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Log-concavity of the cohomology of nilpotent Lie algebras in characteristic two

Grant CAIRNS

Department of Mathematics, La Trobe University, Melbourne, VIC 3086, Australia
E-mail: g.cairns@latrobe.edu.au

Abstract

It is known that the Betti numbers of the Heisenberg Lie algebras are unimodal over fields of characteristic two. This note observes that they are log-concave. An example is given of a nilpotent Lie algebra in characteristic two for which the Betti numbers are unimodal but not log-concave.

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The Heisenberg Lie algebra of dimension $2m+1$ is the Lie algebra $\mathfrak{h}_m$ having the basis $\{x_1,\dots,x_m,y_1,\dots,y_m,z\}$ and nonzero relations $[x_i,y_i]=z$, $1\le i\le m$. For the cohomology with trivial coefficients, the Betti numbers $b_n=\dim H^n(\mathfrak{h}_m)$ have been explicitly computed in all characteristics [1, 3, 4]. Recall that the Betti numbers are *unimodal* if $b_i\le b_j$ for all $0\le i\le j\le m$ and $b_i\ge b_j$ for all $m\le i\le j\le 2m+1$, and they are *concave* (resp., *log-concave*) if b_i is at least as great as the arithmetic (resp., geometric) mean of the pair b_{i-1},b_{i+1} for all $1\le i\le 2m$. So concave implies log-concave which implies unimodal. In characteristic zero, unimodality is quite common. The Heisenberg Lie algebras play a key role in the construction of all known examples of Lie algebras in characteristic zero where the Betti numbers are not unimodal [2]. In fact, the Betti numbers of $\mathfrak{h}_m$ are unimodal only in characteristic two [1]. On the other hand, we know of no nilpotent Lie algebra in characteristic two whose Betti numbers fail to be unimodal. In [1], the question was posed: in characteristic two, do all nilpotent Lie algebras have unimodal Betti numbers? Since log-concavity is a common route taken to prove unimodality, it is natural to ask whether the Betti numbers of the Heisenberg algebras are unimodal in characteristic two. We record the following observation as a theorem, though it is really just a corollary of the works [1, 4].

Theorem 1. *Over fields of characteristic two, the Betti numbers of $\mathfrak{h}_m$ are log-concave; i.e., $b_n^2\ge b_{n-1}b_{n+1}$ for all n.*

Proof. For the rest of this note we fix the characteristic to be two. Emil Sköldberg showed that the Poincaré polynomial $S_m(t)=\sum_n b_nt^n$ is [4]

$$S_m(t)=\frac{\big(1+t^3\big)(1+t)^{2m}+\big(t+t^2\big)(2t)^m}{1+t^2} \tag{1}$$

Though we will not need them, we mention that the individual Betti numbers are given in [1]; for all $i\le m$,

$$b_n=\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^i\binom{2m}{n-2i}+\sum_{i=0}^{\lfloor\frac{n-3}{2}\rfloor}(-1)^i\binom{2m}{n-3-2i}$$

To establish the log-concavity, we observe that the Betti numbers of $\mathfrak{h}_{m+1}$ are essentially determined by those of $\mathfrak{h}_m$, with a curious correction for the middle two terms. Explicitly,

$$S_{m+1}(t)=(1+t)^2S_m(t)-2^m\big(t^{m+1}+t^{m+2}\big) \tag{2}$$

This relation is easily deduced from (1). Using induction, we assume that S_m is log-concave. Since $(1+t)^2$ is log-concave, $(1+t)^2 S_m(t)$ is thus also log-concave (see [5]). So in view of (2), to establish the log-concavity of S_{m+1}, it remains to verify it for the middle terms; that is, for $\mathfrak{h}_{m+1}$ we require that $b_{m+1}^2 \geq b_m b_{m+2}$. But by Poincaré duality, $b_{m+1} = b_{m+2}$, and so we only require $b_{m+1} \geq b_m$, and this is given by the unimodality of the Betti numbers, which was shown in [1]. This completes the proof. $\square$

The following example shows that, despite the above result, log-concavity is not a route for establishing unimodality in the general setting of nilpotent Lie algebras in characteristic two.

Example 2. Let $\mathfrak{g}$ denote the 7-dimensional Lie algebra with basis $x_1, \dots, x_7$ and defining relations:

$$\begin{aligned} \left[x_1, x_i\right] &= x_{i+1}, \quad i = 2, \dots, 6 \\ \left[x_2, x_i\right] &= x_{i+2}, \quad i = 3, 4 \\ \left[x_3, x_4\right] &= x_7 \end{aligned}$$

Clearly $\mathfrak{g}$ is nilpotent (and actually graded and filiform). Direct calculations using Mathematica show that in characteristic two, the Betti numbers are

b_0	b_1	b_2	b_3	b_4	b_5	b_6	b_7
1	2	3	6	6	3	2	1

As $b_2^2 < b_1 b_3$, the Betti numbers are not log-concave.

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