Ashdin Publishing
Journal of Physical Mathematics
Vol. 3 (2011), Article ID P110504, 7 pages
doi:10.4303/jpm/P110504

*Research Article*

# Lie Derivatives along Antisymmetric Tensors, and the M-Theory Superalgebra

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Received 25 March 2011; Accepted 10 May 2011

**Abstract** Free differential algebras (FDAs) provide an algebraic setting for field theories with antisymmetric tensors. The "presentation" of FDAs generalizes the Cartan-Maurer equations of ordinary Lie algebras, by incorporating $p$-form potentials. An *extended Lie derivative* along antisymmetric tensor fields can be defined and used to recover a Lie algebra dual to the FDA that encodes all the symmetries of the theory *including those gauged by the $p$-forms*. The general method is applied to the FDA of $D = 11$ supergravity: the resulting dual Lie superalgebra contains the M-theory supersymmetry anticommutators in presence of 2-branes.

**MSC 2010:** 53Z05, 83F50

## 1 Introduction

Supergravity in eleven dimensions [7,8] is today considered an effective theory (a particular limit of M-theory, for a review see e.g. [20,21]). More than two decades ago, it was formulated [10] as the gauging of a free differential algebra (FDA) [4,5,10,19,22], an algebraic structure that extends the Cartan-Maurer equations of an ordinary Lie algebra $G$ by including $p$-form potentials, besides the usual left-invariant 1-forms corresponding to the Lie group generators of $G$. Thus the 3-form of $D = 11$ supergravity acquires an algebraic interpretation, as well as the $p$-forms present in supergravity theories in various dimensions.

The group-geometric method in [3,4,9,11,15,16] yields lagrangians based on given FDAs. These FDAs encode the symmetries of the resulting field theories.

Only some time later it was realized how to extract from the FDA also the symmetries gauged by the $p$-forms, via a new ("extended") Lie derivative defined along antisymmetric tensors [6]. The extended Lie derivatives, together with the ordinary Lie derivatives of the $G$ Lie algebra contained in the FDA, close on an algebra that can be considered dual to the FDA.

The transformations on the fields generated by the extended Lie derivatives are the symmetries gauged by the antisymmetric tensors, and can be explicitly computed.

In this paper, we generalize the treatment of [6] (limited to 2-forms) to include arbitrary $p$-forms, and apply it to the FDA of $D = 11$ supergravity. The resulting dual Lie superalgebra contains the supersymmetry anticommutators of M-theory coupled to a 2-brane discussed in [12], one of the extended Lie derivatives corresponding to the pseudo-central charge $Z^{m_1 m_2}$.

In fact, a supertranslation algebra containing pseudo-central charges $Z^{m_1 m_2}$ and $Z^{m_1 - m_5}$ had already been found by D'Auria and Fré, who proposed in [10] a method to "resolve" FDAs into ordinary Lie algebras by considering the $p$-forms as composites of 1-form potentials of a larger group, containing the generators of $G$ plus some extra generators. For the FDA of $D = 11$ supergravity, the extra generators were found to be the two pseudo-central charges $Z^{m_1 m_2}$ and $Z^{m_1 - m_5}$ and an additional spinorial charge $Q'$.

Here we obtain a similar (but not identical) algebra: besides $Z^{m_1 m_2}$ we find a vector-spinor charge $Q^m$.

Closer contact with the D'Auria-Fré algebra can be achieved by further extending our treatment to FDAs containing more than one $p$-form. Then, we can apply it to an FDA containing a 3-form and a 6-form, so that both the charges $Z^{M_1 M_2}$ and $Z^{M_1 - M_5}$ enter the stage in the dual Lie algebra. This leads to the same supertranslation algebra of [10], that later was derived [17] in the context of $D = 11$ supergravity coupled to a 2- and a 5-brane.

More recently [23] the D'Auria and Fré resolution of the $D = 11$ FDA has been related to an underlying $E_{11}$ symmetry, and in [18] (where more references can be found on the hidden $E_{11}$ symmetry of $M$-theory) supersymmetry and $E_{11}$ are consistently combined.

A résumé on FDAs and their gauging is given in Section 2. By use of the extended Lie derivatives we obtain the dual formulation of FDAs containing a $p$-form. Both the soft and rigid FDA diffeomorphism algebras are given (the latter being a Lie algebra for constant parameters). This is applied in Section 3 to the FDA of $D = 11$ supergravity. In Section 4, we discuss the possibility of gauging the superalgebra dual of this FDA, thus obtaining a new formulation of $D = 11$ supergravity.

## 2 Free differential algebras and their Lie algebra duals

Rather than the general theory of FDAs (for a detailed review see [4], and [3] for a shorter account), we will treat here the case involving only one $p$-form. It already contains most of the essential features of FDAs. Its "presentation" is given by the generalized Cartan-Maurer equations:

$$d\sigma^A + \frac{1}{2} C^A{}_{BC}\, \sigma^B \sigma^C = 0, \tag{2.1}$$

$$\begin{aligned} &dB^i + C^i{}_{Aj}\, \sigma^A B^j + \frac{1}{(p+1)!} C^i{}_{A_1 \cdots A_{p+1}}\, \sigma^{A_1} \cdots \sigma^{A_{p+1}} \\ &\quad \equiv \nabla B^i + \frac{1}{(p+1)!} C^i{}_{A_1 \cdots A_{p+1}}\, \sigma^{A_1} \cdots \sigma^{A_{p+1}} = 0, \end{aligned} \tag{2.2}$$

where $\sigma^A$ are the usual left-invariant 1-forms associated to a Lie algebra $G$, $B^i$ is a $p$-form in a representation $D^i{}_j$ of $G$, and products between forms are understood to be exterior products.

The Jacobi identities for the generalized structure constants, ensuring the integrability of (2.1), (2.2), that is the nilpotency of the exterior derivative $d^2 = 0$, are

$$C^A{}_{B[C} C^B{}_{DE]} = 0, \tag{2.3}$$

$$C^i{}_{Aj}\, C^j{}_{Bk} - C^i{}_{Bj}\, C^j{}_{Ak} = C^C{}_{AB}\, C^i{}_{Ck}, \tag{2.4}$$

$$2\, C^i{}_{[A_1 j} C^j{}_{A_2 \cdots A_{p+2}]} - (p+1) C^i{}_{B[A_1 \cdots A_p} C^B{}_{A_{p+1} A_{p+2}]} = 0. \tag{2.5}$$

Equations (2.3) are the usual Jacobi identities for the Lie algebra $G$. Equation (2.4) implies that $(C_A)^i{}_j \equiv C^i{}_{Aj}$ is a matrix representation of $G$, while equation (2.5) states that $C^i \equiv C^i{}_{A_1 \cdots A_{p+1}} \sigma^{A_1} \cdots \sigma^{A_{p+1}}$ is a $(p+1)$-cocycle, that is $\nabla C^i = 0$.

### 2.1 Dynamical fields, curvatures and Bianchi identities

The main idea of the group-geometric method [4,9,11,15,16] extended to FDAs is to consider the 1-forms $\sigma^A$ and the $p$-form $B^i$ as the fundamental fields of the geometric theory to be constructed. In the case of ordinary Lie algebras, the dynamical fields are the vielbeins $\mu^A$ of $\tilde{G}$, a smooth deformation of the group manifold $G$ referred to as "soft group manifold". For FDAs the dynamical fields are both the vielbeins $\mu^A$ and the $p$-form field $B^i$: taken together they can be considered the vielbeins of the "soft FDA manifold".

In general $\mu^A$ and $B^i$ do not satisfy any more the Cartan-Maurer equations (2.1), (2.2), so that

$$R^A \equiv d\mu^A + \frac{1}{2} C^A{}_{BC}\, \mu^B \mu^C \neq 0, \tag{2.6}$$

$$R^i = dB^i + C^i{}_{Aj}\, \mu^A B^j + \frac{1}{(p+1)!} C^i{}_{A_1 \cdots A_{p+1}}\, \mu^{A_1} \cdots \mu^{A_{p+1}} \neq 0. \tag{2.7}$$

The extent of the deformation of the FDA is measured by the curvatures: the two-form $R^A$ and the $(p+1)$-form $R^i$. (Note that we use the same symbol $B^i$ for the "flat" and the "soft" $p$-form.) The deformation of the FDA is necessary in order to allow field configurations with nonvanishing curvatures.

Applying the exterior derivative $d$ to the definition of $R^A$ and $R^i$ (2.6), (2.7), using $d^2 = 0$ and the Jacobi identities (2.3)–(2.5), yields the Bianchi identities:

$$dR^A - C^A{}_{BC}\, R^B \mu^C = 0, \tag{2.8}$$

$$dR^i - C^i{}_{Aj} R^A B^j + C^i{}_{Aj}\, \mu^A R^j - \frac{1}{p!} C^i{}_{A_1 \cdots A_{p+1}}\, R^{A_1} \mu^{A_2} \cdots \mu^{A_{p+1}} = 0. \tag{2.9}$$

The curvatures can be expanded on the $\mu^A$, $B^i$ basis of the "soft FDA manifold" as

$$R^A = R^A{}_{BC}\, \mu^B \mu^C + R^A{}_i\, B^i, \tag{2.10}$$

$$R^i = R^i{}_{A_1 \cdots A_{p+1}}\, \mu^{A_1} \cdots \mu^{A_{p+1}} + R^i{}_{Aj}\, \mu^A B^j. \tag{2.11}$$

(Note that the $R^A{}_i\,B^i$ term in (2.10) can be there only for $p = 2$.) The FDA vielbeins $\mu^A$ and $B^i$ are a basis for the FDA "manifold". Coordinates $y$ for this "manifold" run on the corresponding "directions", that is Lie algebra directions and "$p$-form directions". The coordinates running on the $p$-form directions are $p-1$ forms (generalizing the coordinates running on the Lie algebra directions, which are 0-forms).

Eventually, we want *space-time* fields: the only coordinates the fields must depend on are spacetime coordinates, associated with the (bosonic) translation part of the algebra. This is achieved when the curvatures are *horizontal* in the other directions (see later).

How do we find the dynamics of $\mu^A(y)$ and $B^i(y)$? We wish to obtain a geometric theory (i.e. invariant under diffeomorphisms). We need therefore to construct an action invariant under diffeomorphisms, and this is simply achieved by using only diffeomorphic invariant operations as the exterior derivative and the exterior product. The building blocks are the one-form $\mu^A$ and the $p$-form $B^i$, their curvatures $R^A$ and $R^i$: exterior products of them can make up a lagrangian $D$-form, where $D$ is the dimension of space-time.

A detailed account of the procedure, together with various examples of supergravity theories based on FDAs, can be found in [4,3].

### 2.2 Diffeomorphisms and Lie derivative

The variation under diffeomorphisms $y+\varepsilon$ of an arbitrary form $\omega(y)$ on a manifold is given by the Lie derivative of the form along the infinitesimal tangent vector $\epsilon = \varepsilon^M \partial_M$:

$$\delta\omega = \omega(y+\varepsilon) - \omega(y) = d(i_\epsilon \omega) + i_\epsilon d\omega \equiv \ell_\epsilon \omega. \tag{2.12}$$

On $p$-forms $\omega_{(p)} = \omega_{M_1\cdots M_p} dy^{M_1} \wedge \cdots \wedge dy^{M_p}$, the *contraction* $i_{\mathbf{v}}$ along an arbitrary tangent vector $\mathbf{v} = v^M \partial_M$ is defined as

$$i_{\mathbf{v}}\,\omega_{(p)} = p\,v^{M_1} \omega_{M_1 M_2\cdots M_p}\, dy^{M_2} \wedge \cdots \wedge dy^{M_p} \tag{2.13}$$

and maps $p$-forms into $(p-1)$-forms. On the vielbein basis, equation (2.13) becomes

$$i_{\mathbf{v}}\,\omega_{(p)} = p\,v^A \omega_{AB_2\cdots B_p}\, \mu^{B_2} \wedge \cdots \wedge \mu^{B_p}, \tag{2.14}$$

where as usual curved indices $(M, N, \ldots)$ are related to tangent indices $(A, B, \ldots)$ via the vielbein (or inverse vielbein) components $\mu^A_M$ ($\mu^M_A$) (i.e. $\mathbf{v} = v^M\partial_M = v^A \mathbf{t}_A$ where $\mathbf{t}_A \equiv \mu^M_A \partial_M$ etc.). Thus the tangent vectors $\mathbf{t}_A$ are dual to the vielbeins: $\mu^B(\mathbf{t}_A) = \delta^B_A$.

The operator

$$\ell_{\mathbf{v}} \equiv d\, i_{\mathbf{v}} + i_{\mathbf{v}}\, d \tag{2.15}$$

is the *Lie derivative* along the tangent vector $\mathbf{v}$ and maps $p$-forms into $p$-forms.

In the case of a group manifold $G$, we can rewrite the vielbein variation under diffeomorphisms in a suggestive way:

$$\delta\mu^A = d\big(i_\epsilon \mu^A\big) + i_\epsilon d\mu^A = d\varepsilon^A + 2(d\mu^A)_{BC}\,\varepsilon^B \mu^C = (\nabla\varepsilon)^A + i_\epsilon R^A, \tag{2.16}$$

where we have used the definition (2.6) for the curvature, and the $G$-covariant derivative $\nabla$ acts on $\varepsilon^A$ as

$$(\nabla\varepsilon)^A \equiv d\varepsilon^A + C^A{}_{BC}\,\mu^B \varepsilon^A. \tag{2.17}$$

When dealing with FDAs, what is the action of diffeomorphisms on the $p$-form $B^i$? First, we consider diffeomorphisms in the Lie algebra directions. For these, the Lie derivative formula (2.12) holds. We have therefore, with tangent indices:

$$\delta B^i = \ell_{\varepsilon^A \mathbf{t}_A} B^i = d\big(i_{\varepsilon^A \mathbf{t}_A} B^i\big) + i_{\varepsilon^A \mathbf{t}_A} dB^i. \tag{2.18}$$

Since $\mu^A$ and $B^i$ are a basis for the FDA "manifold", the contraction of $B^i$ along a Lie algebra tangent vector $\mathbf{t}_A$ vanishes:

$$i_{\mathbf{t}_A}\mu^B = \delta^B_A, \quad i_{\mathbf{t}_A} B^i = 0,$$

and using the definition of $R^i$ (2.7) the variation (2.18) takes the form

$$\begin{aligned} \delta B^i = i_{\varepsilon^A \mathbf{t}_A} dB^i &= \big(R^i{}_{Aj} - C^i{}_{Aj}\big)\varepsilon^A B^j + \Big((p+1)R^i_{AA_1\cdots A_p} - \frac{1}{p!}C^i_{AA_1\cdots A_p}\Big)\varepsilon^A \mu^{A_1}\cdots\mu^{A_p} \\ &\equiv (\nabla\varepsilon)^i + i_{\varepsilon^A \mathbf{t}_A} R^i. \end{aligned} \tag{2.19}$$

### 2.3 Extended Lie derivatives

Before computing the algebra of Lie derivatives on the FDA fields, we introduce the following.

(i) A new contraction operator $i_{\varepsilon^j \mathbf{t}_j}$, defined by its action on a generic form $\omega = \omega_{i_1 \cdots i_n A_1 \cdots A_m} B^{i_1} \wedge \cdots B^{i_n} \wedge \mu^{A_1} \wedge \cdots \mu^{A_m}$ as

$$i_{\varepsilon^j \mathbf{t}_j} \omega = n\, \varepsilon^j \omega_{j i_2 \cdots i_n A_1 \cdots A_m} B^{i_2} \wedge \cdots B^{i_n} \wedge \mu^{A_1} \wedge \cdots \mu^{A_m}, \tag{2.20}$$

*where $\varepsilon^j$ is a $(p-1)$-form*. This operator still maps $p$-forms into $(p-1)$-forms. We can also define the contraction $i_{\mathbf{t}_j}$, mapping $n$-forms into $(n-p)$-forms, by setting

$$i_{\varepsilon^j \mathbf{t}_j} = \varepsilon^j i_{\mathbf{t}_j}.$$

In particular

$$i_{\mathbf{t}_j} B^i = \delta^i_j, \quad i_{\mathbf{t}_j} \mu^A = 0$$

so that $\mathbf{t}_j$ can be seen as the "tangent vector" dual to $B^j$. Note that $i_{\varepsilon^j \mathbf{t}_j}$ vanishes on forms that do not contain at least one factor $B^i$.

(ii) A new Lie derivative ("extended Lie derivative") given by

$$\ell_{\varepsilon^i \mathbf{t}_i} \equiv i_{\varepsilon^i \mathbf{t}_i} d + d\, i_{\varepsilon^i \mathbf{t}_i}. \tag{2.21}$$

The extended Lie derivative commutes with $d$, satisfies the Leibniz rule and can be verified to act on the fundamental fields as

$$\ell_{\varepsilon^j \mathbf{t}_j} \mu^A = \varepsilon^j R^A{}_j, \tag{2.22}$$

$$\ell_{\varepsilon^j \mathbf{t}_j} B^i = d\varepsilon^i + \big(C^i{}_{Aj} - R^i{}_{Aj}\big) \mu^A \wedge \varepsilon^j \tag{2.23}$$

by applying the definitions of the curvatures (2.6) and (2.7).

### 2.4 The algebra of diffeomorphisms

Using the Bianchi identities (2.8), (2.9), we find that the Lie derivatives *and* the extended Lie derivatives close on the algebra

$$\begin{aligned} \big[\ell_{\varepsilon_1^A \mathbf{t}_A}, \ell_{\varepsilon_2^B \mathbf{t}_B}\big] = {} & \ell_{[\varepsilon_1^A \partial_A \varepsilon_2^C - \varepsilon_2^A \partial_A \varepsilon_1^C + \varepsilon_1^A \varepsilon_2^B (C^C{}_{AB} - 2R^C{}_{AB})]\mathbf{t}_C} \\ & + \ell_{2\varepsilon_1^A \varepsilon_2^B\, (\frac{1}{p!} C^i{}_{ABA_1 \cdots A_{p-1}} - R^i{}_{ABA_1 \cdots A_{p-1}})\, \mu^{A_1} \cdots \mu^{A_{p-1}} \mathbf{t}_i}, \end{aligned} \tag{2.24}$$

$$\big[\ell_{\varepsilon^A \mathbf{t}_A}, \ell_{\varepsilon^j \mathbf{t}_j}\big] = \ell_{[\ell_{\varepsilon^A \mathbf{t}_A} \varepsilon^k + (C^k{}_{Bj} - R^k{}_{Bj}) \varepsilon^B \varepsilon^j]\mathbf{t}_k}, \tag{2.25}$$

$$\big[\ell_{\varepsilon_1^i \mathbf{t}_i}, \ell_{\varepsilon_2^j \mathbf{t}_j}\big] = \ell_{R^B{}_i (\varepsilon_1^i (\varepsilon_2)^j_B - \varepsilon_2^i (\varepsilon_1)^j_B)\mathbf{t}_j}. \tag{2.26}$$

The last commutator between extended derivatives vanishes except in the case $p = 2$ (since only in this case $R^B{}_i$ can be different from 0: then $\varepsilon^i_A$ are the components of the 1-form $\varepsilon^i$, i.e. $\varepsilon^i \equiv \varepsilon^i_A \mu^A$).

Notice that the commutator of two ordinary Lie derivatives *contains an extra piece proportional to an extended Lie derivative*. This result has an important consequence: if the field theory based on the FDA is geometric, that is its action is invariant under diffeomorphisms generated by usual Lie derivatives, then *also the extended Lie derivative must generate a symmetry of the action*, since it appears on the right-hand side of (2.24). Thus, when we construct geometric lagrangians gauging the FDA, we know *a priori* that the resulting theory will have symmetries generated by the extended Lie derivative: the transformations (2.22), (2.23) are invariances of the action.

Equations (2.24)–(2.26) give the algebra of diffeomorphisms on the soft FDA manifold.

**Note.** All the variations under diffeomorphisms (2.16), (2.19), (2.22), (2.23) can be synthetically written as

$$\delta \mu^I = (\nabla \varepsilon)^I + i_{\varepsilon^J \mathbf{t}_J} R^I, \tag{2.27}$$

where $\mu^I = \mu^A, B^i$ and so on. If the curvature $R^I$ is *horizontal* in some directions $J$ (i.e. if $i_{\mathbf{t}_J} R^I = 0$), the diffeomorphisms in these directions become *gauge transformations*, as evident from (2.27). In this case, a finite gauge transformation can remove the dependence on the $y^J$ coordinates, and the fields live on a subspace of the original FDA manifold. This generalizes horizontality of the curvatures on soft group manifolds: a classic example is the Poincaré group manifold, where horizontality in the Lorentz directions implies Lorentz gauge invariance and independence of the fields on the Lorentz coordinates.

### 2.5 Lie algebra dual of the FDA

From the algebra of diffeomorphisms (2.24)–(2.26), we find the commutators of the Lie derivatives on the rigid FDA manifold by taking vanishing curvatures and constant $\varepsilon$ parameters (nonvanishing only for given directions):

$$
\begin{aligned}
\left[\ell_{\mathbf{t}_A}, \ell_{\mathbf{t}_B}\right] &= C^C{}_{AB}\,\ell_{\mathbf{t}_C} + \frac{2}{p!}\,C^i{}_{ABA_1\cdots A_{p-1}}\,\ell_{\sigma^{A_1}\ldots\sigma^{A_{p-1}}\mathbf{t}_i}, \\
\left[\ell_{\mathbf{t}_A}, \ell_{\sigma^{B_1}\ldots\sigma^{B_{p-1}}\mathbf{t}_i}\right] &= \left[C^k{}_{Ai}\,\delta^{B_1\cdots B_{p-1}}_{C_1\cdots C_{p-1}} - (p-1)C^{[B_1}_{AC_1}\,\delta^{B_2\cdots B_{p-1}]}_{C_2\cdots C_{p-1}}\,\delta^k_i\right]\ell_{\sigma^{C_1}\ldots\sigma^{C_{p-1}}\mathbf{t}_k}, \\
\left[\ell_{\sigma^{A_1}\ldots\sigma^{A_{p-1}}\mathbf{t}_i}, \ell_{\sigma^{B_1}\ldots\sigma^{B_{p-1}}\mathbf{t}_j}\right] &= 0.
\end{aligned}
$$

This Lie algebra can be considered the dual of the FDA system given in (2.1), (2.2), and extends the Lie algebra of ordinary Lie derivatives (generating usual diffeomorphisms on the group manifold $G$). Notice the essential presence of the $(p-1)$-form $\sigma_1^A \cdots \sigma^{A_{p-1}}$ in front of the "tangent vectors" $\mathbf{t}_i$.

## 3 The FDA of $D = 11$ supergravity and its dual

We recall the FDA of $D = 11$ supergravity [10]:

$$
\begin{aligned}
&d\omega^{ab} - \omega^{ac}\omega^{cb} = 0\ \big[\,= R^{ab}\big], \quad dV^a - \omega^{ab}V^b - \frac{i}{2}\bar{\psi}\Gamma^a\psi = 0\ \big[\,= R^a\big], \\
&d\psi - \frac{1}{4}\omega^{ab}\Gamma^{ab}\psi = 0\ [= \rho], \qquad\qquad dA - \frac{1}{2}\bar{\psi}\Gamma^{ab}\psi V^a V^b = 0\ \big[\,= R(A)\big].
\end{aligned}
\tag{3.1}
$$

The $D = 11$ Fierz identity $\bar{\psi}\Gamma^{ab}\psi\bar{\psi}\Gamma^a\psi = 0$ ensures the FDA closure ($d^2 = 0$). Its Lie algebra part is the $D = 11$ superPoincaré algebra, whose fundamental fields (corresponding to the Lie algebra generators $P_a$, $J_{ab}$, $Q$) are the vielbein $V^a$, the spin connection $\omega^{ab}$ and the gravitino $\psi$. The 3-form $A$ is in the identity representation of the Lie algebra, and thus no $i$-indices are needed. The structure constants $C^i{}_{A_1\cdots A_{p+1}}$ of (2.2) are in the present case given by $C_{\alpha\beta ab} = -12(C\Gamma_{ab})_{\alpha\beta}$ (no upper index $i$), while the $C^i{}_{Aj}$ vanish.

The equations of motion on the "FDA manifold" have the following solution for the curvatures [10]:

$$
\begin{aligned}
R^{ab} &= R^{ab}{}_{cd}\,V^cV^d + i\left(2\bar{\rho}_{c[a}\Gamma_{b]} - \rho_{ab}\Gamma_c\right)\psi V^c \\
&\quad + F^{abcd}\,\bar{\psi}\Gamma^{cd}\psi + \frac{1}{24}F^{c_1c_2c_3c_4}\,\bar{\psi}\Gamma^{abc_1c_2c_3c_4}\psi,
\end{aligned}
\tag{3.2}
$$

$$
R^a = 0, \tag{3.3}
$$

$$
\rho = \rho_{ab}V^aV^b + \frac{i}{3}\Big(F^{ab_1b_2b_3}\Gamma^{b_1b_2b_3} - \frac{1}{8}F^{b_1b_2b_3b_4}\Gamma^{ab_1b_2b_3b_4}\Big)\psi V^a, \tag{3.4}
$$

$$
R(A) = F^{a_1\cdots a_4}V^{a_1}V^{a_2}V^{a_3}V^{a_4}, \tag{3.5}
$$

where the spacetime components $R^{ab}{}_{cd}$, $\rho_{ab}$, $F^{a_1\cdots a_4}$ of the curvatures satisfy the well-known propagation equations (Einstein, gravitino and Maxwell equations):

$$
\begin{aligned}
&R^{ac}{}_{bc} - \frac{1}{2}\delta^a_b R = 3\,F^{ac_1c_2c_3}F^{bc_1c_2c_3} - \frac{3}{8}\,\delta^a_b\,F^{c_1\cdots c_4}F^{c_1\cdots c_4}, \\
&\Gamma^{abc}\rho_{bc} = 0, \\
&\mathcal{D}_a F^{ab_1b_2b_3} - \frac{1}{2\cdot 4!\cdot 7!}\,\epsilon^{b_1b_2b_3a_1\cdots a_8}\,F^{a_1\cdots a_4}F^{a_5\cdots a_8} = 0.
\end{aligned}
$$

### 3.1 The algebra of diffeomorphisms on the FDA manifold

Using the structure constants extracted from the FDA (3.1) in the general formulas (2.24), (2.25), (2.26), one easily finds the complete diffeomorphism algebra of $D = 11$ supergravity on the FDA manifold. The supertranslation part reads

$$
\begin{aligned}
\left[\ell_{\varepsilon_1^a\mathbf{t}_a}, \ell_{\varepsilon_2^b\mathbf{t}_b}\right] &= \ell_{[\varepsilon_1^a\partial_a\varepsilon_2^c - \varepsilon_2^a\partial_a\varepsilon_1^c]\mathbf{t}_c} - 2\,\ell_{\varepsilon_1^a\varepsilon_2^b R^{cd}{}_{ab}\,\mathbf{t}_{cd}} - 4\,\ell_{\varepsilon_1^a\varepsilon_2^b\bar{\psi}\Gamma^{ab}\psi\,\mathbf{t}}, \\
\left[\ell_{\varepsilon_1^\alpha\mathbf{t}_\alpha}, \ell_{\varepsilon_2^\beta\mathbf{t}_\beta}\right] &= -i\ell_{\bar{\varepsilon}_1\Gamma^c\varepsilon_2\,\mathbf{t}_c} - 2\,\ell_{\varepsilon_1^\alpha\varepsilon_2^\beta\,R^{cd}{}_{\alpha\beta}\,\mathbf{t}_{cd}} - 4\,\ell_{\bar{\varepsilon}_1\Gamma_{ab}\varepsilon_2 V^aV^b\,\mathbf{t}}, \\
\left[\ell_{\varepsilon^a\mathbf{t}_a}, \ell_{\varepsilon^\beta\mathbf{t}_\beta}\right] &= \ell_{(\varepsilon^a\partial_a\varepsilon^\gamma - 2\varepsilon^a\varepsilon^\beta\rho^\gamma{}_{a\beta})\,\mathbf{t}_\gamma} - 8\,\ell_{\varepsilon^a\bar{\varepsilon}\Gamma_{ab}\psi V^b\,\mathbf{t}},
\end{aligned}
$$

where ${R^{cd}}_{\alpha\beta}$ and ${\rho^{\gamma}}_{a\beta}$ are respectively the $\psi\psi$ and the $V\psi$ components of the curvatures $R^{cd}$ and $\rho$, as given in (3.2) and (3.4).

The mixed commutators (between ordinary and extended Lie derivatives) are computed by adapting the general formula (2.25) to the case at hand:

$$\left[\ell_{\varepsilon^A \mathbf{t}_A}, \ell_{\varepsilon \mathbf{t}}\right] = \ell_{(\ell_{\varepsilon^A \mathbf{t}_A}\varepsilon)\mathbf{t}} = \ell_{[\varepsilon^A \partial_A \varepsilon_{BC} + 2\varepsilon_{AC}\partial_B \varepsilon^A + 4\varepsilon_{AB}\varepsilon^D (R^A{}_{CD} - \frac{1}{2}C^A{}_{CD})]\mu^B \mu^C \mathbf{t}},$$

where $\mu^A = V^a, \omega^{ab}, \psi^\alpha$ and the two-form parameter associated to the three-form $A$ is expanded on the $\mu^A$ basis: $\varepsilon = \varepsilon_{AB}\mu^A\mu^B$. For example,

$$\left[\ell_{\varepsilon^a \mathbf{t}_a}, \ell_{\varepsilon_{cd}V^c V^d \mathbf{t}}\right] = \ell_{(\varepsilon^a \partial_a \varepsilon_{bc} + 2\varepsilon_{ac}\partial_b \varepsilon^a) V^b V^c \mathbf{t}}.$$

Finally, commutators between extended Lie derivatives vanish:

$$\left[\ell_{\varepsilon_1 \mathbf{t}_a}, \ell_{\varepsilon_2 \mathbf{t}}\right] = 0.$$

**Note.** The action of the extended Lie derivative is nontrivial only on $A$, where it amounts to a gauge transformation:

$$\ell_{\varepsilon \mathbf{t}} A = d\varepsilon,$$

(cf. equation (2.23)) due to horizontality of $R(A)$ in the $A$-direction.

### 3.2 The dual Lie algebra

Taking constant parameters ($\varepsilon^B = \delta^B_A$ for a fixed $A$, $\varepsilon_{CD} = \delta^{AB}_{CD}$ for fixed $A, B$) and vanishing curvatures, the algebra of Lie derivatives given in the preceding paragraph reduces to the following Lie algebra:

$$\begin{aligned}
\left[P_a, P_b\right] &= -\left(C\Gamma_{ab}\right)_{\alpha\beta} Z^{\alpha\beta}, \\
\left[P_a, Q_\beta\right] &= 2\left(C\Gamma_{ab}\right)_{\alpha\beta} Q^{b\alpha}, \\
\left\{Q_\alpha, Q_\beta\right\} &= i\left(C\Gamma^a\right)_{\alpha\beta} P_a + \left(C\Gamma_{ab}\right)_{\alpha\beta} Z^{ab}, \\
\left[J_{ab}, J_{cd}\right] &= \eta_{a[c}J_{d]b} - \eta_{b[c}J_{d]a}, \\
\left[J_{ab}, P_c\right] &= \eta_{c[a}P_{b]}, \\
\left[J_{ab}, Q_\alpha\right] &= -\frac{1}{4}\left(\Gamma_{ab}\right)_{\alpha\beta} Q_\beta, \\
\left[J_{ab}, Z^{cd}\right] &= 2\delta^{[c}_{[a} Z_{b]}{}^{d]}, \\
\left[J_{ab}, Q^{c\gamma}\right] &= \delta^c_{[a} Q_{b]}{}^{\gamma} - \frac{1}{4}\left(\Gamma_{ab}\right)^{\gamma\beta} Q^{c\beta}, \\
\left[Q_\alpha, Z^{ab}\right] &= 2i\left(C\Gamma^{[a}\right)_{\alpha\beta} Q^{b]\beta},
\end{aligned} \tag{3.6}$$

where only the nonvanishing commutators are given. We have used the familiar symbols for the Lie algebra generators $P_a, Q_\alpha, J_{ab}$ rather than the Lie derivative symbols $\ell_{\mathbf{t}_a}, \ell_{\mathbf{t}_\alpha}, \ell_{\mathbf{t}_{ab}}$. Moreover, we have normalized the generators corresponding to the extended Lie derivatives as

$$Z^{ab} = 4\,\ell_{V^a V^b \mathbf{t}}, \quad Q^{a\alpha} = 4\,\ell_{V^a \psi^\alpha \mathbf{t}}.$$

Notice that when all curvatures vanish, the extended Lie derivative $\ell_{\psi^\alpha \psi^\beta \mathbf{t}}$ has null action on all the FDA fields, (indeed the only nontrivial action $\ell_{\psi^\alpha \psi^\beta \mathbf{t}} A$ is proportional to the spin connection, which vanishes in flat space). Thus we can set $Z^{\alpha\beta} = 4\,\ell_{\psi^\alpha \psi^\beta \mathbf{t}} = 0$, and the commutator $[P_a, P_b]$ can be taken to be vanishing.

The third line of (3.6) reproduces the supersymmetry commutations of M-theory in presence of 2-branes.

Finally, we give the Cartan-Maurer equations of the Lie algebra (3.6):

$$\begin{aligned}
& d\omega^{ab} - \omega^{ac}\omega^{cb} = 0\,\left[= R^{ab}\right], \\
& dV^a - \omega^{ab}V^b - \frac{i}{2}\bar{\psi}\Gamma^a\psi = 0\,\left[= R^a\right], \\
& d\psi - \frac{1}{4}\omega^{ab}\Gamma^{ab}\psi = 0\,\left[= \rho\right], \\
& dB^{ab} - \omega^{ac}B^{cb} + \omega^{bc}B^{ca} - \frac{1}{2}\bar{\psi}\Gamma^{ab}\psi = 0\,\left[= T^{ab}\right], \\
& d\eta^a - \omega^{ac}\eta^c - \frac{1}{4}\omega^{cd}\Gamma_{cd}\eta^a + 2C\Gamma^{ab}\psi V^b - 2i\,C\Gamma^c\psi B^{ac} = 0\,\left[= \Sigma^a\right],
\end{aligned} \tag{3.7}$$

where the bosonic one-form $B^{ab}$ and spinor vector one-form $\eta^{a\alpha}$ correspond to the generators $Z^{ab}$ and $Q^{a\alpha}$. The closure of this algebra (or equivalently the Jacobi identities for the structure constants of the Lie algebra (3.6)) can be easily checked by use of the $D = 11$ Fierz identity:

$$\Gamma^{ab}\psi\bar{\psi}\Gamma^{b}\psi - \Gamma^{b}\psi\bar{\psi}\Gamma^{ab}\psi = 0,$$

the only nontrivial check concerning the $d\eta^{a}$ equation in (3.7).

## 4 Conclusions

Generalizing the results of a previous paper [6], we have further developed an understanding of FDAs in terms of ordinary Lie algebras. In particular, the symmetries gauged by antisymmetric tensors are generated by the extended Lie derivatives introduced in Section 2.

The complete diffeomorphism algebra of FDAs containing a $p$-form has been obtained, both for the soft and rigid FDAs. As in ordinary group manifolds, the diffeomorphism algebra reduces in the rigid case to a Lie algebra.

We have applied these results to $D = 11$ supergravity, and recovered the symmetry algebra of the theory, including the symmetries gauged by the three-form field. Taking its rigid limit yields the Lie algebra of Section 3, containing the supertranslation generators $P_a$, $Q_\alpha$, the Lorentz generators $J_{ab}$, the familiar pseudo-central charge $Z^{ab}$ and an additional spinor-vector charge $Q^{a\alpha}$.

If this algebra can be gauged via the usual procedure of [4,5,10,22] (and there is no a priori reason why it could not) the resulting theory would provide a new formulation of $D = 11$ supergravity in terms of the one-form fields associated to the Lie algebra generators (i.e. vielbein $V^a$, spin connection $\omega^{ab}$, gravitino $\psi$, bosonic one-form $B^{ab}$, spinor-vector one-form $\eta^a$).

We should mention that the D'Auria-Fré algebra of [10] has so far resisted attempts to gauge it: a formulation of $D = 11$ supergravity in terms of the superPoincaré fields, bosonic 1-forms $B^{ab}$, $B^{a_1\cdots a_5}$ and an additional spinor $\eta$ still does not exist. Some recent references on this issue (and on the use of the D'Auria-Fré algebra in M-theory considerations) can be found in [1,2,13,14].

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