p−1), then

$$\left\|F_k(f)\right\|_{L_k^q} \le \frac{\left(\mu_k(T)\right)^{1/q}\left(\mu_k(W)\right)^{1/q}+\varepsilon_T}{1-\varepsilon_W}\|f\|_{L_k^p}.$$

Proof: Let f∈ L_k^p, 1 < p ≤ 2 and let q=p/(p−1). From (4.1), (4.2) and Lemma 3.2 it follows that

$$\left\|F_k(f)-F_k\left(S_W P_T f\right)\right\|_{L_k^q} \le \left\|F_k(f)-F_k\left(S_W f\right)\right\|_{L_k^q}$$

$$+\left\|F_k\left(S_W f\right)-F_k\left(S_W P_T f\right)\right\|_{L_k^q}$$

$$\le \varepsilon_W\left\|F_k(f)\right\|_{L_k^q}+\left\|f-P_T f\right\|_{L_k^p}$$

$$\le \varepsilon_W\left\|F_k(f)\right\|_{L_k^q}+\varepsilon_T\|f\|_{L_k^p}$$

The triangle inequality and the Lemma 3.3 show that

$$\left\|F_k(f)\right\|_{L_k^q} \le \left\|F_k\left(S_W P_T f\right)\right\|_{L_k^q}+\left\|F_k(f)-F_k\left(S_W P_T f\right)\right\|_{L_k^q}$$

$$\le\left[\left(\mu_k(T)\right)^{1/q}\left(\mu_k(W)\right)^{1/q}+\varepsilon_T\right]\|f\|_{L_k^p}+\varepsilon_W\left\|F_k(f)\right\|_{L_k^q}$$

which gives the desired result.

Next, the second continuous-time uncertainty principle of concentration type for the $L_k^1 \cap L_k^p$ theory is given by the following theorem.

Theorem 4.2: Let T and W be a measurable subsets of $\mathbb{R}^d$ and $f \in L_k^1 \cap L_k^p, 1 < p \le 2.$. If f is ε_T -concentrated to T in L_k^1-norm and $F_k(f)$ is ε_W-concentrated to W in L_k^q-norm, q=p/(p−1), then

$$\left\|F_k(f)\right\|_{L_k^q} \le \frac{\left(\mu_k(T)\right)^{1/p}\left(\mu_k(W)\right)^{1/q}}{\left(1-\varepsilon_T\right)\left(1-\varepsilon_W\right)}\|f\|_{L_k^p}$$

Proof: Assume that $\mu_k(T) < \infty$ and $\mu_k(W) < \infty$.

Let $f \in L_k^1 \cap L_k^p, 1 < p \le 2.$ Since $F_k(f)$ is ε_W -concentrated to W in L_k^q-norm, q=p/(p−1), then

$$\left\|F_k(f)\right\|_{L_k^q} \le \varepsilon_W\left\|F_k(f)\right\|_{L_k^q}+\left(\int_W \left|F_k(f)(w)\right|^q d\mu_k(w)\right)^{1/q}$$

$$\le \varepsilon_W\left\|F_k(f)\right\|_{L_k^q}+\left(\mu_k(W)\right)^{1/q}\left\|F_k(f)\right\|_{L_k^\infty}$$

Thus by (2.2),

$$\left\|F_k(f)\right\|_{L_k^q} \le \frac{\left(\mu_k(W)\right)^{1/q}}{1-\varepsilon_W}\|f\|_{L_k^1} \tag{4.3}$$

On the other hand, since f is ε_T -concentrated to T in L_k^1-norm,

$$\|f\|_{L_k^1} \le \varepsilon_T\|f\|_{L_k^1}+\int_T |f(t)| d\mu_k(t)$$

$$\le \varepsilon_T\|f\|_{L_k^1}+\left(\mu_k(T)\right)^{1/p}\|f\|_{L_k^p}$$

Thus

$$\|f\|_{L_k^1} \le \frac{\left(\mu_k(T)\right)^{1/p}}{1-\varepsilon_T}\|f\|_{L_k^p} \tag{4.4}$$

Combining (4.3) and (4.4) we obtain the result of this theorem.

Conclusion 4.3: The first uncertainty principle (Theorem 4.1) depends on the time function f. However, for p=q=2, we obtain

$1-\varepsilon_T-\varepsilon_W \le (\mu_k(T))^{1/2}(\mu_k(W))^{1/2}$ and the inequality is independent on the time function f. Also, the second uncertainty principle (Theorem 4.2) depends on the time function f. In a particular case when p=q=2, we obtain $(1-\varepsilon_T)(1-\varepsilon_W) \le (\mu_k(T))^{1/2}(\mu_k(W))^{1/2}$ and the inequality is independent on the time function f.

These uncertainty principles generalize the results obtained for the Fourier transform and the Dunkl transform in the case p=q=2.

Another uncertainty principle

Let $B_k^p(W), 1 \le p \le 2$, be the set of functions $g \in L_k^p$ that are bandlimited to W (i.e. $g \in B_k^p(W)$ implies $S_W g = g$).

We say that f is ε-bandlimited to W in L_k^p-norm if there is a $g \in B_k^p(W)$ with $\|f-g\|_{L_k^p} \le \varepsilon \|f\|_{L_k^p}$

Then, the space $B_k^p(W)$ satisfies the following property.

Lemma 5.1. Let T and W be a measurable subsets of $\mathbb{R}^d$. For $g \in B_k^p(W)$, $1 \le p \le 2$,

$$\|P_T g\|_{L_k^p} \le (\mu_k(T))^{1/p} (\mu_k(E))^{1/p} \|g\|_{L_k^p}$$

Proof. If $\mu_k(T)=\infty$ or $\mu_k(W)=\infty$, the inequality is clear.

Assume that $\mu_k(T)<\infty$ and $\mu_k(W)<\infty$.

For $g \in B_k^p(W), 1 \le p \le 2$, from Theorem 3.1,

$$g(t) = \int_W E_k(iw,t) F_k(g)(w)\, d\mu_k(w)$$

and by (2.1) and H¨older's inequality,

$$g(t) \le (\mu_k(W))^{1/p} \|F_k(g)\|_{L_k^q} \le (\mu_k(w))^{1/p} \|g\|_{L_k^p},\ q = p/(p-1)$$

Hence,

$$\|P_T g\|_{L_k^p} = \left(\int_T |g(t)|^p\, d\mu_k(t)\right)^{1/p} \le (\mu_k(T))^{1/p} (\mu_k(W))^{1/p} \|g\|_{L_k^p}$$

which yields the result.

Theorem 5.2: Let T and W be a measurable subsets of $\mathbb{R}^d$ and f ∈ L_k^p, $1 \le p \le 2$. If f is ε_W-bandlimited to W in L_k^p-norm, then

$$\|P_T g\|_{L_k^p} \le \left[(1+\varepsilon_W)(\mu_k(T))^{1/p}(\mu_k(W))^{1/p} + \varepsilon_W\right]\|f\|_{L_k^p}$$

Proof: Let f ∈ L_k^p, $1 \le p \le 2$. Since f is ε_W -bandlimited in L_k^p-norm, by definition there is a g in $B_k^p(W)$ with $\|f-g\|_{L_k^p} \le \varepsilon_W \|f\|_{L_k^p}$. For this g, we have

$$\|P_T f\|_{L_k^p} \le \|P_T g\|_{L_k^p} + \|P_T(f-g)\|_{L_k^p} \le \|P_T g\|_{L_k^p} + \varepsilon_W \|f\|_{L_k^p}.$$

Then by Lemma 5.1 and the fact that $\|g\|_{L_k^p} \le (1+\varepsilon_W)\|f\|_{L_k^p}$ we get the result.

Next, the third continuous bandlimited uncertainty principle of concentration type for the L_k^p-norm is given by the following.

Corollary 5.3: Let T and W be measurable subsets of $\mathbb{R}^d$ and f ∈ L_k^p, $1 \le p \le 2$. If f is ε_T-concentrated to T and ε_W-bandlimited to W in L_k^p-norm, then

$$\frac{1-\varepsilon_T-\varepsilon_W}{1+\varepsilon_W} \le (\mu_k(T))^{1/p}(\mu_k(W))^{1/p}$$

Proof: Let f ∈ L_k^p, $1 \le p \le 2$. Since f is ε_T -concentrated to T in L_k^p-norm then by (4.1),

$$\|f\|_{L_k^p} \le \varepsilon_T \|f\|_{L_k^p} + \|P_T f\|_{L_k^p}$$

Thus,

$$\|f\|_{L_k^p} \le \frac{1}{1-\varepsilon_T}\|P_T f\|_{L_k^p}$$

By (5.1) and Theorem 5.2 we deduce the desired inequality of Corollary 5.3.

Conclusion 5.4: The third uncertainty principle (Corollary 5.3) is independent on the bandlimited function f for every $1 \le p \le 2$. This uncertainty principle generalizes the result obtained in when p=q=2.

Acknowledgement

The author is very grateful to the Reviewers of the Journal for their important comments on this work.

References

1. Donoho DL, Stark PB (1989) Uncertainty principles and signal recovery. SIAM J Appl Math 49: 906-931.
2. Kawazoe T, Mejjaoli H (2010) Uncertainty principles for the Dunkl transform. Hiroshima Math J 40: 241-268.
3. Heisenberg W (1949) The physical principles of the quantum theory: Dover, The University of Chicago Press in1930, NewYork.
4. Gabor D (1946) Theory of communication. J Inst Elec Engrg 93: 429-457.
5. Fefferman CL (1983) The uncertainty principle. Bull Amer Math Soc 9: 129-206.
6. Dunkl CF (1992) Hankel transforms associated to finite reflection groups. Contemp Math 138: 123-138.
7. Rosler M (1999) An Uncertainty principle for the Dunkl transform. Bull Austral Math Soc 59: 353-360.
8. Shimeno N (2001) A note on the uncertainty principle for the Dunkl transform. J Math Sci Univ Tokyo 8: 33-42.
9. Soltani F (2013) Heisenberg-Pauli-Weyl uncertainty inequality for the Dunkl transform on Rd. Bull Austral Math Soc 87: 316-325.
10. Soltani F (2013) A general form of Heisenberg-Pauli-Weyl uncertainty inequality for the Dunkl transform. Int Trans Spec Funct 24: 401-409.
11. Soltani F (2014) Pitt's inequality and logarithmic uncertainty principle for the Dunkl transform on R. Acta Math Hungar 143: 480-490.
12. Dunkl DF (1991) Integral kernels with reflection group invariance. Canad J Math 43: 1213-1227.
13. De Jeu MFE (1993) The Dunkl transform. Invent Math 113: 147-162.
14. Rosler M (1999) Positivity of Dunkl's intertwining operator. Duke Math J 98: 445-463.
15. Stein EM (1956) Interpolation of linear operators. Trans Amer Math Soc 83: 482-492.
16. Stein EM, Weiss G (1971) Introduction to Fourier analysis on Euclidean spaces: Princeton Univ Press Princeton NJ.