Lie Triple Derivations of the Lie Algebra of Strictly Block Upper Triangular Matrices

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Abstract

Let \( \mathcal{L} \) be the Lie algebra of all \( n \times n \) strictly block upper triangular matrices over a field \( \mathbb{F} \). In this paper, we explicitly describe all Lie triple derivations of \( \mathcal{L} \) when \( \text{char}(\mathbb{F}) \neq 2 \).

Keywords: Lie triple derivation; Triangular matrix; Lie algebra

Introduction

This paper is a study of the Lie triple derivations of the Lie algebra of strictly block upper triangular matrices over a field \( \mathbb{F} \) when \( \text{char}(\mathbb{F}) \neq 2 \). Let \( (g,\mathbb{F}) \) be a Lie algebra over a field \( \mathbb{F} \) or a ring \( \mathbb{R} \). Recall that an \( \mathbb{F} \)-linear map (resp.an \( \mathbb{R} \)-linear map) \( \delta : g \rightarrow g \) is called a derivation of \( g \) if:

\[
\delta([X,Y]) = [\delta(X),Y] + [X,\delta(Y)]
\]

for all \( X,Y \in g \). A Lie triple derivation is an \( \mathbb{F} \)-linear map (resp.an \( \mathbb{R} \)-linear map) \( f : g \rightarrow g \) which satisfies:

\[
f([X,[Y,Z]]) = [f(X),[Y,Z]] + [X,[f(Y),Z]] + [X,[Y,f(Z)]]
\]

for all \( X,Y,Z \in g \). Clearly, any derivation is a Lie triple derivation. However, the converse statement is, in general, not true [1]. The following elements of \( \text{End}(g) \) are typical examples of Lie triple derivations:

1. For any \( X \in g \), the linear map \( ad X \) defined by \( ad X := [X,Y] \) for all \( Y \in g \) is a derivation which is called an inner derivation. Clearly, an inner derivation is a Lie triple derivation.
2. Any linear map \( f \) that maps \( g \) to the center \( Z(g) \) of \( g \) and \( [g, [g, g]] \) to zero is a Lie triple derivation, called a central Lie triple derivation.

In recent years, significant progress has been made in studying the derivations and Lie triple derivations of matrix Lie algebras over a field or a ring. Wang and Li characterized the Lie triple derivations of the Lie algebra of strictly upper triangular matrices over a commutative ring [2]. Lie, Cao, and Li described the Lie triple and generalized Lie algebra of strictly upper triangular matrices over a commutative ring [3]. Benkovic determined the Lie triple derivations on triangular matrices [4]. More recently, Benkovic described the derivations and Lie triple derivations of upper triangular matrix algebras over a unital algebra [5]. Some other results on the derivations of certain matrix Lie algebras are given in refs. [6-12].

Fix a field \( \mathbb{F} \). Let \( M_{m,n} \) be the set of all \( m \times n \) matrices over \( \mathbb{F} \), and put \( M_{n} := M_{n,n} \). Let \( \mathcal{N} \) (resp. \( B \)) denotes the set of all strictly block upper triangular matrices (resp. block upper triangular matrices) in \( M_{n} \) relative to a given partition. Then \( \mathcal{N} \) and \( B \) are Lie subalgebras of \( gl(n, \mathbb{F}) \), i.e. \( M_{n} \) with the standard Lie bracket. In this paper, we explicitly determine the Lie triple derivations of \( \mathcal{N} \) which are as follows:

- When \( \text{char}(\mathbb{F}) \neq 2 \), Theorem 3.1 shows that every Lie triple derivation of \( \mathcal{N} \) is a sum of the adjoint action of a block upper triangular matrix in \( B \), a central Lie triple derivation, and two special linear maps.

The main motivation of this work comes from Wang and Li's work on the Lie triple derivation of the Lie algebra of strictly upper triangular matrices over a commutative ring [2], and authors' study of derivations of the Lie algebra of strictly block upper triangular matrices over a field [9]. Our work on the Lie triple derivations of \( \mathcal{N} \) not only generalizes the main result of Wang and Li over a field, but also use a new approach that is promising to find the Lie triple derivations of other matrix Lie algebras with appropriate block forms. In disclosing the Lie triple derivation action on \( \mathcal{N} \), we factor out the effects of adjoint action of block upper triangular matrices and those of central Lie triple derivations of \( \mathcal{N} \), explore the remaining Lie triple derivations.

Section 2 gives a basic introduction and determines some linear maps between matrix spaces that will be useful to describe the Lie triple derivations of \( \mathcal{N} \) in section 3. Section 3 describes the Lie triple derivations of \( \mathcal{N} \) over \( \mathbb{F} \) when \( \text{char}(\mathbb{F}) \neq 2 \).

Preliminary

In this section, we describe some linear maps between matrix spaces, and introduce some basic definitions and notations. The result in this section will be useful to prove the main result in Section 3.

Let \( [n] := \{1,2,\ldots,n\} \). Let \( E_{i,j}^{(n)} \in M_{m,n} \) denote the matrix with the only nonzero entry 1 on the \( (p,q) \) position for \( (p,q) \in [m] \times [n] \).

Lemma 2.1

Suppose \( \mathbb{F} \) is an arbitrary field. If \( X \in M_{m,n} \) and \( Y \in M_{n} \) satisfy that:

\[
XA = AY
\]

for all \( A \in M_{m,n} \), then \( X = \lambda I_{m} \) and \( Y = \lambda I_{n} \) for certain \( \lambda \in \mathbb{F} \).

Proof: Suppose \( X = (x_{i,j}) \in M_{m,n} \) and \( Y = (y_{i,j}) \in M_{n} \). For any \( (i,j) \in [m] \times [n] \), by eqn (3),

\[
X E_{i,j}^{(n)} = E_{i,j}^{(n)} Y
\]

Comparing the \( (i,j) \) entry of the matrices in eqn (4), we get \( x_{i,j} = y_{i,j} \). Similarly, comparing the \( (p,q) \) entry for \( p \neq i \), we get \( x_{i,j} = 0 \).

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comparing the \((i, q)\) entry for \(q \neq j\), we get \(0 = y'_{ij}\). Therefore, \(X = \lambda I_n\) and \(Y = \lambda I_n\) for some \(\lambda \in \mathbb{F}\).

**Lemma 2.2**

If linear maps \(\phi : M_{n,p} \rightarrow M_{m,q}\) and \(\psi : M_{n,p} \rightarrow M_{m,q}\) satisfy that:
\[
\phi(AB) = A\phi(B),
\]
for all \(A \in M_{n,p}, B \in M_{m,q}\), then there is \(X \in M_{n,q}\) such that \(\phi(C) = CX\) for \(C \in M_{m,q}\) and \(\psi(D) = DX\) for \(D \in M_{n,q}\).

**Proof:** For any \(j \in [n]\) and \(B \in M_{m,q}\), by eqn (5),
\[
\phi\left(\left(\begin{array}{cc}
\psi(B) & X
\end{array}\right)\right) = \left(\begin{array}{cc}
\psi(B) & X
\end{array}\right)X = X \psi(B).
\]

All such \(E_{i,j}^{m,n}\) span the first row space of \(M_{n,q}\). So \(\phi\) sends the first row of \(M_{n,q}\) to the first row of \(M_{m,q}\). There exists a unique \(X \in M_{n,q}\) such that:
\[
E_{i,j}^{m,n}(\phi(B)) = \phi(E_{i,j}^{m,n}) = E_{i,j}^{m,n}X,\ 	ext{for all } j \in [n], \ B \in M_{n,p}.
\]

where the first equality in eqn (7) is by eqn (6). Therefore, \(\phi(B) = BX\). Hence \(\phi(AB) = A\phi(B) = ABX\) for any \(A \in M_{n,p}\) and \(B \in M_{m,q}\). All such \(AB\) span \(M_{n,q}\). So \(\phi(C) = CX\) for all \(C \in M_{m,q}\).

**Lemma 2.3**

If linear maps \(\phi : M_{n,p} \rightarrow M_{m,q}\) and \(\psi : M_{n,p} \rightarrow M_{m,q}\) satisfy that:
\[
\phi(BA) = \psi(BA).
\]

For all \(A \in M_{n,p}, B \in M_{m,q}\), then there is \(X \in M_{m,q}\) such that \(\phi(C) = CX\) for \(C \in M_{m,q}\) and \(\psi(D) = XD\) for \(D \in M_{n,q}\).

**Proof:** The proof (omitted) is similar to that of Lemma 2.2.

**Lemma 2.4**

If linear maps \(\phi : M_{n,p} \rightarrow M_{m,q}\) and \(\psi : M_{n,p} \rightarrow M_{m,q}\) satisfy that:
\[
\phi(BA) = \psi(BA).
\]

For all \(A \in M_{n,p}, B \in M_{m,q}\), then there is \(X \in M_{m,q}\) such that \(\phi(C) = CX\) for \(C \in M_{m,q}\) and \(\psi(D) = XD\) for \(D \in M_{n,q}\).

**Proof:** For any \(j \in [p]\) and any \(E_{i,j}^{p,q} \in M_{n,q}\), by eqn (9),
\[
\phi\left(\left(\begin{array}{cc}
\psi(B) & X
\end{array}\right)\right) = \left(\begin{array}{cc}
\psi(B) & X
\end{array}\right)X = X \psi(B).
\]

which shows that the only possibly nonzero row of \(\phi(E_{i,j}^{p,q})\) is the first row. So maps the first row of \(M_{n,q}\) to the first row of \(M_{m,q}\). There exists a unique \(X \in M_{n,q}\) such that:
\[
E_{i,j}^{p,q}(\phi(B)) = \phi(E_{i,j}^{p,q}) = E_{i,j}^{m,n}X,\ 	ext{for all } j \in [p], \ E_{i,j}^{p,q} \in M_{n,p}.
\]

where the first equality in eqn (11) is by eqn (10). Therefore, \(\phi(E_{i,j}^{p,q}) = XE_{i,j}^{m,n}\) for all \(E_{i,j}^{p,q} \in M_{n,p}\). So \(\phi(B) = XB\) for \(B \in M_{m,q}\). Then \(\phi(A) = AXB\) for any \(A \in M_{n,p}\) and \(B \in M_{m,q}\). Hence \(\phi(A) = AX\) for all \(A \in M_{n,p}\).

**Lemma 2.5**

If linear maps \(f : M_{p,q} \rightarrow M_{p,q}, g : M_{r,s} \rightarrow M_{r,s}\) and \(h : M_{r,s} \rightarrow M_{r,s}\) satisfy that:
\[
f(AB) = g(AB) + h(AB), \quad \text{for all } A \in M_{p,q}, B \in M_{r,s},
\]
then there exist \(X \in M_{r,s}, Y \in M_{r,s}, Z \in M_{r,s}\) such that:
\[
f(C) = XC + CY, \quad \text{for } C \in M_{r,s},
\]
\[
g(A) = A + AZ, \quad \text{for } A \in M_{p,q},
\]
\[
h(B) = BY - ZB, \quad \text{for } B \in M_{r,s}.
\]
\[ \alpha(A) = XA - AY \quad \text{for } A \in M_{p \times q} \]  
\[ \beta(B) = YB + BZ \quad \text{for } B \in M_{q \times r} \]  
\[ \gamma(C) = CW - ZC \quad \text{for } C \in M_{r \times s} \]

**Proof:** We first define a linear map \( f : M_{p \times r} \to M_{q \times s} \) by:

\[ f'(AB) = \alpha(AB) + A\beta(B) \quad \text{for } A \in M_{p \times r}, B \in M_{q \times s}. \]

By eqns (27) and (32), for \( C \in M_{r \times s} \),

\[ \phi(ABC) = f'(ABC) + AB\gamma(C). \]

Applying Lemma 2.5 to \( f : M_{p \times r} \to M_{q \times s} \) defined by \( f(D) = \phi(D) \) for \( D \in M_{p \times r} \), \( g : M_{q \times s} \to M_{r \times s} \) defined by \( g(F) = f'(F) \) for \( F \in M_{q \times s} \), and \( h : M_{r \times s} \to M_{q \times s} \) defined by \( h(G) = \gamma(G) \) for \( G \in M_{r \times s} \) in eqn (33), we find \( X' \in M_{p \times q}, Y' \in M_{q \times r}, Z' \in M_{r \times s} \) such that:

\[ \alpha(A) = X'A + AY' \quad \text{for } A \in M_{p \times r}, \]
\[ \beta(B) = Y'B - Z'B \quad \text{for } B \in M_{q \times s}. \]

By eqns (33), (36), (35), and (38),

\[ X' + Y'B + AY' - Z'B = (X' - X)A + AB(B) - Z'B. \]

Applying Lemma 2.1 in eqn (39), we find \( \lambda \in \mathbb{F} \), \( I_1 \in M_{p \times q} \), and \( I_2 \in M_{q \times r} \) such that:

\[ X' - X = \lambda I_{p \times q}, \quad Z' - Z = \lambda I_{q \times r}. \]

Therefore, by eqns (37), (38), and (40),

\[ \alpha(A) = (X' - X)A + A\lambda I_{q \times r}, \]
\[ \beta(B) = BZ - (\lambda I_{q \times r} + Y'B). \]

Define \( Y' := \lambda I_{q \times r} + Y'B \), we get eqns (29) and (30).

We make some notations that will be frequently used later. A sequence \((n_1, n_2, \ldots, n_r)\) is called an ordered partition of \( n \) if \( t, n_1, \ldots, n_r \in \mathbb{Z}^+ \) and \( n_1 + n_2 + \ldots + n_r = n \). The \( t \times t \) block form associate with an ordered partition \((n_1, n_2, \ldots, n_r)\) of \( n \) is called block upper triangular (resp. block lower triangular) if the \((i,j)\) block or \((j,i)\) block for \( t \times t \) block matrix form associate with an ordered partition \((n_1, n_2, \ldots, n_r)\) of \( n \).

**Definition 2.7:** Consider the matrices in \( M_\Delta \) corresponding to the \( t \times t \) block form matrix associate with an ordered partition \((n_1, n_2, \ldots, n_r)\) of \( n \).

1. Let \( B(n_1, n_2, \ldots, n_r) \) (abbr. \( B \)) denote the Lie Theory algebra of all block upper triangular matrices in \( M_\Delta \).

2. Let \( \Lambda(n_1, n_2, \ldots, n_r) \) (abbr. \( \Lambda \)) denote the Lie algebra of all strictly block upper triangular matrices in \( M_\Delta \).

3. Call \((i,j) \in \mathbb{Z}^+ \) (i.e. \( 1 \leq i < j \leq t \)) the block index set of \( \Lambda \). Given a subset \( \Delta \) of \((i,j) \in \mathbb{Z}^+ \), let \( \Lambda^\Delta := \bigoplus \Lambda^\delta \).

The normalizer of Lie algebra \( \Lambda \) in \( M_\Delta \) is \( B \). For any \( X \in B \), the adjoint action of \( \Lambda \)

\[ ad X : \Lambda \to \Lambda, \quad Y \mapsto [X,Y] \]

is a derivation (resp. Lie triple derivation) of \( \Lambda \).

**Lie triple derivations of \( \Lambda \) for \( char(\mathbb{F}) \neq 2 \)**

This section of the goal is to describe the Lie triple derivations of the Lie algebra \( \Lambda \) of strictly block upper triangular \( n \times n \) matrices over a field \( \mathbb{F} \) with \( char(\mathbb{F}) \neq 2 \). The cases \( t \leq 4 \) are trivial. So we consider \( t \geq 5 \). Here is the main result for \( t \geq 5 \).

**Theorem 3.1**

Suppose \( char(\mathbb{F}) \neq 2 \). When \( t \geq 5 \), every Lie triple derivation \( f \) of the Lie algebra \( \Lambda = \Lambda(n_1, \ldots, n_r) \) can be written (not unqiuely) as:

\[ f = ad X + \phi_{i,j} + \phi_{i,j}^0 + \phi_{i,j}^1 \]

where the summand components are given below:

1. \( X \in B \).
2. \( \phi_{i,j} \in \text{End}(\Lambda) \) is a central Lie triple derivation in:

\[ \text{Cen}(\Lambda) = \{ \phi \in \text{End}(\Lambda) : ker \phi \supseteq N^{\text{re}} \} \]

3. \( \phi_{i,j}^0 \in \text{End}(\Lambda) \) satisfies that \( ker \phi \supseteq N^{\text{re}} \) and \( Im \phi \subseteq N^{\text{re}} \).
4. \( \phi_{i,j}^1 \in \text{End}(\Lambda) \) satisfies that \( ker \phi \supseteq N^{\text{re}} \) and \( Im \phi \subseteq N^{\text{re}} \).

Before proving Theorem 3.1, we first prove several results on the images \( f(N^0) \) for a Lie triple derivation \( f \) and \( N^0 \subseteq \Lambda \). The next lemma describes the image of \( f \) on \( A^{\text{re}} \) and \( A^{\text{re}} \).

**Lemma 3.2**

Suppose \( char(\mathbb{F}) \neq 2 \). Then for any Lie triple derivation \( f \) of \( \Lambda^0 \):

\[ f(A^{\text{re}}^\Delta) = \sum_{i=0}^{r-1} N_i \]

where \( \sum_{i=0}^{r-1} N_i \) is a direct sum of \( N_i \).

**Proof:** To get eqn (44), first we prove that \( f(A^{\text{re}}^\Delta) = 0 \) for any \( A^{\text{re}} \in N^0, \]

1. Suppose \( f \). Then for any \( A^{\text{re}} \in N^0 \) and \( A^{\text{re}} \).

Given $A^{(3)}$, the matrix $(A^{(3)})_{ij}$ for any $A^{(3)} \in \mathcal{N}_h$ could be any matrix in $\mathcal{N}_h$ with rank no more than rank $(A^{(3)})_j$. Therefore, $f(A^{(3)}_j)=0$. Overall, we have proved eqn (44). The proof of eqn (45) is similar to that of eqn (44).

Now we consider the image of a Lie triple derivation $f$ on $N_t^{23}, N_t^{3-2}$.

**Lemma 3.3**

Let $f$ be a Lie triple derivation of $A$. Then:

\[
f(A^{(2)}) \subseteq A^{r(1)} + \sum_{j=1}^{3} N_t^{2-j} + A^{r(j-1)} + N_t^{j-1},
\]

\[
f(A^{(2)-1}) \subseteq A^{r(2-j)} + \sum_{j=1}^{3} N_t^{2-j} + A^{r(j-1)} + N_t^{j-1}.
\]

**Proof:** To get eqn (49), first we prove that $f(A^{(2)})=0$ for any $A^{(3)} \in \mathcal{N}_h$, $i<j$, $i \neq 2$, and $j \neq 3$, and $(i,j) \notin \{(1,1-1),(1,2),(2,2)\}$. Either $i>2$ or $j<2$.

1. Suppose $j<2$. Then for any $A^{(1)} \in \mathcal{N}_h^{21}$ and $A^{r(2-j)} \in \mathcal{N}_h^{j-1}$:

\[
f(A^{(2)}) = 0 = (A^{(2)-1})_{i,j}.
\]

Now we further discuss eqn (51) in the following two cases:

- When $j=2$. $0=(A^{(2)2})(A^{(3)2})(A^{(4)2})$. Given $A^{(3)}$, the matrix $(A^{(3)})_{ij}$ for any $A^{(3)} \in \mathcal{N}_h$ could be any matrix in $\mathcal{N}_h$ with rank no more then rank $(A^{(3)})_2$. Therefore, $f(A^{(3)})=0$.

- When $j>3$. $0=(A^{(2)j})(A^{(3)j})(A^{(4)j})$ for any $A^{(3)} \in \mathcal{N}_h^{21}$ and $A^{r(2-j)} \in \mathcal{N}_h^{j-1}$. Therefore, $f(A^{(3)})=0$.

2. Suppose $i>2$. Then for any $A^{(2)} \in \mathcal{N}_h^{21}$ and any $A^{(3)} \in \mathcal{N}_h^{3}$:

\[
0 = f(A^{(2)})(A^{(3)})(A^{(4)})_{i,j}.
\]

Now we further discuss eqn (52) in the following two cases:

- When $i=0$. $0=(A^{(2)0})(A^{(3)0})(A^{(4)0})$. Given $A^{(3)}$, the matrix $(A^{(3)})_{ij}$ for any $A^{(3)} \in \mathcal{N}_h$ could be any matrix in $\mathcal{N}_h$, with rank no more then rank $(A^{(3)})_2$. Therefore, $f(A^{(3)})=0$.

- When $i>3$. $0=(A^{(2)i})(A^{(3)i})(A^{(4)i})_{i,j}$ for any $A^{(3)} \in \mathcal{N}_h^{21}$ and $A^{r(2-j)} \in \mathcal{N}_h^{j-1}$. Therefore, $f(A^{(3)})=0$.

Overall, we have proved eqn (49). The proof of eqn (49) is similar to that of eqn (50).

Now we consider the image of a Lie triple derivation $f$ on $N_t^{24}, ..., N_t^{3-3}$.

**Lemma 3.4**

Let $f$ be a Lie triple derivation of $A$ and $3k \leq t \leq 3$. Then:

\[
f(A^{(k)}) \subseteq \sum_{i=1}^{k-1} N_t^{2-i} + \sum_{j=1}^{3} N_t^{2-j} + A^{r(j-1)} + N_t^{j-1} + A^{r(k-1)} + N_t^{k-2}.
\]

**Proof:** Given any $A^{(k)} \in \mathcal{N}_h^{k}$, it suffices to prove that $f(A^{(k)})=0$ for $i,j \in \{1,2,3\}, (i,j) \notin \{(1,0),(1,1),(0,1),(0,2)\}$ to get eqn (53). Either $i>2$ or $j<2$.

1. Suppose $j<2$. Then for any $A^{(1)} \in \mathcal{N}_h^{0}$ and $A^{r(2-j)} \in \mathcal{N}_h^{j-1}$:

\[
f(A^{(2)}) = 0 = (A^{(2)-1})_{i,j}.
\]

Now we further discuss eqn (54) in the following two cases:

- When $k=j$. $0=(A^{(2)})_{j}(A^{(3)})_{j}(A^{(4)})_{j}$. Given $A^{(3)}$, the matrix $(A^{(3)})_{ij}$ for any $A^{(3)} \in \mathcal{N}_h$ could be any matrix in $\mathcal{N}_h$ with rank no more then rank $(A^{(3)})_j$. Therefore, $f(A^{(3)})=0$.

- When $k=j$. $0=(A^{(2)})_{j}(A^{(3)})_{j}(A^{(4)})_{j}$ for any $A^{(3)} \in \mathcal{N}_h^{21}$ and $A^{r(2-j)} \in \mathcal{N}_h^{j-1}$. Therefore, $f(A^{(3)})=0$.

Overall, we have proved eqn (53). Next we consider the image of a Lie triple derivation $f$ on $N_t^{11}, N_t^{24}, ..., N_t^{3-2}$.
Lemma 3.5

Suppose $f$ be a Lie triple derivation of $N$ and $1 \leq k \leq t$. Then:

$$f(A^{t+1}) = \sum_{i=1}^{n} A^{t+1}_{ii} + \sum_{j \neq i} A^{t+1}_{ij} + A^{t+1}_{ji} + A^{t+1}_{ii}$$  \hspace{1cm} (55)

Proof: Given any $A^{k+1}_{ij} \in N^{k+1}_{ij}$, it suffices to prove that $f(A^{k+1}_{ij}) \neq 0$, $f(A^{k+1}_{ji}) \neq 0$, and $f(A^{k+1}_{ii}) \neq 0$ for $i \neq j$, $i \neq k$, $j \neq k+2$, $i \neq (1, t-1), (1, t)$ to get eqn (55).

We first prove that $f(A^{k+1}_{ij}) \neq 0$ for $i \neq j$, $i \neq k$, $j \neq k+2$, $(i, j) \notin \{(1, t-1), (1, t), (2, t)\}$. Either $i > 2$ or $j < 1$.

1. Suppose $j < 1$. Then for any $A^{1,1} \in N^{1,1}$ and $A^{1,1} \in N^{1,1}$,

$$0 = f(A^{1,1}) \neq \sum_{i=1}^{n} A^{1,1}_{ii} + \sum_{j \neq i} A^{1,1}_{ij} + A^{1,1}_{ji} + A^{1,1}_{ii}$$

Therefore, $0 = f(A^{1,1}) = f(A^{1,1})$ for any $A^{1,1} \in N^{1,1}$ and $A^{1,1} \in N^{1,1}$. So $f(A^{1,1}) = 0$.

2. Suppose $i > 2$. Then for any $A^{2,2} \in N^{2,2}$ and any $A^{2,2} \in N^{2,2}$,

$$0 = f(A^{2,2}) \neq [A^{2,2}, [A^{2,2}, A^{2,2}]] + A^{2,2}, A^{2,2}, A^{2,2} + A^{2,2}, [A^{2,2}, A^{2,2}]$$

Therefore, $0 = f(A^{2,2}) = f(A^{2,2})$ for any $A^{2,2} \in N^{2,2}$ and $A^{2,2} \in N^{2,2}$. So $f(A^{2,2}) = 0$.

Next we show that $f(A^{k+1}_{ij}) = 0$ for $(k+2) = (2, (2t-2))$. For any $A^{2,2} \in N^{2,2}$ and $A^{2,2} \in N^{2,2}$,

$$0 = f(A^{2,2}) \neq [A^{2,2}, [A^{2,2}, A^{2,2}]] + A^{2,2}, A^{2,2}, A^{2,2} + A^{2,2}, [A^{2,2}, A^{2,2}]$$

Therefore, $f(A^{2,2}) = 0$. It remains to show that $f(A^{2,2}) = 0$, $f(A^{2,2}) = 0$. Now we show that $f(A^{2,2}) = 0$, (similarly for $f(A^{2,2}) = 0$). For any $A^{2,2} \in N^{2,2}$ and $A^{2,2} \in N^{2,2}$,

$$0 = f(A^{2,2}) \neq [A^{2,2}, [A^{2,2}, A^{2,2}]] + A^{2,2}, A^{2,2}, A^{2,2} + A^{2,2}, [A^{2,2}, A^{2,2}]$$

Therefore, $f(A^{2,2}) = 0$.

Overall, we have proved eqn (55).

The following lemma shows that any $f \in End(N)$ that satisfies $Ker f \supset N^t$, Im $f \subset N^t$ is a Lie triple derivation.

Lemma 3.6

Suppose $f \in End(N)$ satisfies that:

$$f(N) = \begin{cases} 0 & \text{for } N \subset N^t, (i, j) \notin \Omega \text{ otherwise.} \\ \end{cases}$$  \hspace{1cm} (56)

Then $f$ is a Lie triple derivation of $N$.

Proof: The $f$ satisfying eqn (56) also satisfies the Lie triple derivation property:

$$f([N, [N, N]]) = [f(N), [N, N]] + [N, [f(N), N]] + [N, [N, f(N)]]$$

Similarly, we have the following lemma.

Lemma 3.7

Suppose $f \in End(N)$ satisfies that:

$$f(N) = \begin{cases} 0 & \text{for } N^0 \subset N^t, (i, j) \notin \Omega; \\ \end{cases}$$  \hspace{1cm} (57)

Then $f$ is a Lie triple derivation of $N$.

Next we consider the image of a Lie triple derivation $f$ on other $N^t$. Lemma 3.8

Suppose char$(F) = 2$. For a Lie triple derivation $f$ of $N$, $i, j \in [t]$ and $j+1$, the image $f(N^t)$ satisfies that:

$$f(N^t) = \sum_{i=1}^{n} N^t + \sum_{j=1}^{n} N^t$$  \hspace{1cm} (58)

Proof: Let $i = k < t$. We prove eqn (58) by induction on $k$. 1. $k = 0$: $N^t = \sum_{i=1}^{n} N^t, N^t = \sum_{i=1}^{n} N^t$. Then:

$$f([N^t, f(N^t)]) = [f(N^t), f(N^t)] + [N^t, f(N^t)]$$

where the last relation in eqn (59) is by lemma 3.2, 3.3 and 3.4. So we done for $k = 3$.

2. $k = t + 3$: Suppose eqn (58) holds for all $k < t$ where $t > 3$. Now $N^{t+3} = \sum_{i=1}^{n} N^{t+3}, N^{t+3}$. For $A^{t+3} \in N^{t+3}$, $A^{t+3} \in N^{t+3}$, $A^{t+3} \in N^{t+3}$, $A^{t+3} \in N^{t+3}$, $A^{t+3} \in N^{t+3}$, $A^{t+3} \in N^{t+3}$, $A^{t+3} \in N^{t+3}$, $A^{t+3} \in N^{t+3}$, $A^{t+3} \in N^{t+3}$, $A^{t+3} \in N^{t+3}$ where the last relation in eqn (60) is by induction hypothesis, the case $k = 3$, and lemma 3.2, 3.3, 3.4, and 3.5. So eqn (58) is true for $t+3$.

So far all possible nonzero blocks of $f(N^t)$ for $N^t$ have been located. The next lemma explicitly describes most of those nonzero blocks of $f(N^t)$. It essentially implies that the $f$-images on these blocks are the same as the images of an adjoint action of a block upper triangular matrix. Denote the index set:

$$\Omega = \{(i, j) \in \{1 \leq i < j \leq t\} \mid \{(1, t-1), (1, t), (2, t-1), (2, t)\}$$

Lemma 3.9

Let $f$ be a Lie triple derivation of $N$. Then for any $(p, q) \in \Omega$, there exists $X \in N^t$ such that:

$$f(N^p) = -X \cdot N^p \quad \text{for all } N \in N^p \subseteq N$$

$$f(N^q) = X \cdot N^q \quad \text{for all } N \in N^q \subseteq N$$

Proof: Given $(p, q) \in \Omega$, we prove eqns (61) and (62) by the following steps:

1. We prove eqn (62) for $(q) = (t-1, t)$. Then $2 < p < t$. For $A^{t-1} \in N^{t-1}$, $A^{t-1} \in N^{t-1}$, and $A^{t-1} \in N^{t-1}$, $A^{t-1} \in N^{t-1}$, $A^{t-1} \in N^{t-1}$, $A^{t-1} \in N^{t-1}$, $A^{t-1} \in N^{t-1}$, $A^{t-1} \in N^{t-1}$, $A^{t-1} \in N^{t-1}$, $A^{t-1} \in N^{t-1}$ where the last relation in eqn (63) is by induction hypothesis, the case $n = 3$, and lemma 3.2, 3.3, 3.4, and 3.5. So eqn (58) is true for $t+3$. 

So far all possible nonzero blocks of $f(N^t)$ for $N^t$ have been located. The next lemma explicitly describes most of those nonzero blocks of $f(N^t)$. It essentially implies that the $f$-images on these blocks are the same as the images of an adjoint action of a block upper triangular matrix. Denote the index set:

$$\Omega = \{(i, j) \in \{1 \leq i < j \leq t\} \mid \{(1, t-1), (1, t), (2, t-1), (2, t)\}$$

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Therefore,
\[-f(A^{(p)})_{ij}(A^{(-1)})_{ij+1} = (A^{(p)})_{ij}f(A^{(-1)})_{ij} \]  
(63)

Applying lemma 2.4 to \( \varphi : N_{q} \rightarrow N_{q+1} \) defined by \( \varphi(g) = -f(C)^{q}_{ij} \) and \( \varphi(N_{q})_{ij} \rightarrow N_{q+1} \) defined by \( \varphi(D) = f(D^{q})_{ij} \) in eqn (63), we find \( X_{p+1} \in N_{q+1} \) such that \( f(A^{(p)})_{ij} = X_{p+1}(A^{(-1)})_{ij} \) for all \( A^{(p)} \in N_{q} \).

Similarly, we can prove eqn (61) for \((i,p) = (1,2)\) via lemma 2.4.

In other words, for \( 2 < q < t \), there is \( -Y_{q} \in N_{q} \) such that \( f(A^{(q)})_{ij} = (A^{(q)})_{ij}Y_{q} \) for all \( A^{(q)} \in N_{q} \).

We can prove eqn (62) for \((q,j) = (t-2,1)\).

For \( 1 < q < t-2 \), \( A^{(q)} \in N_{q} \), \( A^{(q)} \in N_{q+1} \), and \( A^{(q)} \in N_{q+2} \). By a similar computation as eqn (63), for \( A^{(q)} \in N_{q} \), \( A^{(q)} \in N_{q+1} \), and \( A^{(q)} \in N_{q+2} \),
\[-f(A^{(q)})_{ij}(A^{(-1)})_{ij+1} = (A^{(q)})_{ij}f(A^{(-1)})_{ij} \]  
(64)

Applying lemma 2.4 to \( \varphi : N_{q} \rightarrow N_{q+1} \) defined by \( \varphi(g) = -f(C)^{q}_{ij} \) and \( \varphi(N_{q})_{ij} \rightarrow N_{q+1} \) defined by \( \varphi(D) = f(D^{q})_{ij} \) in eqn (64), we find \( X_{p+1} \in N_{q+1} \) such that \( f(A^{(q)})_{ij} = X_{p+1}(A^{(-1)})_{ij} \) for all \( A^{(q)} \in N_{q+2} \).

Similarly, we can prove eqn (62) for \((i,p) = (1,3)\) via lemma 2.4.

In other words, for \( 3 < q < t \), there is \( -Y_{q} \in N_{q} \) such that \( f(A^{(q)})_{ij} = (A^{(q)})_{ij}Y_{q} \) for all \( A^{(q)} \in N_{q} \).

Next we prove eqn (62) for \((q,j) = (t-3,1)\).

For \( 1 < q < t-3 \), \( A^{(q)} \in N_{q} \), \( A^{(q)} \in N_{q+1} \), \( A^{(q)} \in N_{q+2} \), and \( A^{(q)} \in N_{q+3} \). By a similar computation as eqn (63), for \( A^{(q)} \in N_{q} \), \( A^{(q)} \in N_{q+1} \), and \( A^{(q)} \in N_{q+2} \),
\[-f(A^{(q)})_{ij}(A^{(-1)})_{ij+1} = (A^{(q)})_{ij}f(A^{(-1)})_{ij} \]  
(65)

Applying lemma 2.4 to \( \varphi : N_{q} \rightarrow N_{q+1} \) defined by \( \varphi(g) = -f(C)^{q}_{ij} \) and \( \varphi(N_{q})_{ij} \rightarrow N_{q+1} \) defined by \( \varphi(D) = f(D^{q})_{ij} \) in eqn (65), we find \( X_{p+1} \in N_{q+1} \) such that \( f(A^{(q)})_{ij} = X_{p+1}(A^{(-1)})_{ij} \) for all \( A^{(q)} \in N_{q+3} \).

Similarly, we can prove eqn (62) for \((i,p) = (1,4)\) via lemma 2.4.

In other words, for \( 4 < q < t-1 \), there is \( -Y_{q} \in N_{q} \) such that \( f(A^{(q)})_{ij} = (A^{(q)})_{ij}Y_{q} \) for all \( A^{(q)} \in N_{q} \).

Now we prove eqn (62) for \((q,j) = (t-4,1)\).

For \( 1 < q < t-4 \), \( A^{(q)} \in N_{q} \), \( A^{(q)} \in N_{q+1} \), \( A^{(q)} \in N_{q+2} \), \( A^{(q)} \in N_{q+3} \), and \( A^{(q)} \in N_{q+4} \). By a similar computation as eqn (63), for \( A^{(q)} \in N_{q} \), \( A^{(q)} \in N_{q+1} \), \( A^{(q)} \in N_{q+2} \), \( A^{(q)} \in N_{q+3} \), and \( A^{(q)} \in N_{q+4} \),
\[-f(A^{(q)})_{ij}(A^{(-1)})_{ij+1} = (A^{(q)})_{ij}f(A^{(-1)})_{ij} \]  
(66)

Applying lemma 2.4 to \( \varphi : N_{q} \rightarrow N_{q+1} \) defined by \( \varphi(g) = -f(C)^{q}_{ij} \) and \( \varphi(N_{q})_{ij} \rightarrow N_{q+1} \) defined by \( \varphi(D) = f(D^{q})_{ij} \) in eqn (66), we find \( X_{p+1} \in N_{q+1} \) such that \( f(A^{(q)})_{ij} = X_{p+1}(A^{(-1)})_{ij} \) for all \( A^{(q)} \in N_{q+4} \).

Finally, the equalities eqns (61) and (62) are proved.

Proof of Theorem 3.1. By Lemma 3.9, for \((p,q) \in \Omega\) we can find a matrix \( X_{q} \in N_{q} \), which satisfies eqns (61) and (62). Let \( X_{q} \in N_{q} \) be the matrix such that the \((p,q)\) block is \( X_{q} \) and \( 0 \) elsewhere, and let
\[ X_{q} : \sum_{(i,p) \in \Omega} X_{pq} \in N_{q}, \quad f_{i} : = f - ad X_{q}. \]  
(68)

Then \( f_{i} \) is a Lie triple derivation. The equalities eqns (61) and (62) imply that
\[ f_{i}(A^{(q)})_{ij} = 0 \quad \text{for all } A^{(q)} \subseteq N_{q}, \quad f_{i}(A^{(q)})_{ij} = 0 \quad \text{for all } A^{(q)} \subseteq N_{q}. \]  
(69)

By Lemmas 3.2, 3.3, 3.4, 3.5, and 3.8 when char(\( \mathbb{F} \))=2, for any \( N_{q} \subseteq N \), the only possibly nonzero blocks of \( f_{i}(\cdot) \) and the following:

- \((1,1,1,1)\) blocks when \( j \in \{+1,+2\} \).
- \((j,1,+2)\) Define \( \psi_{1,2}^{(q)} \subseteq End(N) \) such that for \( A \in N \),
\[ \psi_{1,2}^{(q)}(A) = \sum_{i,j} f_{i}(A^{(q)})^{ij} = \sum_{i,j} f(A^{(q)})^{ij}, \]
\[ \psi_{1,2}^{(q)}(A) = \sum_{i,j} f_{i}(A^{(q)})^{ij} = \sum_{i,j} f(A^{(q)})^{ij}. \]

Then Lemmas 3.6, and 3.7 show that \( \psi_{1,2}^{(q)} \) are Lie triple derivation of \( N \). Now we get a Lie triple derivation:
\[ f_{i} : = f_{i} - \psi_{i,1} - \psi_{i,2} = f - ad X_{q} - \psi_{1,1} - \psi_{1,2}. \]

Define a linear map \( \varphi_{i} \subseteq End(N) \) such that for \( A \in N \),
\[ \varphi_{i}(A) = \sum_{i,j} f_{i}(A^{(q)})^{ij} = \sum_{i,j} f(A^{(q)})^{ij}. \]

Then \( \psi_{q} \subseteq Z(N)=1 \) and \( ker\varphi_{i} \subseteq [X_{q},N_{q}^{\prime}] \), so that \( \varphi_{i} \) is a central Lie triple derivation of \( N \). We get a new derivation:
\[ f_{i} : = f_{i} - \varphi_{i} = f - ad X_{q} - \varphi_{1,1} - \varphi_{1,2}. \]

where \( f_{i}(N_{q}) \subseteq N_{q} \).

To get eqn (44), it suffices to prove the following claim regarding \( f_{i} \) there exist \( X_{q} \in \mathbb{F}^{n} \) for \( i \in \{t\} \) such that for each \( k \in \{t-4\} \), the Lie triple derivation:
\[ f_{i}^{(k)} := f_{i} \sum_{j \neq k} X_{q} \]

Satisfies that $f_2^{(k)}(N^{pq}) = 0$ for $1 \leq p < q \leq k+4$. The proof is done by induction on $k$:

1. $k=1$: We proceed the case $k=1$ by the following steps:
   - For any $A^1 \in \mathbb{M}^5$, $A^2 \in \mathbb{M}^5$, and $A^3 \in \mathbb{M}^5$, we have:
     \[ f_2(A^1, A^2, A^3) = f_2(A^1, A^2, A^3) \] (70)
     \[ = [f_2(A^1), A^2, A^3]] + [A^2, [f_2(A^1), A^3]] + [A^3, [A^2, f_2(A^1)]] \]
     \[ - f_2(A^1, A^2, A^3) + (A^3, f_2(A^1, A^2)) + (A^2, f_2(A^1, A^3)) + (A^1, f_2(A^2, A^3)) \]

   Applying Lemma 2.6 in eqn (70), we find $X^{11} \in \mathbb{B}^{11}$, $X^{12} \in \mathbb{B}^{12}$, and $X^{13} \in \mathbb{B}^{13}$ such that:

   \[ f_2(A^1)_{12} = (X^{11}A^{11} - A^{11}X^{11})_{12}, f_2(A^1)_{13} = (X^{11}A^{11} - A^{11}X^{11})_{13}, \]
   \[ \text{and } f_2(A^1)_{13} = (X^{11}A^{11} - A^{11}X^{11})_{14}. \]

   Let $f_2^{(1)} = f_2 - ad X^{11} - ad X^{12} - ad X^{13} - ad X^{14}$.

   Then: $f_2^{(1)}(N^{12}) = 0$, $f_2^{(1)}(N^{13}) = 0$, $f_2^{(1)}(N^{14}) = 0$ and $f_2^{(1)}(N^{1+4}) = 0$.

   - For any $A^1 \in \mathbb{M}^{51}$, $A^2 \in \mathbb{M}^{52}$, and $A^3 \in \mathbb{M}^{53}$, by a similar computation as eqn (70),
     \[ f_2^{(1)}(A^1, A^2, A^3)_{15} = (A^1, f_2(A^2, A^3))_{15}, \]
     \[ \text{and by eqn (71)} \]
     \[ f_2^{(1)}(A^1)_{15} = (A^1, f_2(A^1))_{15}. \]

   Applying Lemma 2.4 in eqn (74), we find $Y^{16} \in \mathbb{B}^{16}$ such that:

   $f_2^{(1)}(A^1, A^2, A^3)_{15} = (A^1, f_2(A^2, A^3))_{15}$
   $f_2^{(1)}(A^1)_{15} = (A^1, f_2(A^1))_{15}$.

   Let $f_2^{(2)} = f_2 - ad X^{16}$. It is easy to show that $f_2^{(2)}(N^{16}) = 0$ for $1 \leq p < q \leq 5$. So $k=1$ is done.

2. $k=2$: Similar to eqn (70), for any $A^1 \in \mathbb{M}^{11}$, $A^2 \in \mathbb{M}^{12}$, and $A^3 \in \mathbb{M}^{13}$

   \[ f_2^{(2)}(A^1, A^2, A^3)_{15} = (A^1, f_2(A^2, A^3))_{15}, \]
   \[ \text{and by eqn (73),} \]

   Applying Lemma 2.2 in eqn (73), we find $X^{16} \in \mathbb{B}^{16}$ such that:

   $f_2^{(2)}(A^1, A^2, A^3)_{15} = (A^1, f_2(A^2, A^3))_{15}$
   $f_2^{(2)}(A^1)_{15} = (A^1, f_2(A^1))_{15}$. \]

   Let $f_2^{(3)} = f_2 - ad X^{16}$. It is easy to show that $f_2^{(3)}(N^{16}) = 0$ for $1 \leq p < q \leq 6$. So $k=2$ is done.

3. $k=3$: Suppose the claim holds for all $k \leq \ell$ where $\ell > 2$ is given. In other words, there exist $X^i \in \mathbb{B}^i$ for all $i \in [\ell]$ such that:

   $f_2^{(\ell-1)} = f_2 \sum_{i=\ell}^{\ell} \text{ad } X^i$ satisfies that $f_2^{(\ell-1)}(N^{pq}) = 0$ for $1 \leq p < q \leq \ell+3$.

   Similar to eqn (70), for any $A^{1,2,3} \in N^{1,2,3}$, $A^{1,2,3} \in N^{1,2,3}$, and $A^{1,2,3} \in N^{1,2,3}$.

   \[ f_2^{(\ell-1)}(A^{1,2,3}, A^{1,2,3})_{1,2,3} = (A^{1,2,3}, f_2(A^{1,2,3}))_{1,2,3} \]
   \[ \text{Applying Lemma 2.2 in the last equation, we find } -X^{16} \in \mathbb{B}^{16} \text{ such that:} \]

   \[ f_2^{(\ell-1)}(A^{1,2,3})_{1,2,3} = (A^{1,2,3}, X^{16})_{1,2,3} \]

   $f_2^{(\ell)} = f_2^{(\ell-1)} - ad X^{16}$. It is easy to show that $f_2^{(\ell)}(N^{pq}) = 0$ for $1 \leq p < q \leq \ell+4$. So $k=3$ is proved.

Overall, the claim is completely proved; in particular, $f_2^{(\ell)}(N) = 0$.
Let $X := X_0 + \sum_{i=1}^\ell X^i$, then we get (43).

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