Local envelopes on CR manifolds

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Abstract

We study the problem whether CR functions on a sufficiently pseudoconcave CR manifold $M$ extend locally across a hypersurface of $M$. The sharpness of the main result will be discussed by way of a counter-example.


1 Introduction

It is a very classical fact that for a strictly (pseudo)convex real hypersurface $H$ in $\mathbb{C}^n$, $n \geq 2$, holomorphic functions extend from the concave side across $H$. In the present note we will study the corresponding question for CR functions on embedded CR manifolds from a strictly local point of view.

All manifolds will be assumed to be $C^\infty$-smooth. Recall that a submanifold $M$ of $\mathbb{C}^n$ is called CR manifold if the dimension of the complex tangent space $T_p^cM = T_pM \cap JT_pM$ does not depend on $p \in M$ ($J = J_p$ denoting multiplication by the complex unit of $T_p\mathbb{C}^n$). In this case, the complex tangent spaces form a bundle $T^cM \subset TM$, whose complex rank $m$ is called CR dimension of $M$, shortly $m = \text{CRdim}M$. A CR manifold $M \subset \mathbb{C}^n$ is called generic if its CR dimension is as small as real/complex linear algebra allows, i.e. if $m = n - \text{codim}M$. A $C^1$-function $f$ on $M$ is called CR function if $df|_{\pi^{-1}M}$ is $J$-linear. Locally one may express this by a system of $m$ independent linear first-order differential equations, allowing to interpret the CR property in distributional sense. The space of continuous CR distributions on $M$ will be denoted by $CR(M)$.

The nonintegrability of $T^cM$ is measured by the nonvanishing of the Levi form. For $X \in T_p^cM$ define the vector-valued Levi form by $\mathcal{L}(X) = [J\dot{X}, \dot{X}] \mod T_p^cM$, where $\dot{X}$ is an arbitrary smooth section of $T_p^cM$ extending $X$. It is easily verified that the expression is tensorial and yields a well defined mapping $\mathcal{L} : T_p^cM \to T_pM/T_p^cM$. Let $\Sigma_p = (T_p^cM)^\perp = \{\eta \in T_p^cM : \eta|_{T_p^cM} \equiv 0\}$ be the fiber of the characteristic bundle $\Sigma$ of $M$. For nonzero $\eta \in \Sigma_p$, we define the directional Levi form by $\mathcal{L}(\eta, X) = \langle \eta, \mathcal{L}(X) \rangle$. Now a generic CR manifold $M$ is called strictly/weakly $q$-concave if for every nonzero $\eta \in \Sigma$ the hermitian form $\mathcal{L}(\eta, \cdot)$ has at least $q$ negative/nonpositive eigenvalues. Finally $M$ is called strictly pseudoconvex if $\mathcal{L}(\eta, \cdot)$ is strictly definite for some nonzero $\eta \in \Sigma$ (see [9] for more on CR geometry).

Throughout we will work in the following setting: $M$ will denote a smooth generic CR manifold passing through the origin and $H$ a smooth real hypersurface of $\mathbb{C}^n$ intersecting $M$ transversally in the origin. Hence $H_M = H \cap M$ is a smooth hypersurface of $M$ near 0. For simplicity we will assume that the intersection is even $J$-generic, meaning that $T^c_0H$ and $T^0_0H$ are transverse, or equivalently, that $H_M$ is itself a generic CR manifold near 0. For a distinguished local side $H^+$ of $H$, we will consider subdomains $U^+ = M \cap H^+$ whose boundary $\partial U^+$ contains

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a neighborhood of 0 in $H_M$. We ask whether CR functions on $U^+$ extend to a uniform $M$-neighborhood of 0.

Such extension cannot hold for strictly pseudoconvex $M$. In this case $M$ can be, after a convenient holomorphic coordinate change, locally imbedded into some strictly convex hypersurface. This gives us plentiful functions with isolated peak points, destroying any hope for extension (independently of the shape of $H$). If we assume $H$ to be strictly pseudoconvex, a similar reason excludes extension from domains $U^+$ lying on the convex side. Our aim is to strive for weak assumptions on $M$ guaranteeing extension under the hypothesis $H$ is strictly pseudoconvex and $U^+$ lies on the concave side.

It can be seen that $M$ is weakly 1-concave precisely if it is nowhere strictly pseudoconvex [4]. It is known that in this case one has extension phenomena for certain Dirichlet-type problems [3, 4, 10]. Interestingly, weak 1-concavity of $M$ is not enough for our Cauchy-type problem (see Section 3). Our main result is the following.

**Theorem 1.1.** Let $M \subset \mathbb{C}^n$ be a smooth generic weakly 2-concave CR manifold of CR dimension $m$ intersecting a smooth strictly pseudoconvex hypersurface $H \subset \mathbb{C}^n$ $J$-generically in the origin. Let $U^+ \subset M$ be a relative domain, lying on the pseudoconcave side of $H$ and containing in its closure a neighborhood of the origin in $H_M$. Then there is an open neighborhood $V$ of the origin in $M$ such that every continuous CR functions on $U^+$ uniquely extends to a continuous CR function on $U^+ \cup V$.

In the strictly 2-concave case, this was proved in [8] by means of adapted integral formulas. Our approach will be very different, focusing on the geometry of related envelopes of holomorphy. In the weakly 2-concave case, Theorem 1.1 is even new for hypersurfaces. Here the reader may consult [10] for refinements for $J$-degenerate intersections.

In Section 3 we will see that Theorem 1.1 fails if $M$ is only weakly 1-concave. Note that the CR orbits of $M$ near 0 may be very complicated (see [9]). It is worth observing that in our situation we need no assumption on CR orbits. Compare this to global results in [2, 10], where the situation is very different. Finally we remark that it should be a subtle task to sharpen the condition on $H_M$ significantly. Our arguments extend to the case where $H$ is weakly pseudoconvex but satisfies a certain finite-type condition at 0 (see Remark 2.1). However, even for extendability of holomorphic functions from a given side of a real hypersurface of $\mathbb{C}^n$, finding a geometric characterization is a long-standing open problem.

### 2 Proof of the main result

After a quadratic holomorphic coordinate change, we may assume that $H$ is strictly convex near the origin. After a unitary rotation, $M$ writes as a smooth graph $y'' = h(z', x'')$, where $z' = (z_1, \ldots, z_m)$, $z'' = (z_{m+1}, \ldots, z_n) = (x_{m+1} + iy_{m+1}, \ldots, x_n + iy_n)$, $h(0) = 0$, $dh(0) = 0$. The strategy is first to prove an extension result for holomorphic functions, and to conclude then by approximation techniques.

**Part 1: Holomorphic extension.** First we assume that we are to extend functions holomorphic in a thin ambient domain $V^+ \subset \mathbb{C}^n$ containing $U^+$. Let $(X, \pi)$ be the envelope of holomorphy of $V^+$. Recall that $X$ is an $n$-dimensional complex manifold, $\pi : X \to \mathbb{C}^n$ a locally biholomorphic map, and $V^+$ can be viewed as a subdomain of $X$ via a canonical embedding $\alpha : V^+ \hookrightarrow X$ satisfying the lifting property $\pi \circ \alpha = \text{id}_{V^+}$. The fact that $X$ is the maximal domain over $\mathbb{C}^n$ to which all holomorphic functions on $V^+$ extend simultaneously translates as follows: (i) $f \mapsto f \circ \alpha$ is a topological isomorphism from $\mathcal{O}(X)$ onto $\mathcal{O}(V^+)$ (extension) and (ii) $X$ is a Stein manifold (maximality). Since $X$ is Stein there is a strictly plurisubharmonic function $\rho \in \mathcal{C}^{\infty}_R(X)$ such that $\{ \rho < r \}$ is relatively compact in $X$ for all $r \in \mathbb{R}$ (see [6], [7] [9] for envelopes). Holomorphic extension from $V^+$ to a neighborhood of 0 is the content of the
following claim: The mapping $\alpha$ extends as a lifting to an $M$-neighborhood $V$ of the origin whose size depends on $U^+$, but not on the particular shape of $V^+$.

Let $h(z)$ be a complex linear defining function of $T_0^{\ast}H$ such that $T_0H = \{ \text{Re}(h) = 0 \}$ and $\text{Re}(h)$ increases along the direction pointing into the convex side. For $\epsilon > 0$ small, we consider the family $B_\epsilon = \{ h(z) = \epsilon, |z| < \epsilon \}$. If $\delta > 0$ is very small, then convexity of $H$ and $J$-genericity of $M \cap H$ yield for $|\epsilon| < \delta$: (i) $M_c = B_\epsilon \cap M$ is a weakly 1-concave generic CR submanifold of $B_\epsilon$ of CR dimension $m - 1 > 0$ (topologically an $(\dim M - 2)$-ball), (ii) $M_c \setminus U^+$ is either empty, an isolated point or a compact ball. The latter means in particular that the boundaries of $M_c$ stay in $U^+$. We may furthermore assume $M_c \subset U^+$ for $-\delta < c < 0$. The idea is now to use a version of the continuity principle for subfamilies of the $M_c$.

To prove the claim it suffices to show that, for $|\tilde{c}| < \delta$, the union $\bigcup_{c \in [-\delta/2, \tilde{c}]} M_c$ lifts to $X$ \([-\delta/2, \tilde{c}]\) denoting the straight segment in $\mathbb{C}$ between $-\delta/2$ and $\tilde{c}$. If this is not the case, then there is a maximal half-open segment $[-\delta/2, \tilde{c})$, where $\tilde{c} < c$, such that $\bigcup_{c \in [-\delta/2, \tilde{c}]} M_c$ lifts to $X$. Maximality of $\tilde{c}$ implies that $\sup \rho \circ \alpha |_{M_c} \to \infty$ if $[-\delta, \tilde{c}) \ni z \to \tilde{c}$. Since the distance of the boundaries of the $M_c$ to $\partial X$ is positive, this implies that $\rho \circ \alpha |_{M_c}$ is nonconstant and has a maximum in the interior whenever $c$ is close to $\tilde{c}$. This contradicts the subsequent maximum principle, and the claim follows.

**Lemma 2.1.** Let $\mathcal{D}$ be a relatively compact domain in a smooth generic weakly 1-concave CR manifold $\mathcal{M} \subset \mathbb{C}^n$. If $\psi$ is a smooth strictly plurisubharmonic function defined near $\overline{\mathcal{D}}$, then we have $\max_{\overline{\mathcal{D}}} \psi \leq \max_{\partial \mathcal{D}} \psi$.

This follows from [5]. For the sake of completeness we provide a short argument: If $\max_{\overline{\mathcal{D}}} \psi > \max_{\partial \mathcal{D}} \psi$, the same holds for a generic Morse perturbation $\psi$, which we may choose such that $\psi$ has no critical points on $\mathcal{M}$ and $\psi |_{\mathcal{M}}$ is also Morse. Then $\psi |_{\overline{\mathcal{D}}}$ has somewhere a quadratic maximum $z_0$. Thus $\mathcal{M}$ touches the strictly pseudoconvex hypersurface $\{ \psi = \psi(z_0) \}$ in $z_0$ from the convex side, and is therefore itself strictly pseudoconvex near $z_0$. The lemma follows.

In the sequel, we will need a simple a-priori estimate: Pick $\eta > 0$ such that the intersection $B^+$ of $B_\eta(0)$ with the concave side of $M$ is contained in $U^+$ and that $(\partial B^+ \setminus H_M) \subset U^+$. Applying the claim to the points of $\partial B^+ \cap H_M$, we see that the restriction of any $f \in \mathcal{O}(U^+)$ to $B^+$ is bounded. Applying the claim with $B^+$ instead of $U^+$, we obtain extension from $U^+$ to $U^+ \cup V$ together with an estimate $\sup V |f| \leq \sup_{B^+} |f|$ ($\tilde{f}$ denoting the extension). The estimate immediately follows from the inclusion $\tilde{f}(V) \subset f(B^+)$. In fact, if we had $c \in \tilde{f}(V) \setminus f(B^+)$, then $(f(z) - c)^{-1}$ would still be holomorphic near $B^+$ without being extendable along $V$.

**Part 2: Approximation.** CR extension will now be derived by an application of the Baoendi-Treves approximation theorem ([1], see also [11]). Since $H_M$ is generic near the origin, there is a smooth totally real $n$-dimensional submanifold $R \subset H_M$. We may include $R$ into a smooth foliation $R_s = \{ s_1 = \tilde{s}_1, \ldots, s_m = \tilde{s}_m \}$ of an $M$-neighborhood of the origin such that (i) $s_1, \ldots, s_m$ are smooth real functions with independent differentials, (ii) the parameter $\tilde{s}$ ranges over some ball $U_s$ around the origin in $\mathbb{R}^m$ and (iii) $s_1$ is a local defining function of $H_M$ positive on the $(+)$-side of $H_M$. Suplementing functions, we obtain real coordinates $s_1, \ldots, s_m, t_1, \ldots, t_n$ on an $M$-neighborhood of 0. By [1], there are arbitrarily small open balls $U_s$ and $B_1 \subset B_2 \subset \mathbb{R}^m$, all centered in 0, such that continuous CR functions on $\{ s_1 > 0 \} \times B_2$ can be approximated by the restrictions of polynomials in $z_1, \ldots, z_n$, locally uniformly on compact subsets of $\{ s_1 > 0 \} \times B_1$.

For $U^+$ given as in Theorem 1.1, we may arrange that $\{ s_1 > 0 \} \times B_2 \subset U^+$. Pick furthermore a slightly smaller ball $U'_s \subset U_s$. The constructions in Part 1 depend continuously on the data. If $\lambda > 0$ is sufficiently small, we can find an $M$-neighborhood $V$ of 0 such that every function $f$ holomorphic near $\{ s \in U'_s : s_1 > \lambda \} \times B_2$ possesses a holomorphic extension $\tilde{f}$ to an ambient neighborhood of $V \cup \{ s \in U'_s : s_1 > \lambda \} \times B_2$ satisfying $\sup V |\tilde{f}| \leq \sup_{\{ s_1 > \lambda \} \times B_1} |f|$.
Let now $P_j$ be polynomials approximating $g \in CR(U^+)$. Then $P_j\big|_{\{s \in U'_s : s_1 > \lambda\} \times B_1}$ converges uniformly, hence also its restriction to $V$ by the a-priori estimate. Thus the limit defines a continuous CR function $g_V$ on $V$. Since $P_j$ approaches $g$ locally uniformly on $\{s \in U'_s : s_1 > 0\} \times B_1$, $g$ and $g_V$ glue into the desired extension. Uniqueness follows from general structure theorems [11] or from a closer inspection of the approximation process. The proof of Theorem 1.1 is complete.

Remark 2.1. a) It is not very essential to work with continuous CR functions. If $g$ is a CR distribution on $U^+$, we may use a method from [1], [11], to represent it on $\{s \in U'_s : s_1 > 0\} \times B_2$ as $g = \Delta_M^k f$, where $f$ is a continuous CR function on $\{s \in U'_s : s_1 > 0\} \times B_2$, $\Delta_M$ is a CR variant of the Laplace operator and $k$ is a sufficiently large integer. Now one first extends $f$ by Theorem 1.1 and obtains the desired extension as $\Delta_M^k f$. We omit the details.

b) The argument still works if $H$ is only weakly pseudoconvex but possesses a supporting holomorphic hyperplane touching it (from the concave side) with finite-order contact at the origin. But, as mentioned in the introduction, it should be hard to obtain a sharp result.

c) One can reduce the number of strictly pseudoconvex directions required for $H$ if one assumes weak $q$-concavity for $M$ with $q > 2$ (compare [8]).

3 Weakly 1-concave counter-example

Our example will be a modification of the weakly but not strongly 1-concave hypersurface

$$M_0 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : y_3 = |z_2|^2\}$$

Note that $M_0$ is foliated by complex lines and the Levi form $\mathcal{L}$ has one zero eigenvalue at every $z \in M_0$. Pick a smooth function $g : \mathbb{R} \to \mathbb{R}^+_0$ which vanishes identically for $t \leq 0$ and is strictly convex for $t > 0$. We claim that the hypersurface

$$M = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : y_3 = |z_2|^2 - g(x_1 - |z_2|^2)\}$$

is weakly 1-concave in a neighborhood of the origin. To see this, we observe first that the term $|z_2|^2$ implies that the Levi form$^2$ of $y_3 - |z_2|^2 - g(x_1 - |z_2|^2)$ is positive in the $z_2$-direction, which is contained in $T_0^* M$. Hence we have a positive direction at any $z \in M$ close to the origin. Secondly, we note that the slices

$$M_c = \{(z_1, z_3) \in \mathbb{C}^2 : y_3 = |c|^2 - g(x_1 - |c|^2)\} \cong M \cap \{z_2 = c\}$$

are concave graphs over the real $(z_1, x_3)$-hyperplane in $\mathbb{C}^2_{z_1, z_3}$. Consequently the Levi form of $y_3 - |z_2|^2 - g(x_1 - |z_2|^2)$ must have a nontrivial nonpositive eigenvector tangent to $M$ at any $z \in M$. This implies the claim.

Next we verify that the hypersurface $H_M = M \cap \{x_1 = |z_2|^2\}$ can be embedded into a strictly pseudoconvex hypersurface $H$ transverse to $M$. Note that the simplest candidate $H_0 = \{x_1 = |z_2|^2\}$ is only weakly pseudoconvex. Instead we try to construct $H$ as a graph $x_1 = h(y_1, z_2, z_3)$ satisfying $dh(0) = 0$. The desired $h$ is hence prescribed along $\pi_{y_1, z_2, z_3}(H_M)$. These partial data already imply that $H$ will have positive Levi curvature in the $z_2$-direction. But now it is standard that we can produce a strictly pseudoconvex $H$ by bending $H_0$ near the origin strongly enough along the $y_3$-direction into the pseudoconvex side (without changing $H_M$).

$^2$For hypersurfaces the characteristic bundle is one-dimensional. Hence the total and the directional Levi forms coincide essentially.

$^3$The Levi form of a function is $\mathcal{L}_\phi(X) = \frac{1}{2} \partial \overline{\partial} \phi(X, \overline{X})$. The Levi form of a regular level set $M = \{\phi = c\}$ is the restriction of $\mathcal{L}_\phi$ to $T^* M \cong T_1^* M$. 
As a matter of fact, the complex hyperplanes $E_t = \{z_3 = it\}$, $t < 0$, do not intersect $M^+ = M \cap \{x_1 < |z_2|^2\}$. On the other hand, the intersection $E_t \cap M$ contains points in an arbitrarily given neighborhood of the origin, if $t < 0$ is sufficiently close to 0. Hence the functions $f_t = (z_3 - it)^{-1}$, $t < 0$, show that there is no local CR extension from $M^+$ to a uniform neighborhood of the origin.

We conclude with a remark on the envelope of $V^+$.

**Remark 3.1.** Fix a domain $U^+ \subset M$ as in Theorem 1.1, and consider ambient open neighborhoods $V^+$ of $U^+$. We observe that there is always holomorphic extension from $V^+$ through some part of $H$. To this end, we construct small Bishop discs attached to the generic CR manifold $H_M$ (see [9] for the disc method). Because of the strict pseudoconvexity of $H$ the interior of the discs will lie in the pseudoconvex side of $H$. If we deform $H_M$ together with the attached discs into $V^+$, we obtain a one-sheeted part of $X$ (the envelope of holomorphy of $V^+$) which passes through $H$ into the pseudoconvex side and contains the origin in its closure.

Note that the size of this part of $X$ depends sensitively on the thickness of $V^+$. However the disc argument shows that for every $V^+$ the projection of the envelope to $\mathbb{C}^n$ contains points on the pseudoconvex side of distance to $H$ bounded from below by some uniform positive constant.

Of course the above arguments show that there is no $M$-neighborhood of the origin lifting simultaneously to all possible $X$. Intuitively speaking, the trouble is that the $X$ lose the contact to $M$ at the origin.

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**References**


