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Jordan δ-Derivations of Associative Algebras

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Abstract

We described the structure of jordan δ-derivations and jordan δ-prederivations of unital associative algebras. We gave examples of nonzero jordan $\frac{1}{2}$-derivations, but not $\frac{1}{2}$-derivations.

Keywords: δ-derivation; Jordan δ-derivation; Associative algebra; Triangular algebras

Introduction

Let Jordan δ-derivation be a generalization of the notion of jordan derivation [1,2] and δ-derivation [3-14]. Jordan δ-derivation is a linear mappings j, for a fixed element of δ from the main field, satisfies the following condition

$$j(x^2)=\delta(j(x)x+xj(x)). \qquad (1)$$

Note that various generalizations of Jordan derivations have been widely studied [15-17]. If algebra A is a (anti) commutetive algebra, then jordan δ-derivation of A is a δ-derivation of A.

In this paper we consider jordan δ-derivations of associative unital algebras. Naturally, we are interested in the nonzero mappings with $\neq 0,1$, 1 and algebras over field with characteristic $p\neq 2$. In the main body of work, we using the following standard notation

$$[a,b]=ab-ba, a\circ b=ab+ba.$$

Jordan δ-derivations of Associative Algebras

In this chapter, we consider jordan δ-derivations of associative unital algebras. And prove, that jordan δ-derivation of simple associative unital algebra is a δ- derivation. Also, we give the example of non-trivial jordan $\frac{1}{2}$-derivations.

Lemma: Let A be an unital associative algebra and j be a jordan δ-derivation, then $\delta=\frac{1}{2}$ and $j(x)=\frac{1}{2}(xa+ax)$, where $[x, [x, a]] = 0$ for any $x \in A$.

Proof: Let $x = 1$ in condition (1), then $j(1) = 0$ or $\delta=\frac{1}{2}$. If $j(1) = 0$, then for $x = y + 1$ in (1), we get

$$j(y\cdot 1+1\cdot y)=\delta(j(y)\cdot 1+1\cdot j(y)+j(1)\cdot y+y\cdot j(1)).$$

That is, if $j(1) = 0$, then $j(y) = 0$.

If $\delta=\frac{1}{2}$ and j(1) = a, then $j(x)=\frac{1}{2}(xa+ax)$. Using the identity (1), obtain

$$2x^2\circ a=(x\circ a)x+x(x\circ a)$$

and

$$x^2a+ax^2=2xax.$$

That is $[x, [x, a]] = 0$. Lemma is proved.

It is easy to see, that mapping $j(x)=\frac{1}{2}(xa+ax)$, where $[x, [x, a]] = 0$ for any $x \in A$, is a jordan $\frac{1}{2}$-derivation. Using Kaygorodov et al. [6] $\frac{1}{2}$-derivation of unital associative algebra A is a mapping R_a, where R_a-multiplication by the element in the center of the algebra A.

Below we give an example of an unital associative algebra with a Jordan $\frac{1}{2}$- derivation, different from $\frac{1}{2}$-derivation.

Example: Consider the algebra of upper triangular matrices of size 3×3 with zero diagonal over a non-commutative algebra B. Let $A^\#$ be an algebra with an adjoined identity for the algebra A. Then, easy to see, that for any elements $X, Y \in A^\#$, right $[X, Y] = me_{13}$ for some $m \in B$. So, for $a = t(e_{12} + e_{21})$ and $t \in B$, will be $[A^{\#},a]\neq 0$, but $[X, [X, a]] = 0$. So, using corollary from Lemma, mapping $j(x)=\frac{1}{2}(ax+xa)$ is a jordan $\frac{1}{2}$ - derivation of algebra $A^\#$, but not $\frac{1}{2}$ - derivation of algebra $A^\#$ and $a\notin Z(A^{\#})$.

Theorem 1: Jordan δ-derivation of simple unital associative algebra A is a δ-derivation.

Proof: Note, that case of $\delta = 1$ was study in Herstein et al. Cusack et al. [1,2]. It is clear, that the case $\delta=\frac{1}{2}$ is more interesting. Using Herstein et al. [18], $L=A^{(-)}/Z(A)$ is a simple Lie algebra. Clearly, that $[[a, x], x] = 0$ and $[[x, a], a] = 0$. Using roots system of simple Lie algebra [19], we can obtain, that $a \in Z(A)$, so $[A, a] = 0$. Which implies that the mapping j is a $\frac{1}{2}$- derivation. Theorem is proved.

Jordan δ-pre-derivations of Associative Algebras

Linear mapping ζ be a prederivation of algebra A, if for any elements $x, y, z \in A$:

$$\zeta(xyz) = (x)yz + x\zeta(y)z + xy\zeta(z).$$

Prederivations considered in Burde and Bajo et al. [20, 21]. Jordan δ-prederivation ς is a linear mapping, satisfies the following condition

$$\varsigma(x^3)=\delta(\varsigma(x)xx+x\varsigma(x)x+xx\varsigma(x)). \qquad (2)$$

The main purpose of this section is showing that Jordan δ-prederivation of unital associative algebra is a jordan derivation or

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jordan $\frac{1}{2}$ - derivation.

Theorem 2: Let ς be a jordan δ-prederivation of unital associative algebra A, then ς is a jordan $\frac{1}{2}$ - derivation or jordan derivation.

Proof: Note, that if ς is a jordan δ-prederivation, then $\varsigma(1)=3\delta\varsigma(1)$. So, $\varsigma(1)=0$ or $\delta=\frac{1}{3}$. If $\delta=\frac{1}{3}$, then

$$\varsigma(x^3+3x^2+3x+1)=\frac{1}{3}(x^2+2x+1)\varsigma(x+1)+(x+1)\varsigma(x+1)(x+1)+\varsigma(x+1)(x^2+2x+1)).$$

That is, we have

$$9\varsigma(x^2)+6\varsigma(x)=3x\circ\varsigma(x)+3\varsigma(1)\circ x+x^2\circ\varsigma(x)+x\varsigma(x)x.$$

Replace x by $x + 1$, then obtain

$$2\varsigma(x)=x\circ\varsigma(1)=x\circ a.$$

So, using (2), we obtain

$$x^3\circ a=\frac{1}{3}(x^2(x\circ a)+x(x\circ a)x+(x\circ a)x^2).$$

That is

$$x^3a+ax^3=x^2ax+xax^2.$$

We easily obtain

$$[x^2,[x,a]]=0.$$

Replace x by $x + 1$, then obtain $[x, [x, a]] = 0$. Using Lemma, we obtain that ς is a jordan $\frac{1}{2}$-derivation. The case $\varsigma(1) = 0$ is treated similarly, and the basic calculations are omitted. In this case, we obtain that ς is a jordan derivation (for $\delta = 1$) or zero mapping. Theorem is proved.

Jordan δ-derivations of Triangular Algebras

Let A and B be unital associative algebras over a field R and M be an unital (A, B)-bimodule, which is a left A-module and right B-module. The R-algebra

$$T=Tri(A,B,M)=\left\{\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} : a\in A, b\in B, m\in M\right\}$$

under the usual matrix operations will be called a triangular algebra. This kind of algebras was first introduced by Chase [22]. Actively studied the derivations and their generalization to triangular algebras [15- 17, 23].

Triangular algebra is an unital associative algebra and triangular algebras satisfy the conditions of Lemma. So, if j is a jordan δ-derivation of algebra T, then $\delta=\frac{1}{2}$ and there is C, which $j(X)=\frac{1}{2}(CX+XC)$, where $C=\begin{pmatrix} a & m* \\ 0 & b \end{pmatrix}$ for any $X \in T$.

Also,

$$\left[\left[\begin{pmatrix} a & m_* \\ 0 & b \end{pmatrix},\begin{pmatrix} x & m \\ 0 & y \end{pmatrix}\right],\begin{pmatrix} x & m \\ 0 & y \end{pmatrix}\right]=0$$

for any $x\in A, y\in B, m\in M$. Easy to see, mapping $j_A: A\to A$, satisfing condition $j_A(x)=\frac{1}{2}(ax+xa)$ and $j_B: B\to B$, satisfing condition $j_B(x)=\frac{1}{2}(bx+xb)$, are jordan $\frac{1}{2}$ -derivations, respectively, of algebras A and B. Also, for $m = 0, y = 0$ and $x = 1_A$ we can get

$$m_* = 0. \tag{3}$$

On the other hand, for $x = 0$ and $y = 1_B$, we can get

$$mb = am. \tag{4}$$

Theorem 3: Let A and B be a central simple algebras, then jordan δ-derivation of triangular algebra T is a δ-derivation.

Proof: T is an unital algebra and we can consider case of $\delta=\frac{1}{2}$. Algebras A and B are central simple algebras, then $a=\alpha\cdot 1_A$ and $b=\beta\cdot 1_B$. Using (4), we obtain $a=\alpha\cdot 1_A, b=\alpha\cdot 1_B$. So, jordan $\frac{1}{2}$ - derivation of T is a $\frac{1}{2}$ - derivation.

Theorem is proved.

Theorem 4: Let A be a central simple algebra and M be a faithful module right B-module, then jordan δ-derivation of triangular T is a δ-derivation.

Proof: T is an unital algebra and we can consider case of $\delta=\frac{1}{2}$. Algebra A is a central simple algebra, then $a=\alpha\cdot 1_A$. Using (4), we obtain $\alpha m = mb$. The module M is a faithful module, we have $b=\alpha\cdot 1_B$. So, jordan $\frac{1}{2}$ - derivation is a $\frac{1}{2}$ -derivation. Theorem is proved.

Comment: Noted, using the example if non-trivial jordan $\frac{1}{2}$ -derivation, but not $\frac{1}{2}$ -derivation, of unital associative algebra, we can construct new example of non-trivial jordan $\frac{1}{2}$ -derivation of triangular algebra. For example, we can consider triangular algebra $Tri(A^\#, A^\#, A^\#)$, where $A^\#$ is a bimodule ovar $A^\#$. In conclusion, the author expresses his gratitude to Prof. Pavel Kolesnikov for interest and constructive comments.

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