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List of Graph Obtained from Folding a Graph

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Abstract

In this paper we examining the relation between graph folding of a given graph and folding of new graphs obtained from this graph by some techniques like dual, gear, subdivision, web, crown, simplex, crossed prism and clique sum graphs. In each case, we obtained the necessary and sufficient conditions, if exist, for these new graphs to be folded. A simplex graph κ (G) of an undirected graph G is itself a graph with a vertex for each clique in G . Two vertices of κ (G) are joined by an edge whenever the corresponding two cliques differ in the presence or absence of a single vertex. The single vertices are called the zero vertices

Keywords: Dual • Gear • Subdivision • Web • Crown • Simplex • Crossed prism • Clique sum graphs• Graph folding

Introduction

By a graph we mean a simple and finite connected graph that is a graph without multiple edges or loops. Let G be a graph, then:

The dual graph G^* of a graph G is obtained by placing a vertex in every face of G and an edge joining every two vertices in neighboring faces [1].

A gear graph, denoted G_n, is a graph obtained by inserting an extra vertex between each pair of adjacent vertices on the perimeter of a wheel graph W_n. Thus G_n has 2n+1 vertices and 3n edges [2].

If the edge e joins vertices v and w, then the subdivision of e replace e by a new vertex u and two new edges vu and uw [3].

The web graph $W_{n,r}$ is a graph consisting of r concentric copies of the cycle graphs C_n , with corresponding vertices connected by edges [4].

Crown graph on 2n vertices is an undirected graph with two sets of vertices {u_1, u_2, ..., u_n } and { $v_1, v_2, ..., v_n$ } and with an edge from u_i to v_j whenever $i \neq j$ [5]. The crown graph can be viewed as a complete bipartite graph from which edges $u_i v_j$, i =1,...,n have been removed.

A simplex graph κ (G) of an undirected graph G is itself a graph, with a vertex for each clique in G. Two vertices of κ (G) are joined by an edge whenever the corresponding two cliques differ in the presence or absence of a single vertex. The single vertices are called the zero vertices [6].

A crossed prism graph for positive even n is a graph obtained by taking two disjoint cycle graphs C_n and adding edges (v_k, v_{2k+1}) and (v_{k+1}, v_{2k}) for k =1,3,...,$(n-1)$ [7]. We will denote this graph by CP_n .

If two graphs G and H each contain cliques of equal size, the clique-sum of G and H is formed from their disjoint union by identifying pairs of vertices in these two cliques to form a single shared clique, without deleting any of the clique edges [8].

Let G 1 and G 2 be two simple graphs and $f: G_1 \to G_2$ a continuous map. Then f is called a graph map, if

For each vertex $v \in V(G_1), f(v)$ is a vertex in $V(G_2)$.

For each edge $e \in E(G_1), \dim(f(e)) \leq \dim(e)$ [9].

(10) A graph map $f: G_1 \to G_2$ is called a graph folding if and only if f maps vertices to vertices and edges to edges [10].

If the edges and vertices of a face σ_i of G_1 are mapped to the edges and vertices of a face σ_j of G_2, then we write $f(\sigma_i) = \sigma_j$.

Materials and Methods

Graph folding of the dual graph

Theorem: Let G_1 and G_2 be graphs and $f: G_1 \to G_2$ a graph folding. Consider the graph map $g: G_1^* \to G_2^*$ defined by

(i)For all $v_i^* \in V(G_1^*)$, *iff* $f(\sigma_i) = \sigma$, where σi is a face of G_1.

(ii) *iff* $(\sigma_i) = \sigma_j$ and $f(\sigma_j) = \sigma_j$, where σ_i , σ_j are neighboring faces, then $\{v_i^*, v_j^*\} = \{v_k^*, v_j^*\}$, where v_k^* is the vertex of the face σ_k which is neighboring to σ_j but not neighboring to σ_i .

(iii) *if* $f(\sigma_i) = \sigma_j$ and $f(\sigma_k) = \sigma_l$, then $(v_i^*) = (v_l^*)$ and $(v_k^*) = (v_j^*)$ such that each of σ_i , σ_k and σ_j, σ_l , are neighboring faces .

Proof: Let $f: G_1 \to G_2$ be a graph folding. Suppose σ_i , σ_j and σ_l are faces of the graph G_1 such that

$f(\sigma_i) = \sigma_j$ and $f(\sigma_j) = \sigma_j$, where σ_i , σ_j are neighbouring faces. If σ_k is neighbouring to σ_j and not neighbouring to σ_i then there is no edges joining v_i^* and v_k^* in G_1^* , but each of $\{v_i^*, v_j^*\}$ and $\{v_k^*, v_j^*\}$ is an edge of G_1^* .Thus by the given definition g maps edges to edges. Now, let $f(\sigma_i) = \sigma_j$ and $f(\sigma_k) = \sigma_l$. Then by the given definition of g it maps the vertex v_i^* to v_l^* and the vertex v_k^* to v_j^*. Now since each of the faces σ_i , σ_k and σ_j , σ_l are neighbouring then each of $\{v_i^*, v_k^*\}$ and $\{v_i^*, v_j^*\}$ is an edge of G_1^*, i.e., the map maps edges to edges. And consequently g is a graph folding of the dual graph of G_1.

Example: Consider the graphs G_1 and G_2 .Let $f: G_1 \to G_2$ be a graph folding defined by $f(v_6) = (v_1)$ and $f\{(v_6, v_2), (v_6, v_3), (v_6, v_4), (v_6, v_5)\} = \{(v_1, v_2), (v_1, v_3), (v_1, v_4), (v_1, v_5)\}$ i.e., $f\{\sigma_5, \sigma_6, \sigma_7, \sigma_8\} = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ .The graph map $g: G_1^* \to G_2^*$ defined by $g\{v_5^*, v_6^*, v_7^*, v_8^*\} = \{v_3^*, v_4^*, v_1^*, v_2^*\}$ and $g\{(v_5^*, v_1^*), (v_5^*, v_6^*), (v_5^*, v_7^*), (v_6^*, v_2^*), (v_6^*, v_8^*), (v_7^*, v_3^*), (v_7^*, v_8^*), (v_8^*, v_8^*)\} = \{(v_3^*, v_1^*)(v_3^*, v_4^*)(v_3^*, v_1^*)(v_4^*, v_2^*)(v_4^*, v_2^*)(v_1^*, v_3^*)(v_1^*, v_2^*)(v_2^*, v_4^*)\}$ is a graph folding, see (Figure 1). The omitted vertices and edges or faces are mapped to themselves through this paper.

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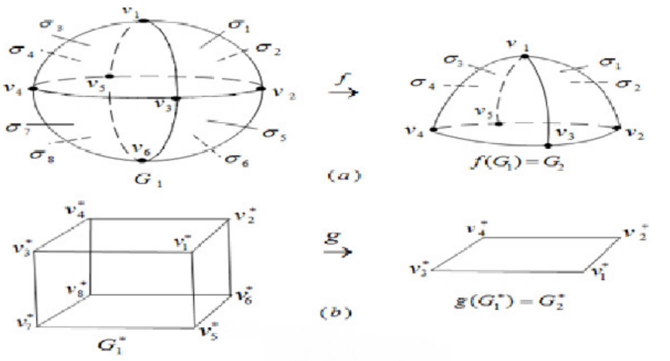

Figure 1. Graph folding of a graph and its dual graph.

Graph folding of the gear and the subdivision graphs

If we subdivided each edge of a graph G, we get a new graph G_s we will call it the subdivision graph. It should be noted that any graph folding of the wheel graph W_n maps the hup into itself.

Theorem: Let W_n be a wheel graph and G_n be the corresponding gear graph. Let $f: W_n \to W_n$ be a graph folding. Then the graph map $g: G_n \to G_n$ defined by

(i) $g\{v_i, v_r\} = \{v_k, v_s\}$ And $g\{v_r, v_j\} = \{v_s, v_l\}$ *iff* $\{v_i, v_j\} = \{v_k, v_l\}$ where v_r, v_s are the extra vertices inserting between the adjacent vertices v_i, v_j and v_k, v_l respectively.

(ii)For the hub v, $g(v) = v$.

Proof: Let $f: W_n \to W_n$ be a graph folding and consider the edges $\{v_i, v_j\}, \{v_k, v_l\} \in E(W_n)$ such that $f\{v_i, v_j\} = \{v_k, v_l\}$, i.e., $f\{v_i\} = \{v_k\}$ and $f\{v_j\} = \{v_l\}$. Now let v_r, v_s be the new vertices inserted between the vertices of the edges $\{v_i, v_j\}$ and $\{v_k, v_l\}$ respectively. Then we have four new edges $\{v_i, v_r\}, \{v_r, v_j\}, \{v_k, v_s\}$ and $\{v_s, v_l\} \in E(G_n)$ but $g\{v_i, v_r\} = \{v_k, v_s\}$ and $g\{v_r, v_j\} = \{v_s, v_l\}$ i.e., the map g maps edges of G_n to another edges of G_n. Also, for all $v_i, v_k \in (W_n)$ *if* $f(v_i) = v_k$, then g maps the edge $\{v_i, v\}$ to the edge $\{v_k, v\}$ where v is the hub, and consequently g is a graph folding of the gear graph G_n.

Example: Consider the wheel graph W_7 and the corresponding gear graph G_7. Let $f: W_7 \to W_7$ be a graph folding defined by $f\{v_2, v_3\} = \{v_6, v_5\}$ And $f(v_2, v_1), (v_2, v_3), (v_2, v_7), (v_3, v_4), (v_3, v_7) = \{(v_6, v_1), (v_6, v_5), (v_6, v_7), (v_5, v_4), (v_5, v_7)\}$. Then the graph map $g: G_7 \to G_7$ defined by $g\{v_8, v_2, v_9, v_3, v_{10}\} = \{v_{13}, v_6, v_{12}, v_5, v_{11}\}$ is a graph folding, see (Figure 2). In this case g maps the edges $(v_1, v_8), (v_8, v_2), (v_2, v_7), (v_2, v_9), (v_9, v_3), (v_3, v_7), (v_3, v_{10}), (v_{10}, v_4)$ to the edges $(v_1, v_{13}), (v_{13}, v_6), (v_6, v_7), (v_6, v_{12}), (v_{12}, v_5), (v_5, v_7), (v_5, v_{11}), (v_{11}, v_4)$ respectively.

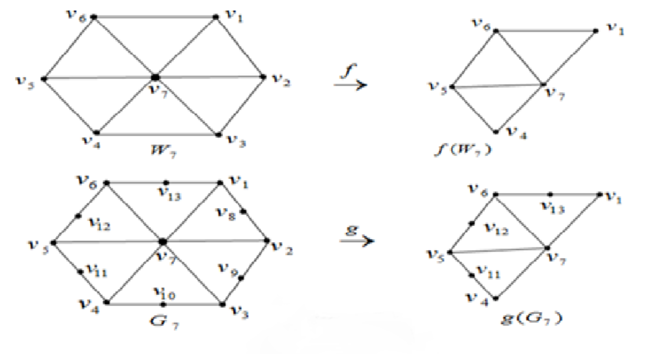

Figure 2. Graph folding of the wheel graph W7 and the gear graph G7.

Theorem: Let G be a graph and Gs the subdivision graph of G. Let $f: G \to G$ be a graph folding defined by: for all $\{v_i, v_j\} \in E(G), f\{v_i, v_j\} = \{v_k, v_l\} \in E(G)$. Then the graph map $g: G_s \to G_s$ defined by:

(i) Mapping the edges vu and uw to themselves *iff* f maps the edge vw to itself.

(ii) $g\{v_i, u_r\} = \{v_k, u_s\}$ and $g\{u_r, v_j\} = \{u_s, v_l\}$ where u_r, u_s are the new vertices replaced for the edges $\{v_i, v_j\}$ and $\{v_k, v_l\}$ respectively, is a graph folding.

The proof is obvious

Example: Consider the graph G and its subdivision G_s shown in (Figure 3).Let $f: G \to G$ be a graph folding defined by $f(v_1) = v_3$ and $f\{e_1, e_2\} = \{e_4, e_3\}$. Then the graph map $g: G_s \to G_s$ defined by $g\{v_1, u_1, u_2\} = \{v_3, u_4, u_3\}$ and $g\{e_1', e_2', e_7', e_8'\} = \{e_4', e_3', e_6', e_5'\}$ is a graph folding.

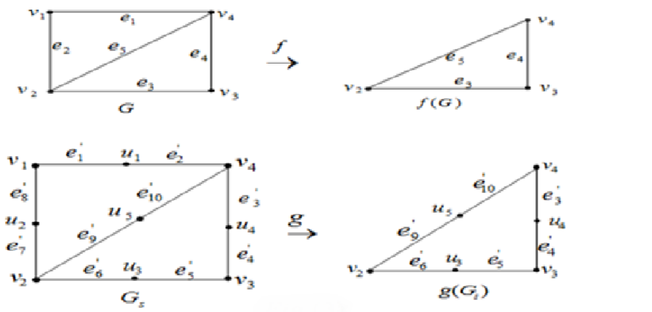

Figure 3. Graph folding of a graph and its subdivision graph.

Results and Discussion

Graph folding of the web and crown graphs

Theorem: Let Cn be a cycle graph, where $V(C_n) = \{v_1, \ldots, v_n\}, n$ is even and $W_{n,r}$ be the web graph where $V(W_{n,r}) = \{v_1, \ldots, v_n, v_{n+1}, \ldots, v_{2n}, v_{2n+1}, \ldots, v_{3n}, \ldots, v_{(r-1)n+1}, \ldots, v_{rn}\}$. Let $f: C_n \to C_n$ be a graph folding defined by for all $v_i \in (C_n), (v_i) = v_j \in V(C_n)$.

Then the graph map g: $Wn, \to Wn$, defined by

(i) For all $v_i \in (C_n), g(v_i) = f(v_i) = v_j$

(ii)For all $v_{i+(s-1)n} \in V(W_{n,r}), s = 2, \ldots, r$ then $g(v_{i+(s-1)n}) = v_{j=(s-1)n}$

Proof: Let Cn be a cycle graph with even vertices and $f: C_n \to C_n$ a graph folding defined by for all $vi \in$ $v_i \in (C_n), (v_i) = v_j \in V(C_n)$. Consider the vertices $v_i, v_j, v_k, v_l \in (C_n)$ such that f maps the edge $\{v_i, v_k\}$ to the edge $\{v_j, v_l\}$. For the vertices $v_i \in V(C_n), g(v_i) = f(v_j) = v_j$, and hence g is a graph folding. If s=2, then $g\{v_i, v_{i+n}\} = \{v_j, v_{j+n}\}$ and $g\{v_{i+n}, v_{k+n}\} = \{v_{j+n}, v_{l+n}\}$,i.e., g maps edges to edges. The same procedure can be done if s=3, 4,…, r. Thus g is a graph folding. For illustration see (Figure 4).

Example: Consider the cycle graph C_4. Let f:$C4 \to C4$ be a graph folding defined by $f(v_2) = (v_4)$ and $f\{(v_2, v_1), (v_2, v_3)\} = \{(v_4, v_1), (v_4, v_3)\}$. The graph map $g: W_{4,3} \to W_{4,3}$ defined by $g\{v_2, v_6, v_{10}\} = \{v_4, v_8, v_{12}\}$ and $g\{(v_2, v_1), (v_2, v_3), (v_2, v_6), (v_6, v_5), (v_6, v_7), (v_6, v_{10}), (v_{10}, v_9), (v_{10}, v_{11})\} = \{)\} = \{(v_4, v_1), (v_4, v_3), (v_4, v_8), (v_8, v_5), (v_8, v_7), (v_8, v_{12}), (v_{12}, v_9), (v_{12}, v_{11})\}$ is a graph folding, (Figure 5).

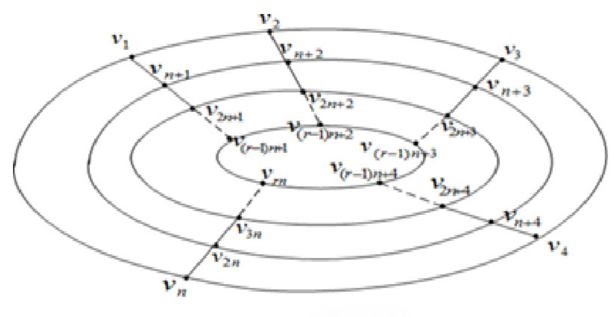

Figure 4. g is a graph folding.

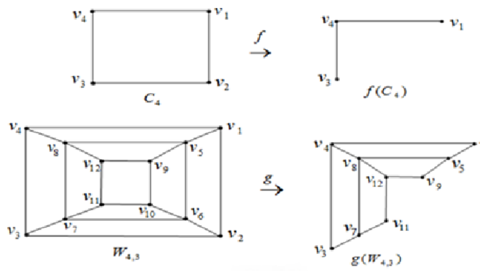

Figure 5. Graph folding of the cycle graph C4 and the web graph W4,3.

Graph folding of the simplex and crossed prism graphs

Theorem: Let G be a graph and $f:G\to G$ a graph folding. Then the graph map $g:(G)\to(G)$ defined by

(i) For a zero vertex $v\in V(k(G))$, $iff\,(v_i)=v_k$, then $g\{v_i,v\}=\{v_k,v\}$, where $v_i,v_k\in V(G)$.

(ii) If $\{v_i,v_j\}$ and $\{v_k,v_l\}$ are cliques of G such that $f\{v_i,v_j\}=\{v_k,v_l\}$, then $g\{v_i,v_r\}=\{v_k,v_s\}$ and $g\{v_r,v_j\}=\{v_s,v_l\}$ where v_r,v_s are the new vertices of the cliques $\{v_i,v_j\}$ and $\{v_k,v_l\}$ respectively.

(iii) If $\sigma=\{v_i,v_j,v_k\}$ and $\delta=\{v_l,v_m,v_n\}$ are cliques of G such that $f\{v_i,v_j,v_k\}=\{v_l,v_m,v_n\}$, then $g\{v_i,v_{r_\mu}\}=\{w,v_{s_\mu}\}$, $\mu=1,2,3$, where u,w are the new vertices of the two cliques σ,δ and v_{r_μ}, v_{s_μ} are the new vertices of the edges of σ and δ, respectively and so on.

Proof: Let G be a graph and $f:G\to G$ a graph folding.

(i) Consider the vertices $v_i\in(G)$ such that $(v_i)=v_k$. Let v be a zero vertex of $((G))$, then by the given definition of g it maps the vertex v onto itself. Then we get new edges $\{v_i,v\},\{v_k,v\}\in E(k(G))$, but $g\{v_i,v\}=\{v_k,v\}$ i.e., g maps edges to edges of $k((G))$.

(ii) Consider the cliques $\{v_i,v_j\}$ and $\{v_k,v_l\}$ of G such that $\{v_i,v_j\}=\{v_k,v_l\}$. Let v_r and v_s be the new vertices of the two cliques respectively, then we have new four edges $\{v_i,v_r\},\{v_r,v_j\},\{v_k,v_s\},\{v_s,v_l\}\in E(k(G))$ but $g\{v_i,v_r\}=\{v_k,v_s\}$ and $\{v_r,v_j\}=\{v_s,v_l\}$, i.e., g maps edges to edges.

(iii) Finally, let $\sigma=\{v_i,v_j,v_k\}$ and $\delta=\{v_l,v_m,v_n\}$ be cliques of G such that $f\{v_i,v_j,v_k\}=\{v_l,v_m,v_n\}$. And consider the new vertices u, of the two cliques σ, δ respectively. Then we have new edges $\{u,v_{r_\mu}\}\{w,v_{s_\mu}\}\in((G)),\mu=1,2,3$ where v_{r_μ} and v_{s_μ} are the new vertices of the edges of the cliques σ and δ respectively. The map g then maps the new edges of the boundary of σ to the new edges of the boundary of δ according to the rule (ii), and $g\{u,v_{r_\mu}\}=\{w,v_{s_\mu}\}$, where $\{u,v_{r_\mu}\},\{w,v_{s_\mu}\}\in E(k(G))$ Hence g is a graph folding of the simplex graph (G).

Example: Let G be the graph and $f:G\to G$ the graph folding defined by $f\{v_3,v_6,v_8\}=\{v_1,v_4,v_7\}$ and $f\{(v_5,v_6),(v_2,v_3),(v_3,v_6),(v_6,v_8),(v_3,v_8)\}=\{(v_5,v_4),(v_2,v_1),(v_1,v_4),(v_4,v_7),(v_1,v_7)\}$. Then the graph map $g:k(G)\to k(G)$ defined by $g\{v_3,v_6,v_8,u_7,u_8,u_9,u_{10},u_{11},w_2\}=\{v_1,v_4,v_7,u_4,u_5,u_3,u_2,u_1,w_1\}$ and $g\{\{(v_3,v),(v_6,v),(v_8,v),(v_5,u_8),(u_8,v_6),(v_2,u_7),(u_7,v_3)(v_3,u_9),(u_9,v_6),(v_6,u_{10}),(u_{10},v_8),(v_3,u_{11}),(u_{11},v_8),(w_2,u_9),(w_2,u_{10}),(w_2,u_{11})\}$ $=\{(v_1,v),(v_4,v),(v_7,v),(v_5,u_5),(u_5,v_4),(v_2,u_4),(u_4,v_1),(v_1,u_3),(u_3,v_4),(v_4,u_2),(u_2,v_7),(v_1,u_1),(u_1,v_7),(w_1,u_3),(w_1,u_2),(w_1,u_1)\}$ is a graph folding, see (Figures 6 and 7).

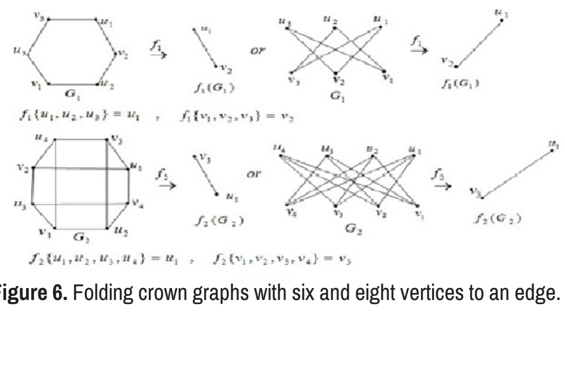

Figure 6. Folding crown graphs with six and eight vertices to an edge.

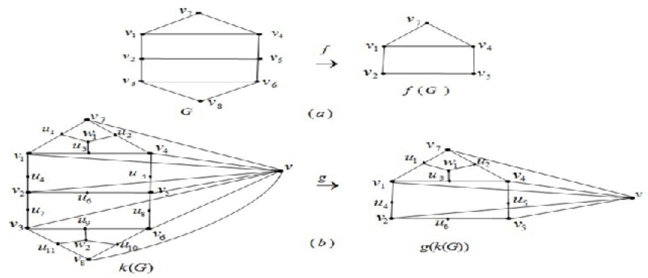

Figure 7. Graph folding of the simplex graph.

Example: Consider the cycle graph C_4 and let $f_1:C_4\to C_4$ be a graph folding defined by $f_1(v_1)=(v_3)$ and $f_1\{(v_1,v_2),(v_1,v_4)\}=\{(v_3,v_2),(v_3,v_4)\}$. The graph map $g_1:\ CP_4\to CP_4$ defined by $g_1\{v_1,u_1\}=\{v_3,u_3\}$ is not a graph folding since $g_1(v_1,u_2)=(v_3,u_2)\notin E(CP_4)$. While if $f_2:C_4\to C_4$ is a graph folding defined by $f_2\{v_1,v_2\}=\{v_3,v_4\}$ $andf_2\{(v_1,v_2),(v_1,v_4),(v_2,v_3)\}=\{(v_3,v_4),(v_3,v_4),(v_4,v_3)\}$. Then the graph map $g_2:\ CP_4\to CP_4$ defined $g_2\{v_1,v_2,u_1,u_2\}=\{v_3,v_4,u_3,u_4\}andg_2$ $\{(v_1,v_2),(v_1,v_4),(v_2,v_3),(u_1,u_2),(u_1,u_4),(u_2,u_3),(v_1,u_2),(v_2,u_1)\}=\{(v_3,v_4),(v_3,v_4),(v_4,v_3),(u_3,u_4),(u_3,u_4),(u_4,u_3),(v_3,u_4),(v_4,u_3)\}$ is a graph folding, see (Figure 8).

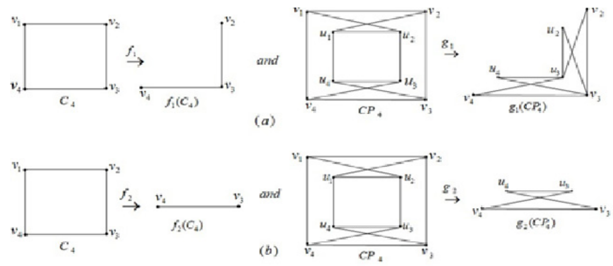

Figure 8. The crossed prism graph may or may not be folded.

Graph folding of the clique-sum graph

We will denote the clique-sum of the two graphs G and H by $GcliH$.

Definition: Let G_1,G_2,G_3 and G_4 be graphs. Let $f:G_1\to G_3$ and $g:G_2\to G_4$ be graph maps. Then we can define a map from the clique sum of G_1 and G_2 to the clique sum of G_3 and G_4 denoted by $fclig:G_1cliG_2\to G_3cliG_4$ as follows:

$(fclig)(e)\ =\{f(e)\ =g(e)\ =e$

This map we call it the clique-sum map of the maps f and.

Theorem: Let G_1,G_2,G_3 and G_4 be graphs. Let $f:G_1\to G_3$ and $g:G_2\to G_4$ be graph foldings.

Then the clique-sum map $fclig:G_1cliG_2\to G_3cliG_4$ is a graph folding.

Proof: Suppose f and g are graph foldings. Now, let $e\in G_1cliG_2$, then either $e\in G_1$ or $e\in G_2$. In these two cases and since each of f and g is a graph folding, then $(fclig)(e)\in(G_3cliG_4)$. As a result of $(fclig)$ maps counters to edges, the clique- sum map is still a graph folding (Figure 9).

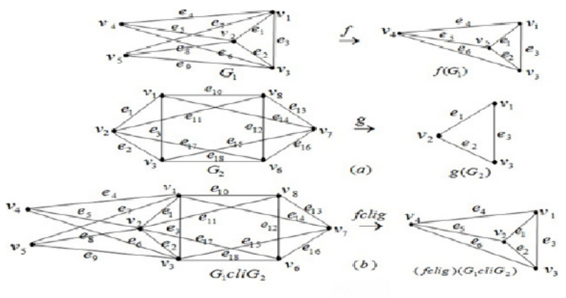

Figure 9. Graph folding of the clique-sum graph.

Conclusion

Consider the two graphs G_1 and G_2 shown .Let $f: G_1 \to G_1$ be a graph folding defined by $f\{v_5\}=\{v_4\}$, $f\{e_7,e_8,e_9\}=\{e_4,e_5,e_6\}$, $g\{v_6,v_7,v_8\}=\{v_1,v_2,v_3\}$ $and g\{e_{17},e_{18},e_{12},e_{16},e_{10},e_{11},e_{13},e_{14},e_{15}\}=\{e_1,e_3, e_3,e_1,e_3,e_2,e_2,e_1,e_2\}$. Then the clique sum map $fclig$: $G_1cliG_2 \to G_1cliG_2 defined by (fclig)\{v_5,v_6,v_7,v_8\}=\{v_4,v_1,v_2,v_3\}$ $and (fclig)\{e_7,e_8,e_9,e_{17},e_{18},e_{12},e_{16},e_{10},e_{11},e_{13},e_{14},e_{15}\}=\{e_4,e_5,e_6,e_1,e_3,e_3,e_1,e_3,e_2,e_2,e_1,e_2\}$ is a graph folding.

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