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Invasive Species Model with Linear Rat Harvesting on Easter Island

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Abstract

The invasive species model describes the connections between three species: People, trees and rats. In 2008, Basener, Brooks, Radin and Wiandt presented an article in that, they created a mathematical model for such dynamical system. In this work we changed the model and investigated the equilibrium points and stability of the invasive species model with harvesting. We have shown that the system has a conditionally stable equilibrium point, in this case the three populations live together. We made numerical simulations, too, and saw the amount of the rats decrease because of the harvesting.

Keywords: Dynamical systems; Mathematical biology; Invasive species model; Nonlinear ODE system; Stability

MSC: 37N25, 37N30, 65M12, 92B05, 93D05

Introduction

The history of the civilization on Easter Island has long interested archaeologists. The Easter Island is located in the Pacific Ocean, at the southeastern point of the Polynesian Triangle. It is famous for the culture and monumental stone statues, so called Moai. We know from archeological records, that at the time of the initial settlement, the island had many species of trees, e.g.,, palm species which grew up to 15 meters or more. In 1786 comte de La Perouse's visited to the island and found only 2000 inhabitants and no trees. People used the trees for construction and transportation of statues. An other factor was the extinction of plants species, was the appearance of the Polynesian rat. Studies have shown the dramatically effect of the rats in the ecosystem. These factors caused the population to collapse.

In 2008, Basener, Brooks, Radin and Wiandt presented an article in that, they created a mathematical model [1]. The invasive species model is a system of three differential equations, which describes the relations between the people, trees and rats. In this model, it is assumed that the amount of the resources available for the people is proportional to the number of trees. The growth rate of the human population is defined by the logistic equation. Analogically, the growth rate of the rat population is defined by the similar logistic equation where the carrying capacity is the amount of the trees. In the equation for the rat population we assume that the rats eat the seeds of the trees and the humans also decrease the amount of the trees. The stability property of this model is investigated both theoretically and numerically.

In this paper we suggest a natural modification of this model. Namely, we will investigate the case where the amount of the rats is decreased due to some external factor, e.g., exterminations by the people. We investigated the effect of this new added factor to the stability property of the model. We show that the system has a conditionally stable equilibrium point, and in this case the three populations live together. We also made numerical simulations with explicit numerical solvers which supports the theoretical result, namely, that the amount of the rats decreases because of the harvesting. The paper is organized as follows.

We gave the description of the invasive species model, given by Basener, Brooks, Radin and Wiandt. We define the equilibrium points for the system and investigate their stability. We analyze their stability property, by using Roughgarden theorem. We construct the discrete models by using the explicit Euler method on uniform mesh. We examine four different cases and the numerical results confirm our theoretical results. The paper is finished with some conclusion. We added an Appendix to the paper which includes some technically complex calculations in order to check the conditions of the stability.

Invasive Species Model

The invasive species model describes the growth rate of the human population, the tree resource and the rat population. Trees are the primary resources for people, they build houses and canoes for the fishing and transporting the statues. The trees represent the primary resource for the rats, too.

In this model, we assume that the amount of the resources available for the people is proportional to the number of trees. The growth rate of the human population is defined by the logistic equation, in which the carrying capacity is the amount of trees:

$$\frac{dP}{dt} = aP\left(1-\frac{P}{T}\right). \tag{1}$$

The seeds of the trees are nutriment for the rat population. The growth rate of the rat population is defined by the logistic equation and the carrying capacity is the amount of the trees, like in the model of the human population:

$$\frac{dR}{dt} = cR\left(1-\frac{R}{T}\right). \tag{2}$$

The rats eat the seeds of the tress and the humans decrease the amount of the trees, too:

$$\frac{dT}{dt} = \frac{b}{1+fR}T\left(1-\frac{T}{M}\right)-hP. \tag{3}$$

The system of equations (1-3) yields a system of nonlinear ordinary

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differential equation. The parameter a shows the growth rate of the human population, parameter c represent the growth rate of the rat population and parameter b is the growth rate of the trees. The parameter f shows the effect of the rats, and h is the harvest by the human population. The parameter M denotes the carrying capacity of the trees. We measure P in people. The parameter T means the units of the amount of trees that would support one human. We measure R in the number of rats that would be supported by one tree unit.

In article [1] the equilibrium points of the system (1-3) and their stability are analyzed.

Definition 1: The points P^*, T^* and R^* are equilibrium points for the system (1-3), when they are solutions of the following system of algebraic equations:

$$aP^*\left(1-\frac{P^*}{T^*}\right)=0,$$
$$cR^*\left(1-\frac{R^*}{T^*}\right)=0, \tag{4}$$
$$\frac{b}{1+fR^*}T^*\left(1-\frac{T^*}{M}\right)-hP^*=0.$$

Hence, the system (4) results in the following equilibrium points:

$$\mathcal{P}_1(0,M,0) \tag{5}$$

$$\mathcal{P}_2(0,M,M) \tag{6}$$

$$\mathcal{P}_3\left(\frac{(b-h)M}{b},\frac{(b-h)M}{b},0\right) \tag{7}$$

$$\mathcal{P}_4\left(\frac{(b-h)M}{b+fhM},\frac{(b-h)M}{b+fhM},\frac{(b-h)M}{b+fhM}\right). \tag{8}$$

Our aim is to analyse the stability of these points.

Definition 2: An equilibrium solution $\mathcal{P}_i^*=\mathcal{P}_i(P^*,T^*,R^*)$ to an autonomous system of first order ordinary differential equations is called:

1. Stable if for every ε > 0, there exists δ > 0 such that every $\mathcal{Q}_i$ initial values $\|\mathcal{P}_i^*-\mathcal{Q}_i\|<\delta$, than denote the solutions $u_i^*(t)=\Phi(t,\mathcal{P}_i^*)$ and $u_i(t)=\Phi(t,\mathcal{Q}_i)$

$\|u_i^*(t)-u_i(t)\|<\varepsilon$ ·

2. Asymptotically stable if stable and $\lim_{t\to\infty}\|u_i^*(t)-u_i(t)\|=0$.

3. Unstable if is not stable.

The analysis of the stability can be done with help of the eigenvalues of the Jacobian matrix denoted by λ_i.

Theorem 1: An equilibrium point is

1. Stable, if all $\mathrm{Re}(\lambda_i) \le 0$.

2. Asymptotically stable, if all $\mathrm{Re}(\lambda_i) < 0$.

3. Unstable, if $\mathrm{Re}(\lambda_i) > 0$.

As it is well-known, an equilibrium of a three-dimensional continuous dynamical system is asymptotically stable if and only if the real parts of the three eigenvalues of the Jacobian, evaluated at the equilibrium, are negative.

The linear stability conditions are hard to check requires because it needs the knowledge of the eigenvalues of the Jacobian. The following statement gives an equaivalent formulation of this condition.

Theorem 2: The linear stability is equivalent to the conditions [2].

$$tr(J)<0$$

$$det(J)<0$$

$$\sum M(J)>0$$

$$\sum M(J)tr(J)-det(J)<0,$$

Where tr(J) denotes the trace, det(J) the determinant of the Jacobian matrix and ΣM(J) means the sum of the principal minors.

Let us analyse the stability properties of the different cases.

1. In the first case, when the amount of the trees is the carrying capacity and there aren't people and rats on the island, the point $\mathcal{P}_1$ is linearly unstable. Near the equilibrium point the amount of the people and rats decrease, and the trees increase.

2. In the case of $\mathcal{P}_2$, the trees and rats are at the carrying capacity and there are no humans. The result is similar to that of the first case, so $\mathcal{P}_2$ is linearly unstable.

3. If the equilibrium point is $\mathcal{P}_3$, there are no rats, only the human population and trees. In this situation the point $\mathcal{P}_3$ is unstable.

4. The last case is the most interesting one, where all the three populations (people, trees and rats) live together. The point $\mathcal{P}_4$ is conditionally stable and the three populations tend to the same equilibrium point.

The above results are given in article [1], where the authors investigated the stability of these cases.

Invasive Species Model with Harvesting

In this section we present a new model, which is obtained by some generalization of the system (1-3). As we have seen in Basener et al. [1] and Basener et al.[3], the human population and the rat population decrease the amount of the trees. We will investigate the case where the rats are continuously harvesting. It can be interesting, if we assume that, the amount of the rats is decreased, e.g., exterminations by the people. We assume that the harvesting function is given as - gR, this gives the number of individuals harvested per unit of the time. Hence, the systems of equations (1-3) is modified into a form which depends on both the growth rate and the harvesting rate:

$$\frac{dR}{dt}=cR\left(1-\frac{R}{T}\right)-gR. \tag{9}$$

We assume that we can decrease the amount of the rats, and the harvesting is proportional to the number of the rats. In this case instead of equation (2), we use the equation (9). With the harvesting, we would like to achieve, that the amount of the trees and the human population increase. In the presence of harvesting our model turns into the system

$$\frac{dP}{dt}=aP\left(1-\frac{P}{T}\right)$$
$$\frac{dT}{dt}=\frac{b}{1+fR}T\left(1-\frac{T}{M}\right)-hP \tag{10}$$
$$\frac{dR}{dt}=cR\left(1-\frac{R}{T}\right)-gR$$

Where a, b, M, f, h, c and g are given positive parameters. We study

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the existence and the stability of the equilibrium points of this model, which has great importance from ecological point of view.

The system has four equilibrium points, which we get by solving the following algebraic system of equations:

$$aP^*\left(1-\frac{P^*}{T^*}\right)=0,$$

$$cR^*\left(1-\frac{R^*}{T^*}\right)-gR^*=0,$$

$$\frac{b}{1+fR^*}T^*\left(1-\frac{T^*}{M}\right)-hP^*=0.$$

Hence, the equilibrium points of the system (10) are the following:

$$\mathcal{P}_5(0,M,0) \quad (11)$$

$$\mathcal{P}_6\left(0,M,M\left(1-\frac{g}{c}\right)\right) \quad (12)$$

$$\mathcal{P}_7\left(\frac{M(b-h)}{b},\frac{M(b-h)}{b},0\right) \quad (13)$$

$$\mathcal{P}_8\left(E,E,E\left(1-\frac{g}{c}\right)\right), \quad (14)$$

where we used the notation:

$$E=\frac{cM(b-h)}{cb+fhM(c-g)}. \quad (15)$$

Stability of the equilibrium points

The Jacobian-matrix of the system (10) is:

$$J(P,T,R)=\begin{pmatrix} a-\frac{2aP}{T} & \frac{aP^2}{T^2} & 0 \\ -h & \frac{b(M-2T)}{(1+fR)M} & \frac{bfT(T-M)}{(1+fR)^2 M} \\ 0 & \frac{cR^2}{T^2} & c-g-\frac{2cR}{T} \end{pmatrix}.$$

For studying the stability of the points, we use this matrix at the equilibrium points.

The stability of $\mathcal{P}_5$: The first case shows the state of the island when there are no people and rats, and the trees are at their carrying capacity. At the equilibrium point $\mathcal{P}_5$ the Jacobian has the form

$$J(\mathcal{P}_5)=\begin{pmatrix} a & 0 & 0 \\ -h & -b & 0 \\ 0 & 0 & c-g \end{pmatrix}.$$

Hence, the eigenvalues of this matrix are:

$$\lambda_1=a,$$

$$\lambda_2=-b,$$

$$\lambda_3=c-g.$$

Since $\lambda_1 > 0$, the equilibrium point $\mathcal{P}_5$ is unstable.

The stability of $\mathcal{P}_6$: In this case we analyze the island before the arrival the Polynesian settlements. There are no people on the island, only the rat population and trees. The trees are at their carrying capacity and the amount of the rats depends on the growth rate c and the harvesting rate g. At the equilibrium point $\mathcal{P}_6$ the Jacobian matrix is the following:

$$J(\mathcal{P}_6)=\begin{pmatrix} a & 0 & 0 \\ -h & \frac{-b}{1+fM\left(1-\frac{g}{c}\right)} & 0 \\ 0 & c\left(1-\frac{g}{c}\right)^2 & c-g-2c\left(1-\frac{g}{c}\right) \end{pmatrix}.$$

Hence, the eigenvalues of this matrix are:

$$\lambda_1=a,$$

$$\lambda_2=\frac{-b}{1+fM\left(1-\frac{g}{c}\right)},$$

$$\lambda_3=c-g-2c\left(1-\frac{g}{c}\right).$$

Since $\lambda_1 > 0$, the equilibrium point $\mathcal{P}_6$ is unstable.

The stability of $\mathcal{P}_7$: In this situation there are no rats on the island, only the human population and trees. We investigate the stability of the equilibrium point $\mathcal{P}_7$, the Jacobian matrix has the form

$$J(\mathcal{P}_7)=\begin{pmatrix} -a & a & 0 \\ -h & 2h-b & -f(b-h)h\frac{M}{b} \\ 0 & 0 & c-g \end{pmatrix}.$$

The eigenvalues of this matrix are:

$$\lambda_1=c-g$$

$$\lambda_{2,3}=\frac{-a-b+2h\pm\sqrt{(a-b)^2-4hb+4h^2}}{2}.$$

If the eigenvalues λ_2 and λ_3 are negative and c<g, which means that the harvesting rate of rats is larger than the growth rate of the rats, the equilibrium point $\mathcal{P}_7$ is stable.

The stability of $\mathcal{P}_8$: The second case is more interesting both mathematically and ecologically, because it is the equilibrium that corresponds to the coexistence of all three biological populations: the people, the trees and the rats. For this point the Jacobian matrix has the form

$$J(\mathcal{P}_8)=\begin{pmatrix} -a & a & 0 \\ -h & \frac{b(M-2E)}{M\left(1+fE\left(1-\frac{g}{c}\right)\right)} & \frac{bfE(E-M)}{(1+fR)^2 M} \\ 0 & c\left(1-\frac{g}{c}\right)^2 & c-g-2c\left(1-\frac{g}{c}\right) \end{pmatrix}.$$

We use Theorem (4) and we get the conditions of the stability (5). The equilibrium point $\mathcal{P}_8$ is conditionally stable.

Numerical Simulations

In this section we describe the numerical solution of the system by using the well-known explicit Euler method. We define a sequence of the meshes on the solution domain [0, L].

Let

$$\omega_\tau = \left\{ t_n = n\tau, n = 0,1,\ldots,N+1, \tau = \frac{L}{N+1} \right\} \tag{16}$$

be an equisdistant mesh, where N ∈ N and τ is the step-size of the mesh. We use the explicit Euler method to solve (10) and we get the following system:

$$\frac{P_{n+1} - P_n}{\tau} = aP_n\left(1 - \frac{P_n}{T_n}\right) \tag{17}$$

$$\frac{T_{n+1} - T_n}{\tau} = \frac{b}{1 + fR_n} T_n\left(1 - \frac{T_n}{M}\right) - hP_n \tag{18}$$

$$\frac{R_{n+1} - R_n}{\tau} = cP_n\left(1 - \frac{R_n}{T_n}\right) - gR_n, \tag{19}$$

Where P_n, T_n, R_n denote the approximation to the solution of the system at time t_n, and a, b, c, f, h, g, M are given constants. We denote by P(0), T(0) and and R(0) the initial conditions of the system: respectively

$$P(0) = P_0, \quad T(0) = T_0, \quad R(0) = R_0 \tag{20}$$

are. Formulas (17)-(20) define a one-step iteration, where in the knowledge of (P_n, T_n, R_n) we can directly calculate (P_{n+1}, T_{n+1}, R_{n+1}).

First we make numerical simulations for the case where, the harvesting rate of the rats is zeros (g=0). We choose the parameters according to the article radin 2008a. Assume that the growth rate of the human population a=0.03, the growth rate of the rats is c=10 and b=1. The carrying capacity of trees is M=12000 and the parameter h is 0.25. When f=0.001 and g=0, the system (1-3) is stable [4].

Figure 1 shows the numerical solution the system of differential

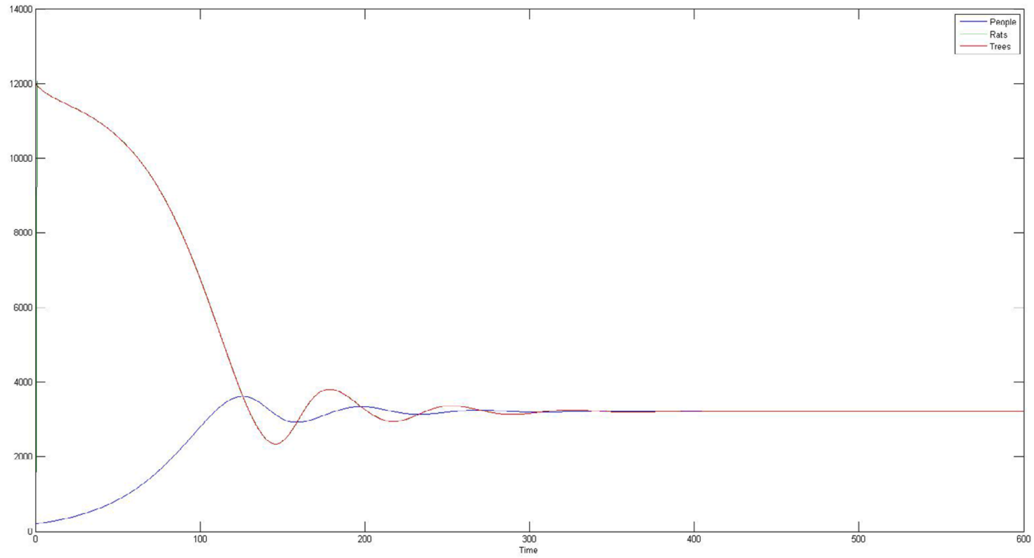

Figure 1: The numerical solution when the harvest rate of the rats is g=0.

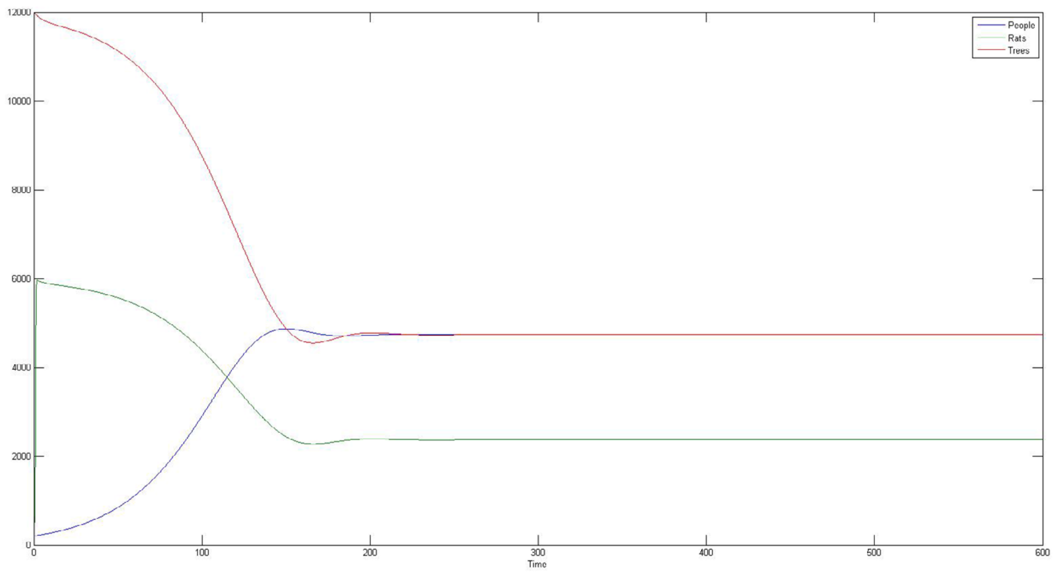

Figure 2: The numerical solution when the harvest rate of the rats is g=5.

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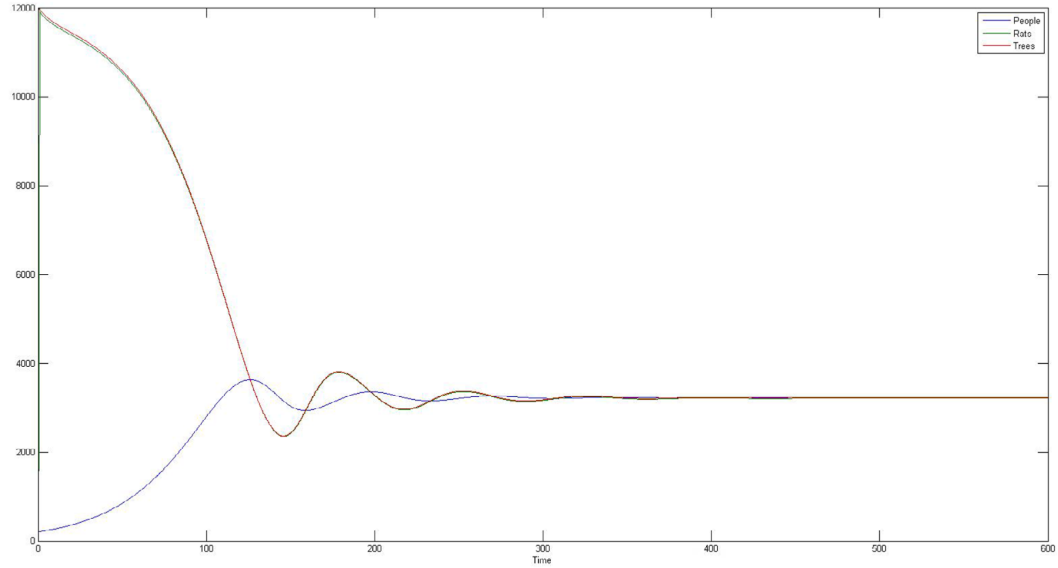

Figure 3: The numerical solution when the harvest rate of the rats is g=0.5.

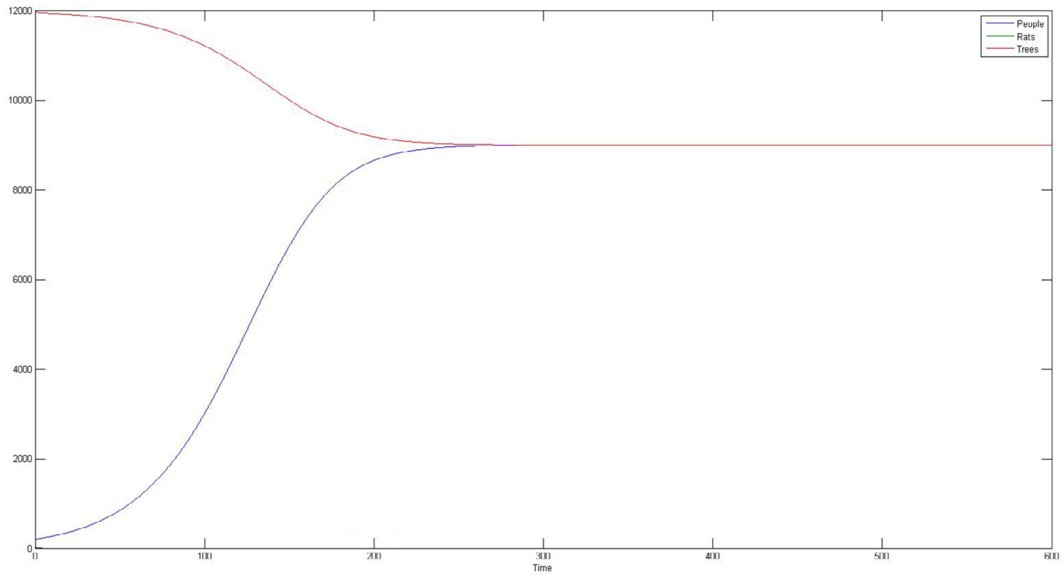

Figure 4: The numerical solution when the harvest rate of the rats is g=15.

equations. The equilibrium point is linearly stable, people, trees and rats tend to the same value.

We make numerical simulations to investigate the stability of the system. We set the growth rate of the human population to a=0.03, the growth rate of the rats to c=10 and b=1. The carrying capacity of trees is M=12000 and the parameter h is 0.25. When f=0.001 and g=5, the equilibrium point $\mathcal{P}_8$ is stable.

Figure 2 shows the numerical solution of the system of differential equations. The equilibrium point is linearly stable, people and trees tend to the same value. The harvest rate of the rat population is large, hence the size of the population decreases.

In the following case we set the growth rate of the human population to a=0.03, the growth rate of the rats to c=10 and b=1. The carrying capacity of trees is M=12000 and the parameter h is 0.25. When f=0.001 and g=0.5, the equilibrium point $\mathcal{P}_8$ is stable.

Figure 3 shows the numerical solution of the system of differential equations. The equilibrium point is linearly stable, people and trees tend to the same value. The harvest rate of the rat population is small, hence the number of the rats is close to that of the other two populations.

In the next case, we set the growth rate of the human population to a=0.03, the growth rate of the rats is c=10 and b=1. The carrying capacity of trees is M=12000 and the parameter h is 0.25. When f=0.001 and g=15, the equilibrium point $\mathcal{P}_7$ is stable.

Figure 4 shows the numerical solution of the system of differential equations. The equilibrium point is stable, people and trees tend to the same value. The harvest rate of the rat population is larger than the growth rate of the rats, hence the number of the rats tends to zero.

Conclusion

The invasive species model describes the connections between three species: people, trees and rats. The model is interesting both mathematically and from a biological point of view. We changed the model and investigated the equilibrium points and stability of the invasive species model with harvesting. We have shown that the system has a conditionally stable equilibrium point, in this case the three populations live together. We made numerical simulations, too, and saw the amount of the rats decrease because of the harvesting.

In the future we plan to extend the model in different other direction, too, by involving some further effects, e.g., the diffusion of the seeds of then trees. We also plan to develop the numerical model in different ways. Namely, by using the operator splitting method to increase the efficiency of the model. The use of implicit numerical models may increase the numerical stability of the discrete model therefore we also aim to involve such approach into the numerical modelling.

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