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Local Convergence for a Regula Falsi-Type Method under Weak Convergence

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Abstract

We present a local convergence analysis for a regula falsi-type method for solving nonlinear equations. In the earlier studies such as hypotheses on the second derivative have been used to show convergence of this method. In this paper we show convergence under hypotheses only on the first derivative. Moreover, we provide a radius of convergence and computable error bounds on the distances involved. Numerical examples are also given in this study.

Keywords: Regula falsi method; Radius of convergence; Local convergence

Introduction

In this study we are concerned with the problem of approximating a locally unique solution x^* of equation

$$F(x) = 0, \tag{1.1}$$

where $F : D \subseteq S \to S$ is a nonlinear function, D is a convex subset of S and S is R or C. Newton-like methods are used for finding solution of (1.1), these mathods are usually studied based on: semi-local and local convergence. The semi-local convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls [1-2]. The classical regula falsi method [3-7] is an efficient way of generating a sequence approximating x^*. However, there are some disadvantages, since one end-point is kept after step, if a concave or convex region of F(x) has been reached. Moreover, the asymptotic convergence rate of iterative sequence

$\{(x_n - x^*)\}$ is low in general. To overcome these problems, [8] proposed the regula falsi-type method defined for each n = 0, 1, 2, · · ·by

$$x_{n+1} = x_n - 2B_n^{-1}F(x_n) \tag{1.2}$$

where x_0 is an initial point and

$$B_n = \frac{F(x_n + F(x_n)) - F(x_n)}{F(x_n)} \pm \sqrt{\left(\frac{F(x_n + F(x_n)) - F(x_n)}{F(x_n)}\right)^2 + 4p^2F^2(x_n)}$$

The quadratic convergence of method (1.2) was shown in [6] under the assumption that F″ is bounded. However, this is a restrictive condition in many cases. As a motivational example, let us define function f on D = (−1, 2) by

$$f(x) = \begin{cases} x2\ln x + x4 - x3, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Then, we have that

$$f'(x) = 2x\ln x + 4x^3 - 3x^2 + x$$

And

$$f''(x) = 2\ln x + 12x^2 - 6x + 3$$

Hence, function f″ is unbounded on D. Therefore, the results in [9-12] cannot apply to solve equation (1.1). In the present study, we use hypotheses only on the first derivative in our local convergence analysis. Moreover, we provide a radius of convergence and computable error bounds on the distances

$|x_n - x^*|$ not given in [13,14].

In order to include more general methods, we shall study instead of (1.2), method defined for each n = 0, 1, 2, · · · by

$$x_{n+1} = x_n - (1+\gamma)A_n^{-1}F(x_n) \tag{1.3}$$

where x_0 is an initial point and

$$A_n = \frac{F(x_n + F(x_n)) - F(x_n)}{F(x_n)} \pm \alpha\sqrt{\left(\frac{F(x_n + F(x_n)) - F(x_n)}{F(x_n)}\right)^2 + 4p^2F^2(x_n)}$$

where ∈ S −{−1} and _ ∈ S are given parameters. Notice that if α =ɤ = 1, we obtain Chen's method (1.2) and if _ = = 0, we obtain Steffensen's method [15]. Notice in particular that the method in (27) is a special case of method (1.2). Other choices of _, are possible [16-21]. The rest of the paper is organized as follows: Section 2 contains the local convergence analysis of method (1.3). The numerical examples are given in the concluding Section 3.

Local Convergence

We present the local convergence analysis of method (1.3) in this section. Let $U(v, \rho)$, $\bar{U}(v, \rho)$ the open and closed balls in S, respectively, with center $v \in S$ and of radius $\rho > 0$.

It is convenient for the local convergence analysis of method (1.3) that follows to define some functions and parameters. Let $L_0 > 0$, $L > 0, M > 0$, $\alpha \geq 0$, $\gamma \in R - \{-1\}$ and $p \in R$ be given parameters. Define

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functions on the interval [0,+∞) by

$$g_0(t) = L_0(1+\frac{M_0}{2})t + |\alpha| M\sqrt{1+4p^2t^2}$$

$$h_0(t) = g_0(t) - 1,$$

And

$$g(t) = \frac{LM_0}{2})t + |\alpha| M\sqrt{1+4p^2t^2}$$

Suppose that

$$|\alpha| < \frac{1}{M} \tag{2.1}$$

Then, we have by (2.1) that $h_0(0) = |\alpha| M - 1 < 0$. We also get that $h_0(t) \to +\infty$ as $t \to +\infty$. Hence, function h_0 has zero in the interval (0, +∞) by the Intermediate value theorem. Denote by r_0 the smallest such zero. We have that

$r_0 < \frac{1}{L_0}$. Indeed, if $r_0 > \frac{1}{L_0}$ then

$$1 = g_0(r_0) \geq g_0(\frac{1}{L_0}) = L_0(1+\frac{M}{2})\frac{1}{L_0} + |\alpha| M\sqrt{1+\frac{4p^2}{L_0^2}} > 1$$

which is a contradiction. Moreover, define functions on the interval [0, r0] by

$$g_1(t) = \frac{Lt}{2(1-L_0t)} + (|\gamma| + \frac{g(t)}{1-L_0t}\frac{M}{1-g_0(t)}$$

And

$$h_1(t) = (1-g0(t)Lt + 2((1-L_0t)|\gamma| + g(t)) - 2(1-L_0t)(1-g_0(t))$$

Suppose that

$$|\gamma| < 1-2|\alpha| M \tag{2.2}$$

And

$$|\gamma| < \frac{1}{2M} \tag{2.3}$$

Then, we have by (2.2), (2.3) that g0(0) = g(0) = $|\alpha| M$ and

$$h_1(0) = 2(|\gamma| + g(0) - (1-g_0)) = 2(|\gamma| M) < 0$$

Moreover, we have that $h_1(r_0) = 2(|\gamma|(1-L_0r_0) + g(r_0)) > 0$ since $g_0(r_0) = 0$ $r_0 < \frac{1}{L_0}$ and g(r0) > 0. Hence function h1 has zeros in the interval (0, r0). Denote by r1 the smallest such zero. Define parameter

$$r = \frac{2}{2L_0 + L} < \frac{1}{L_0} \tag{2.4}$$

Then, we have that

$$h_1(r) = (1-g_0(r))(L_r - 2(1-L_0r)) + 2(|\gamma|(1-L_0r) + g(r))$$

$$= (|\gamma|(1-L_0r) + g(r)) > 0$$

Since $Lr - 2(1-L_0r) = 0$ by (2.4) L0r < 1 and g(r) > 0. Hence, we obtain that

$$r_1 < r \tag{2.5}$$

Then, we conclude that for each $t \in [0, r_1)$

$$0 \leq g_0(t) < 1 \tag{2.6}$$

$$g(t) \geq 0 \tag{2.7}$$

And

$$0 \leq g_1(t) < 1 \tag{2.8}$$

hold.

Next, using the above notation we can show the local convergence result for method (1.3).

Theorem 2.1 Let $F: D \subseteq S \to S$ be a differentiable function. Suppose that there exist $x^* \in D$ parameters L0 > 0,L > 0,M > 0, $\alpha \in S, \gamma \in S - \{-1\}$ and $p \in R$ such that for each $x, y \in D$

$$|\alpha| < \frac{1}{2M}$$

$$|\alpha| < 1 - 2|\alpha| M$$

$$F(x^*) = 0, F'(x^*) \neq 0 \tag{2.9}$$

$$|F(x^*)^{-1}(F'(x) - F'(x^*))| \leq L_0|x - x^*| \tag{2.10}$$

$$|F(x^*)^{-1}(F'(x) - F'(y))| \leq L|x - y| \tag{2.11}$$

$$|F(x^*)| \leq M_0 \tag{2.12}$$

$$|F(x^*)^{-1}F'(x) \leq M \tag{2.13}$$

And

$$\bar{U}(x^*, (1+M_0)r_1) \subseteq D \tag{2.14}$$

where r_1 is given by (2.1). Then, sequence $\{x_n\}$ generated for $x_0 \in U(x^*, r_1)$ by method (1.3) is well defined, remains in $U(x^*, r_1)$ for each n = 0, 1, 2, · · · and converges to x^*. Moreover, the following estimates hold

$$|A_n^{-1}F'(x^*)| \leq \frac{1}{1-g_0(|x_0 - x^*|)} \tag{2.15}$$

And

$$|x_{n+1} - x^*| \leq g_1(|x_n - 0x^*|)|x_n - x^*| < |x_n - x^*| < |x_n - x^*| < r_1 \tag{2.16}$$

where the "g" functions are defined above Theorem 2.1. Furthermore, suppose that there exists $T \in [r, \frac{2}{L_0})$ such that $\bar{U}(x^*, T) \subset D$ then the limit point x^* is the only solution of equation F(x) = 0 in $\bar{U}(x^*, T)$

Proof. By hypothesis $x_0 \in U(x^*, r_1)$ the definition of r1 and (2.10) we obtain that

$$|F'(x^*)^{-1}(F'(x_0) - F'(x^*))| \leq L_0|x_0 - x^*| < L_0 r_1 < 1 \tag{2.17}$$

It follows from (2.17) and the Banach Lemma on invertible functions (2, 3, 21, 25) that $F'(x_0)$ is invertible and

$$|F'(x_0)^{-1}F'(x^*)| \leq \frac{1}{1-L_0|x_0 - x^*|} < \frac{1}{1-L_0r_1} \tag{2.18}$$

Notice that by (2.9), (2.12) and (2.14) we get that

$$|x_0 + F(x_0) - x^*| = |x_0 - x^* + F(x_0) - F(x^*)|$$

$$\leq |x_0 - x^* + F(x_0) - F(x^*)|$$

$$\leq r_1 + |\int_0^1 F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)d\theta|$$

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$$\leq r_1 + |M_0|x_0 - x^*| < (1+M_0)r_1$$

So $x_0 + F(x_0) \in U(x^*,(1+M)r_1)$ where we also used that $x^* + \theta(x_0 - x^*) \in U(x^*,r_1) \subset U(x^*,(1+M)r_1)$ since

$$|x^* + \theta(x_0 - x^*) - x^*| \leq \theta|x_0 - x^*| \leq |x_0 - x^*| < r_1$$

We can write

$$A_0 - F'(x_0) = \frac{F(x_0 + F(x_0)) - F(x_0)}{F(x_0)} - F'(x0)$$

$$\pm\alpha\sqrt{(\frac{F(x_0+F(x_0)) - F(x_0)}{F(x_0)})^2 + 4p^2 + F^2(x_0)}$$

$$= \int_0^1 \frac{F'(x_0 + F(x_0) - x_0))(x_0 + F(x_0) - x_0)\,d\theta}{F(x_0)} - F'(x_0)$$

$$\pm\alpha\sqrt{(\frac{F(x_0+F(x_0)) - F(x_0)}{F(x_0)})^2 + 4p^2 + F^2(x_0)}$$

$$= \int_0^1 [F'(x_0 + \theta F(x_0)) - F'(x_0)]d\theta$$

$$\pm\alpha\sqrt{(\int_0^1 \frac{F'(x_0+\theta F(x_0))F(x_0)\,d\theta}{F(x_0)})^2 + 4p^2F^2(x_0)} \tag{2.19}$$

Using (2.11), the definition of function g, (2.13) and (2.19), we get that

$$|F'(x^*)^{-1}(A_0 - F'(x_0))|$$

$$\leq |\int_0^1 F'(x^*)^{-1}[F'(x_0 + \theta F(x_0))]d\theta]$$

$$+|\alpha|\sqrt{|\int_0^1 F'(x^*)^{-1}F'(x_0+\theta F(x_0))\theta|^2 + 4p^2|F'(x^*)^{-1}F(x_0)|^2}$$

$$\leq \frac{L}{2}|F(x_0)| + |\alpha|\sqrt{M^2 + 4p^2M^2|x_0 - x^*|^2}8$$

$$\leq \frac{L}{2}|\int_0^1 F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)\,d\theta| + |\alpha|M\sqrt{1+4p^2|x_0 - x^*|^2}$$

$$\leq \frac{L}{2}M_0|x_0 - x^*| + |\alpha|M\sqrt{1+4p^2|x_0 - x^*|^2} = g(|x_0 - x^*|) \tag{2.20}$$

As in (2.20), we can get using (2.6)

$$|F'(x^*)^{-1}(A_0 - F'(x^*))|$$

$$\leq \int_0^1 F'(x^*)^{-1}[F'(x_0 + \theta F(x_0)) - F'(x^*)]d\theta|$$

$$+|\alpha|M\sqrt{1+4p^2|x_0 - x^*|^2}$$

$$\leq L_0\int_0^1 |x_0 - x^* + \theta F(x_0)|\,d\theta + |\alpha|M\sqrt{1+4p^2|x_0 - x^*|^2}$$

$$\leq L_0(|x_0 - x^*| + \int_0^1 \theta F(x_0)|\,d\theta + |\alpha|M\sqrt{1+4p^2|x_0 - x^*|^2}$$

$$\leq L_0(|x_0 - x^*| + \frac{M_0}{2}|x_{1+4p2} - x^*|) + |\alpha|M\sqrt{1+4p^2|x_0 - x^*|^2}$$

$$= g_0(|x_0 - x^*|) < g_0(r_1) < 1$$

It follows from (2.21) that A_0 is invertible and

$$|A_0^{-1}F'(x^*)| \leq \frac{1}{1 - g_0(|x_0 - x^*|)}$$

which shows (2.15) for n = 0 and that x1 is well defined. Using method (1.3) for n = 0 we can write

$$\begin{aligned} x_1 - x^* &= x_0 - x^* - F'(x_0)^{-1}F(x_0) + F'(x_0)^{-1} \\ &(A_0 - F'(x_0))A_0^{-1}F(x_0) - \gamma A_0^{-1}F(x_0) \end{aligned} \tag{2.22}$$

Then, we have by (2.11) and (2.18) the estimate

$$\begin{aligned} &|x_0 - x^* - F'(x_0)^{-1}F(x_0)| \leq |F'(x_0)^{-1}F'(x^*)| \\ &\times\| \int_0^1 F'(x^*)^{-1}[F'(x^* + \theta(x_0 - x^*)) - F'(x_0)]d\theta(x_0 - x^*)| \\ &\leq \frac{L|x_0 - x^*|^2}{2(1 - L_0|x_0 - x^*|)} \end{aligned} \tag{2.23}$$

Then, it follows from (2.8), (2.13), (2.15) (for n = 0), (2.20), (2.22) and (2.23) that

$$|x_1 - x^*| \leq |x_0 - x^* - F'(x_0)^{-1}F(x_0)|$$

$$+|F'(x_0)^{-1}F'(x^*)\|F'(x^*)^{-1}(A_0 - F'(x_0))\|A_0^{-1}F'(x^*)|$$

$$\times|F'(x^*)^{-1}F'(x_0)| + |A_0^{-1}F'(x^*)^{-1}\|F'(x^*)^{-1}F(x_0)|$$

$$\leq \frac{L|x_0 - x^*|^2}{2(1 - L_0|x_0 - x^*|)}$$

$$+(|\gamma| + \frac{g(|x_0 - x^*|)}{1 - L_0|x_0 - x^*|})\frac{M|x_0 - x^*|}{1 - g_0(|x_0 - x^*|)}$$

$$\leq g_1(|x_0 - x^*|)|x_0 - x^*|)|x_0 - x^*| < g_1(r_1)|x_0 - x^*| = |x_0 - x^*| < r_1$$

which shows (2.16) for n=0. By simply replacing x0, x1 by x_k, x_{k+1} in the preceding estimates we arrive at (2.15) and (2.16). Using the estimate $|x_k+1 - x^*| < |x_k - x^*| < r_1$, we deduce that $x_{k+1} \in U(x^*,r_1)$ and lim $k \to \infty x_k = x^*$ To show the uniqueness part, let $Q = \int_0^1 F'(y^* + \theta(x^* - y^*))d\theta$ for some $y^* \in \bar{U}(x^*,T)$ with $F(y^*) = 0$. Using (2.6) we get that

$$|F'(x^*)^{-1}(Q - F'(x^*))| \leq \int_0^1 L_0|y^* + \theta(x^* - y^*) - x^*|\,d\theta$$

$$| \leq \int_0^1 (1-\theta)|x^* - y^*|\,d\theta \leq \frac{L_0}{2}T < 1$$

It follows from (2.23) that Q is invertible. Finally, from the identity $0 = F(x^*) - F(y^*) = Q(x^* - y^*)$, we deduce that $x^* = y^*$.

Remark 2.2 1. In view of (2.10) and the estimate

$$\|F'(x^*)^{-1}F'(x)\| = \|F'(x^*)^{-1}(F'(x) - F'(x^*)) + I\|$$

$$\leq 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq 1 + L_0\|x - x^*\|$$

condition (2.13) can be dropped and M can be replaced by

$M(t) = 1 + L_0 t$.

2. The results obtained here can be used for operators F satisfying autonomous differential equations [22] of the form

$$F'(x) = P(F(x))$$

where P is a continuous operator. Then, since $F'(x^*) = P(F(x^*)) = P(0)$ we can apply the results without actually knowing x^*. For example, let $F(x)=e^x - 1$. Then, we can choose: $P(x)=x + 1$.

3. It is worth noticing that method (1.2) or method (1.3) are not changing when we use the conditions of Theorem 2.1 instead of the stronger conditions used in [23-26]. Moreover, we can compute the

R=0.3249
R_0=0.1603
R1=0.0189
$\xi_1 = 0.9999$
$\xi = 1$

Table 1: The parameters.

R=0.6667
R_0=0.3859
R1=0.0669
$\xi = 1$
$\xi = 1$

Table 2: The parameters of Define Function.

computational order of convergence (COC) defined by

$$\xi = In\left(\frac{\| x_{n+1} - x^* \|}{\| x_n - x^* \|}\right) / \text{In}\left(\frac{\| x_n - x^* \|}{\| x_{n+1} - x^* \|}\right)$$

or the approximate computational order of convergence

$$\xi = In\left(\frac{\| x_{n+1} - x_n \|}{\| x_n - xL_{n-1} \|}\right) / \text{In}\left(\frac{\| x_n - x_{n-1} \|}{\| x_{n-1} - x_{n-2} \|}\right)$$

This way we obtain in practice the order of convergence in a way that avoids the bounds involving estimates using estimates higher than the first Frechet derivative of operator F [27].

Numerical Examples

We present two numerical examples in this section.

Example 3.1 Let D = [−1, 1]. Define function f of D by

$$f(x) = e^x - 1. \tag{3.1}$$

Using (3.1) and x^*=0, we get that L_0=e−1<L=M=M_0= $e, \alpha = 0.1226, \gamma = 0.1667\, p = 1$ The parameters are given in Table 1.

Example 3.2 Let $D = [-\infty, +\infty]$ Define function f of D by

$$f(x) = \sin(x). \tag{3.2}$$

Then we have for x^*=0 that L0=L=M=M0=1, $\alpha = 0.3333, \gamma 0.1667, p = 1$ The parameters are given in Table 2.

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