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Introduction to Higher Integral of Differential Equations

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Abstract

In this paper, we introduce higher integral of differential equations. Also, we solve some higher integral of differential equations. Moreover, we show all solutions of some higher integral of differential equations are also solutions of those differential equations.

Keywords: Higher integral; Coefficient functions

Introduction

In this paper, we introduce higher integral of differential equations. In addition, we derive and solve some higher integral of differential equations of order.

A differential equation is an equation that relates an unknown function and one or more of its derivatives of with respect to one or more independent variables [1]. If the unknown function depends only on a single independent variable, such a differential equation is ordinary differential equation. If the unknown function depends only on many independent variables, such a differential equation is partial differential equation. An ordinary differential is linear if it is linear in the unknown function and its derivatives that involve in it. The order of an ordinary differential equation is the order of the highest derivative that appears in the equation [2]. Moreover, differential equations are classified into two main categories. The first one is ordinary differential equations and the other is partial differential equations.

A solution of a differential equation in the unknown function y and the independent variable x on the interval I is a function *y(x)* that satisfies the differential equation identical for all x in I [1]. A solution of a differential equation with arbitrary parameters is called a general solution. A solution of a differential equation that is free of arbitrary parameters is called a particular solution [1]. A solution in which the de-pendent variable is expressed solely in terms of the independent variable and constants is said to be an explicit solution. A relation *G(x,y)* is said to be an implicit solution of an ordinary differential equation on an interval I, provided there exists at least one function f that satisfies the relation as well as the DE on I [1]. Moreover, solution of differential equations is classified as trivial and non-trivial solutions, general and particular solutions and explicit and implicit solutions.

Now we define exact $n-th$ order linear ordinary differential equations. An exact $n-th$ order linear ordinary differential equation is a linear ordinary differential equation which is derived by differentiating a linear ordinary differential equation of order $n-1$. The first integral of an exact $n-th$ order linear ordinary differential equation is a linear ordinary differential equation of order $n-1$ that gives an exact $n-th$ order linear ordinary differential equation by its differentiation [3]. Finally, we introduce and solve $n-th$ order linear ordinary differential equations if their $(n-1)^{th}$ integrals exist.

Motivation

Research questions: Here we would like to raise two basic questions.

1) Does solution method for solving higher order linear differential equations in general exist?

2) Does solution method for solving $(n-1)^{th}$ stage exact differential equations of order n in general exist?

There does not exist any method to solve second order linear differential equations except in a few rather restrictive cases [4]. Thus, there does not exist any method to solve higher order linear differential equations in general except in a few rather restrictive cases. We would like to develop theories on $(n-1)^{th}$ stage exact differential equations of order n to answer the second question.

Research method

Mainly, we use mathematical induction method to derive the $(n-1)^{th}$ integral of differential equations of order n. In addition, we apply integration techniques that help to show the validity of induction step. Activity Find the general solution of $2y-2xy^{(1)}+x^2y^{(2)}-x^3y^{(3)}=x^4\exp x$

Linear First Order Differential Equation and Its Solution

The linear first order ordinary differential equation with unknown dependent variable y and independent variable x is defined by

$$a_0(x)y+a_1(x)y^{(1)}=g(x)\cdot \qquad (4.1)$$

The general solution of the equation in equation 4.1 is given by

$$y=\frac{\int\frac{\mu(x)g(x)}{a_1}dx}{\mu(x)}, \qquad (4.2)$$

where $\mu(x)=\exp(\int(\frac{a_0(x)}{a_1(x)})dx)$ [5]

Lemma 4.1: Let *f (x)* and *g(x)* be continuously differentiable functions of x. Then

$$\int f(x)g^{(n)}(x)dx=\int g(x)f^{(n)}(x)(-1)^n dx+\sum_{k=1}^{n}[g^{(n-k)}(x)f^{(k-1)}(x)(-1)^{k+1}] \qquad (4.3)$$

$\forall n=1,2,3,\cdots$

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Proof:

we apply mathematical induction method on n to prove this lemma.

step 1

for *n = 1*, the result is true by integration by parts method.

step 2

suppose that

$$\int f(x)g^{(n)}(x)dx = \int g(x)f^{(n)}(x)(-1)^n dx + \sum_{k=1}^{n}[g^{(n-k)}(x)f^{(k-1)}(x)(-1)^{k+1}] \quad (4.4)$$

step 3

$$\int f(x)g^{(n+1)}(x)dx = \int f(x)(g^{(1)})^{(n)}(x)dx$$

$$= \int g^{(1)}(x)f^{(n)}(x)(-1)^n dx$$

$$+ \sum_{k=1}^{n}[g^{(n+1-k)}(x)f^{(k-1)}(x)(-1)^{k+1}]$$

$$= g(x)f^{(n)}(x) - \int g(x)f^{(n+1)}(x)(-1)^n dx$$

$$+ \sum_{k=1}^{n}[g^{(n+1-k)}(x)f^{(k-1)}(x)(-1)^{k+1}]$$

$$= \int g(x)f^{(n+1)}(x)(-1)^{n+1} dx$$

$$+ \sum_{k=1}^{n+1}[g^{(n+1-k)}(x)f^{(k-1)}(x)(-1)^{k+1}]$$

Hence proved

Exact Higher Order Differential Equations

First integral of exact higher order differential equations

Definition 1 (Exact higher order differential equations): The $n-th$ order linear ordinary differential equation $G_0(x, y, y^{(1)}, \cdots, y^{(n-1)}, y^{(n)}) = 0$ is said to be exact differential equation [3] if there exists *(n – 1) – th* order linear ordinary differential equation $G_1(x, y, y^{(1)}, \cdots, y^{(n-2)}, y^{(n-1)}) = 0$ such that

$$\frac{d}{dx}(G_1(x, y, y^{(1)}, \cdots, y^{(n-2)}, y^{(n-1)})) = G_0(x, y, y^{(1)}, \cdots, y^{(n-1)}, y^{(n)}) .$$

Here $G_1(x, y, y^{(1)}, \cdots, y^{(n-2)}, y^{(n-1)}) = 0$ is called the first integral of $G_0(x, y, y^{(1)}, \cdots, y^{(n-1)}, y^{(n)}) = 0$.

Definition 2 (The exactness condition of higher order differential equations): The exactness condition of *n – th* order linear ordinary differential equation $a_{11}y + a_{12}y^{(1)} + \quad + a_{1(n)}y^{(n \ 1)} + a_{1(n \ 1)}y^{(n)} = h(x)$ is $\sum_{j=1}^{n+1}[a_{(1)(j)}^{(j-1)}(-1)^{j+1}] = 0$ [3].

Lemma 5.1 (The first integral of exact higher order linear ordinary differential equation): The first integral of exact *n – th* order linear ordinary differential equation [3]

$a_{11}y + a_{12}y^{(1)} + \cdots + a_{1(n)}y^{(n-1)} + a_{1(n+1)}y^{(n)} = h(x)$ is

$b_{11}y + b_{12}y^{(1)} + \cdots + b_{1(n)}y^{(n-1)} = \int h(x)dx + c$, where

$$b_{1j} = \sum_{k=1}^{n+1}[a_{(1)(j+k)}^{(k-1)}(-1)^{k+1}] = 0 \forall j = 1, 2, \cdots, n \cdot$$

Multistage exact higher order differential equations

Definition 3 (The *k-th* stage exact higher order differential equations): The $n-th$ order linear ordinary differential equation

$G_0(x, y, y^{(1)}, \cdots, y^{(n-1)}, y^{(n)}) = 0$ is said to be the $k-th$ stage exact differential equation

if $G_{i+1}(x, y, y^{(1)}, \cdots, y^{(n-i-2)}, y^{(n-i-1)}) = 0$ is the first integral of $G_i(x, y, y^{(1)}, \cdots, y^{(n-i-1)}, y^{(n-i)}) = 0$ ∀i = 0, $\forall i = 0, 1, 2, \cdots, k-1$

and

$$G_0(x, y, y^{(1)}, \cdots, y^{(n-1)}, y^{(n)}) = \frac{d^k}{dx^k}[G_k(x, y, y^{(1)}, \cdots, y^{(n-k-1)}, y^{(n-k)})] .$$

Here $G_k(x, y, y^{(1)}, \cdots, y^{(n-k-1)}, y^{(n-k)}) = 0$ is called the $k-th$ integral of $G_0(x, y, y^{(1)}, \cdots, y^{(n-1)}, y^{(n)}) = 0 \cdot$

Definition 4 (Higher integral of functions of single variable): The n-th integral of continuously integrable function h(x), denoted by $h^{<n>}$, is defined by $h^{<n>} = \int h^{<n-1>} dx$. Here note that $h^{<0>} = h(x) \cdot$

Note: A differential equation of order n has its $(n-1)^{th}$ integral if and only if it is $(n-1)^{th}$ stage exact differential equation.

Definition 5 (Recursive coefficient functions of multistage exact higher order differential equations): Let $a_{1j}(x)(j = 1, 2, ..., n)$ be continuously differentiable coefficient functions of the $(n-1)^{th}$ Stage Exact $(n-1)^{th}$ order linear nonhomogeneous DE

$$a_{11}y + a_{12}y^{(1)} + \cdots + a_{1(n-1)}y^{(n-2)} + a_{1(n)}y^{(n-1)} = h(x) \quad (5.1)$$

Then we define the continuously differentiable coefficient function $a_{ij}(x)$ of the reduced $(n-i)^{th}$ order linear nonhomogeneous DE

$$a_{i1}y + a_{i2}y^{(1)} + \cdots + a_{in-i}y^{(n-i-1)} + a_{i(n+1-i)}y^{(n-i)} = \sum_{k=2}^{i}(\frac{c_k x^{i-k}}{(i-k)!}) + h^{<i-1>} \quad (5.2)$$

by

$$a_{ij}(x) = \sum_{k=1}^{n+2-i-j}[a_{(i-1)(j+k)}^{(k-1)}(-1)^{k+1}] \quad (5.3)$$

$(i = 2, ..., n)$ and $(j = 1, 2, ..., n+1-i)$

Lemma 5.2:

$$a_{(i+1)(j-1)}(x) + a_{(i+1)}(j)^{(1)}(x) = a_{ij}(x)$$

Proof:

It follows from the definition of $a_{ij}(x)$ that

$$a_{(i+1)(j-1)}(x) + a_{(i+1)}(j)^{(1)}(x) = a_{ij}(x) .$$

Lemma 5.3:

$$a_{(i+1)(j)}(x)(for n = r+1) = a_{(i+1)(j)}(x)(for n = r) + a_{(i)(r+2-i)}^{(r+1-i-j)}(-1)^{r+3-i-j}$$

Proof:

It follows from the definition of $a_{ij}(x)$ that

$$a_{(i+1)(j)}(x)(for n = r+1) = a_{(i+1)(j)}(x)(for n = r) + a_{(i)(r+2-i)}^{(r+1-i-j)}(-1)^{r+3-i-j}$$

Lemma 5.4:

$$a_{(i+1)(r+1-i)}(x) = a_{(i)(r+2-i)}(x)$$

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Proof:

For $n = r + 1$, it follows from the definition of $a_{ij}(x)$ that $a_{(i+1)(r+1-i)}(x) = a_{(i)(r+2-i)}(x)\cdot$

Theorem 5.5:

$$\int [\sum_{j=1}^{n-i+1} (a_{(i)(j)} y^{(j-1)})]dx$$

$$= \sum_{j=1}^{n-i} (a_{(i+1)(j)} y^{(j-1)})$$

$$+ \int [\sum_{j=1}^{n+1-i} (a_{(i)(j)}^{(j-1)} (-1)^{j+1})] y dx \forall i = 1, 2, \cdots, n-2$$

Proof:

We apply mathematical induction method on n to prove this theorem.

Step 1

We shall show this holds for $n = 5$. $i = 1$ when $n = 5$. Let's consider the following expression.

$$\int [\sum_{j=1}^{3} (a_{(1)(j)} y^{(j-1)})] =$$

$$= \sum_{j=1}^{3} [\int (a_{(1)(j)} y^{(j-1)})$$

$$= \int (a_{(1)(1)} y) dx + \int (a_{(1)(2)} y^{(1)}) dx + \int (a_{(1)(3)} y^{(2)}) dx$$

$$= \int (a_{(1)(1)} y) dx + a_{(1)(2)} y - \int (a_{(1)(2)}^{(1)} y) dx$$

$$+ a_{(1)(3)} y^{(1)} - \int (a_{(1)(3)}^{(1)} y^{(1)}) dx$$

$$= \int (a_{(1)(1)} y) dx + a_{(1)(2)} y - \int (a_{(1)(2)}^{(1)} y) dx$$

$$+ a_{(1)(3)} y^{(1)} - [a_{(1)(3)}^{(1)} y - \int (a_{(1)(3)}^{(2)} y) dx$$

$$= \int (a_{(1)(1)} y) dx + a_{(1)(2)} y - \int (a_{(1)(2)}^{(1)} y) dx$$

$$+ a_{(1)(3)} y^{(1)} - a_{(1)(3)}^{(1)} y + \int (a_{(1)(3)}^{(2)} y) dx$$

$$= \int [(a_{(1)(1)} - a_{(1)(2)}^{(1)} + a_{(1)(3)}^{(2)}) y] dx + a_{(1)(2)} y$$

$$+ a_{(1)(3)} y^{(1)} - a_{(1)(3)}^{(1)} y$$

$$= \int [(a_{(1)(1)} - a_{(1)(2)}^{(1)} + a_{(1)(3)}^{(2)}) y] dx + [a_{(1)(2)} - a_{(1)(3)}^{(1)}] y$$

$$+ a_{(1)(3)} y^{(1)}$$

$$= \int [(a_{(1)(1)} - a_{(1)(2)}^{(1)} + a_{(1)(3)}^{(2)}) y] dx + a_{(2)(1)} y$$

$$+ a_{(2)(2)} y^{(1)}$$

Thus, for $n=3$ this theorem holds.

Step 2

Suppose that the theorem is true for $n=r$.

That is,

$$\int [\sum_{j=1}^{r-i+1} (a_{(i)(j)} y^{(j-1)})]$$

$$= \sum_{j=1}^{r-i} (a_{(i+1)(j)} y^{(j-1)})$$

$$+ \int [\sum_{j=1}^{r+1-i} (a_{(i)(j)}^{(j-1)} (-1)^{j+1})] y dx \forall i = 1, 2, \cdots, r-2$$

step 3

$$\int [\sum_{j=1}^{r-i+2} (a_{(i)(j)} y^{(j-1)})]$$

$$= \int [a_{(i)(r+2-i)} y^{(r+1-i)} + \sum_{j=1}^{r-i+1} (a_{(i)(j)} y^{(j-1)})] dx$$

$$= \int [a_{(i)(r+2-i)} y^{(r+1-i)}], dx + \int [\sum_{j=1}^{r-i+1} (a_{(i)(j)} y^{(j-1)})] dx$$

$$= \sum_{j=1}^{r-i} (a_{(i+1)(j)} y^{(j-1)})$$

$$+ \int [\sum_{j=1}^{r+1-i} (a_{(i)(j)}^{(j-1)} (-1)^{j+1})] y dx$$

$$+ \int [(a_{(i)(r+2-i)}^{(r+1-i)} (-1)^{r+1-i}) y] dx$$

$$+ \sum_{j=1}^{r+1-i} [(a_{(i)(r+2-i)}^{(j-1)} (-1)^{j+1}) y^{(r+1-i-j)}]$$

$$= \sum_{j=1}^{r-i} (a_{(i+1)(j)} y^{(j-1)})$$

$$+ \int [\sum_{j=1}^{r+2-i} (a_{(i)(j)}^{(j-1)} (-1)^{j+1})] y dx$$

$$+ \sum_{j=1}^{r+1-i} [(a_{(i)(r+2-i)}^{(j-1)} (-1)^{j+1}) y^{(r+1-i-j)}] \text{ by Lemma 2.4 and step2}$$

$$= \sum_{j=1}^{r-i} (a_{(i+1)(j)} y^{(j-1)})$$

$$+ \int [\sum_{j=1}^{r+2-i} (a_{(i)(j)}^{(j-1)} (-1)^{j+1})] y dx$$

$$+ \sum_{j=1}^{r+1-i} [(a_{(i)(r+2-i)}^{(r+1-i-j)} (-1)^{r+1-i-j}) y^{(j-1)}] \text{ by letting } s-1 = r+1-i-j$$

$$= \sum_{j=1}^{r-i} (a_{(i+1)(j)} y^{(j-1)})$$

$$+ \int [\sum_{j=1}^{r+2-i} (a_{(i)(j)}^{(j-1)} (-1)^{j+1})] y dx$$

$$+ \sum_{j=1}^{r-i} [(a_{(i)(r+2-i)}^{(r+1-i-j)} (-1)^{r+1-i-j}) y^{(j-1)}]$$

$$+ a_{(i)(r+2-i)} y^{(r-i)}$$

$$= \sum_{j=1}^{r-i}[(a_{(i+1)(j)}y^{(j-1)} + (a_{(i)(r+2-i)}^{(r+1-i-j)}(-1)^{r+1-i-j})y^{(j-1)})]$$

$$+\int[\sum_{j=1}^{r+2-i}(a_{(i)(j)}^{(j-1)}(-1)^{j+1})]ydx$$

$$+a_{(i)(r+2-i)}y^{(r-i)}$$

$$=\sum_{j=1}^{r-i}[(a_{(i+1)(j)}y^{(j-1)})]$$

$$+\int[\sum_{j=1}^{r+2-i}(a_{(i)(j)}^{(j-1)}(-1)^{j+1})]ydx$$

$$+a_{(i)(r+2-i)}y^{(r-i)} by Lemma 3.3$$

$$=\sum_{j=1}^{r-i}[(a_{(i+1)(j)}y^{(j-1)})]$$

$$+\int[\sum_{j=1}^{r+2-i}(a_{(i)(j)}^{(j-1)}(-1)^{j+1})]ydx$$

$$+a_{(i+1)(r+1-i)}y^{(r-i)} by Lemma\ 3.4$$

$$=\sum_{j=1}^{r+1-i}[(a_{(i+1)(j)}y^{(j-1)})]$$

$$+\int[\sum_{j=1}^{r+2-i}(a_{(i)(j)}^{(j-1)}(-1)^{j+1})]ydx$$

Hence proved

Corollary 5.6: If $\sum_{j=1}^{n+1-i}(a_{(i)(j)}^{(j-1)}(-1)^{j+1})=0 \quad \forall i=1,2,\cdots,n-2$, then

$$\int[\sum_{j=1}^{n-i+1}(a_{(i)(j)}y^{(j-1)})]dx=\sum_{j=1}^{n-i}(a_{(i+1)(j)}y^{(j-1)})+c$$

$\forall i=1,2,\cdots,n-2$

Definition 6 (Multistage exactness conditions of multistage exact higher order differential equations): The $k-th$ stage exactness condition of $n-th$ order linear ordinary differential equation $a_{11}y+a_{12}y^{(1)}+\cdots+a_{1(n)}y^{(n-1)}+a_{1(n+1)}y^{(n)}=h(x)$ is

$$\sum_{j=1}^{n+2-k}[a_{(k)(j)}^{(j-1)}(-1)^{j+1}]=0\cdot$$

Theorem 5.7: If $\sum_{j=1}^{n+1-i}(a_{(i)(j)}^{(j-1)}(-1)^{j+1})=0\ \forall i=1,2,\cdots,n-2$, then

$\sum_{j=1}^{n}(a_{1j}y^{(j-1)})$=h(x) can be reduced to a

$$a_{(n-1)(1)}y+a_{(n-1)(2)}y^{(1)}=I^{(n-2)}(h(x))+\sum_{k=2}^{n-1}(\frac{c_kx^{n-k-1}}{(n-k-1)!})$$

Proof:

From corollary 5.6, we have

$$\int[\sum_{j=1}^{n-i+1}(a_{(i)(j)}y^{(j-1)})]dx=\sum_{j=1}^{n-i}(a_{(i+1)(j)}y^{(j-1)})+c\ \forall i=1,2,\cdots,n-2 \quad (5.4)$$

First we show that

$$\sum_{j=1}^{n-i+1}(a_{(i)(j)}y^{(j-1)})=\sum_{k=2}^{i}(\frac{c_kx^{i-k}}{(i-k)!})+I^{(i-1)}(h(x))$$

$\forall i=1,2,\cdots,n-2$.

Now we apply mathematical induction on *i*.

Step 1

For i = 2, $\sum_{j=1}^{n-1}(a_{(2)(j)}y^{(j-1)})=\int[\sum_{j=1}^{n}(a_{(1)(j)}y^{(j-1)})]dx+c=\int h(x)dx+c$

Step 2

Suppose that

$$\sum_{j=1}^{n-i+1}(a_{(i)(j)}y^{(j-1)})=\sum_{k=2}^{i}(\frac{c_kx^{i-k}}{(i-k)!})+I^{(i-1)}(h(x)) \text{ step 3}$$

consider

$$\sum_{j=1}^{n-i}(a_{(i+1)(j)}y^{(j-1)})=\int[\sum_{j=1}^{n-i+1}(a_{(i)(j)}y^{(j-1)})]dx+c$$

$$=\int[\sum_{k=2}^{i}(\frac{c_kx^{i-k}}{(i-k)!})dx+I^{(i-1)}(h(x))]+c$$

$$=\int[\sum_{k=2}^{i}(\frac{c_kx^{i-k}}{(i-k)!})]dx+\int(I^{(i-1)}(h(x)))dx+c$$

$$=\sum_{k=2}^{i}[\int(\frac{c_kx^{i-k}}{(i-k)!})dx]+(I^{(i)}(h(x)))+c$$

$$=\sum_{k=2}^{i}[(\frac{c_kx^{i-k+1}}{(i-k+1)!})]+(I^{(i)}(h(x)))+c$$

$$=\sum_{k=2}^{i+1}[(\frac{c_kx^{i-k+1}}{(i-k+1)!})]+(I^{(i)}(h(x))), where\ c=c_{i+1}$$

Therefore,

$$\sum_{j=1}^{n-i+1}(a_{(i)(j)}y^{(j-1)})=\sum_{k=2}^{i}(\frac{c_kx^{i-k}}{(i-k)!})+I^{(i-1)}(h(x))\forall i=1,2,\cdots,n-2 \quad (5.5)$$

Consider the following equation.

$$\int[\sum_{j=1}^{n-i+1}(a_{(i)(j)}y^{(j-1)})]dx$$

$$=\int[\sum_{k=2}^{i}(\frac{c_kx^{i-k}}{(i-k)!})+I^{(i-1)}(h(x))]dx$$

$$=\sum_{k=2}^{i}[\int(\frac{c_kx^{i-k}}{(i-k)!})dx]+\int[I^{(i-1)}(h(x))]dx$$

$$=\sum_{k=2}^{i}[(\frac{c_kx^{i-k+1}}{(i-k+1)!})]+I^{(i)}(h(x))$$

Therefore,

$$\int[\sum_{j=1}^{n-i+1}(a_{(i)(j)}y^{(j-1)})]dx=\sum_{k=2}^{i}[(\frac{c_kx^{i-k+1}}{(i-k+1)!})]+I^{(i)}(h(x))\ \forall i=1,2,\cdots,n-2 \quad (5.6)$$

From equations in equation 5.4 and 5.6, we have

$$\sum_{j=1}^{n-i}(a_{(i+1)(j)}y^{(j-1)})+c=\sum_{k=2}^{i}[(\frac{c_kx^{i-k+1}}{(i-k+1)!})]+I^{(i)}(h(x))\ \forall i=1,2,\cdots,n-2 \quad (5.7)$$

In particular, for $i = n-2$ we have

$$a_{(n-1)(1)}y + a_{(n-1)(2)}y^{(1)} = I^{(n-2)}(h(x)) + \sum_{k=2}^{n-1}\left(\frac{c_k x^{n-k-1}}{(n-k-1)!}\right)$$

Hence proved

Corollary 5.8 (The k–th integral of multistage exact higher order differential equations): The k – th integral of multistage exact higher order differential equation

$a_{11}y + a_{12}y^{(1)} + \cdots + a_{1(n)}y^{(n-1)} + a_{1(n+1)}y^{(n)} = h(x)$ is $\sum_{j=1}^{n-k+1}(a_{(k+1)(j)}y^{(j-1)}) = \sum_{s=2}^{k+1}\left(\frac{c_s x^{k-s+1}}{(k-s+1)!}\right) + h^{<k>}$

Note The $(n-1)^{th}$ integral of multistage exact higher order differential equation

$a_{11}y + a_{12}y^{(1)} + \cdots + a_{1(n)}y^{(n-1)} + a_{1(n+1)}y^{(n)} = h(x)$ is

$$\sum_{j=1}^{2}(a_{(n)(j)}y^{(j-1)}) = \sum_{s=2}^{n}\left(\frac{c_s x^{n-s}}{(n-s)!}\right) + h^{<n-1>}.$$

Derivation of $(n-1)^{th}$ Integral of Differential Equations of Order n

Lemma 6.1: $a_{i(n+1-i)} = a_{1n}, \forall i = 2,3,\cdots,n$

Proof:

From definition of a_{ij}, we have

$a_{i(n+1-i)} = a_{(i+1)(n-i)}, \forall i = 2,3,\cdots,n-1$.

We apply mathematical induction method on i to prove this lemma.

Step 1

For $i = 2$, $a_{2(n-1)} = a_{1n}$, by definition of a_{ij}.

Step 2

suppose that $a_{r(n+1-r)} = a_{1n}$.

step 3

consider $a_{(r+1)(n-r)}$.

$a_{(r+1)(n-r)} = a_{(r)(n+1-r)}$, by definition of a_{ij}

$= a_{1n}$ by step 2

Hence proved

Here observe that $a_{n2} = a_{1(n+1)}$

Lemma 6.2: $a_{(n-i)i} = a_{(n-1)1} + (i-1)a_{1n}^{(1)}, \forall i = 1,2,3,\cdots,n-1$

Proof:

We apply mathematical induction method on i to prove this lemma.

step 1

For $i = 1$, the result is true.

step 2

suppose that $a_{(n-r)r} = a_{(n-1)1} + (r-1)a_{1n}^{(1)}$.

step 3

consider $a_{(n-r-1)(r+1)}$.

$a_{(n-r-1)(r+1)} = a_{(r)(n-r)} + a_{1n}^{(1)}$ *by definition of* a_{ij}

$= a_{(n-1)1} + (r-1)a_{1n}^{(1)} + a_{1n}^{(1)}$

$= a_{(n-1)1} + ra_{1n}^{(1)}$

Hence proved

Here observe that $a_{(n)1} = a_{1(n)} + (1-n)a_{1(n+1)}^{(1)}$.

Since the $(n-1)^{th}$ integral of multistage exact higher order differential equation

$a_{11}y + a_{12}y^{(1)} + \cdots + a_{1(n)}y^{(n-1)} + a_{1(n+1)}y^{(n)} = h(x)$ is

$$\sum_{j=1}^{2}(a_{(n)(j)}y^{(j-1)}) = \sum_{s=2}^{n}\left(\frac{c_s x^{n-s}}{(n-s)!}\right) + h^{<n-1>},$$

we have $a_{(n)(1)}y + a_{(n)(2)}y^{(1)} = \sum_{s=2}^{n}\left(\frac{c_s x^{n-s}}{(n-s)!}\right) + h^{<n-1>}$

Therefore $(a_{1(n)} + (1-n)a_{1(n+1)}^{(1)})y + a_{1(n+1)}y^{(1)} = \sum_{s=2}^{n}\left(\frac{c_s x^{n-s}}{(n-s)!}\right) + h^{<n-1>}$

is the $(n-1)^{th}$ integral of multistage exact higher order differential equation $a_{11}y + a_{12}y^{(1)} + \cdots + a_{1(n)}y^{(n-1)} + a_{1(n+1)}y^{(n)} = h(x)$.

Solution of $(n-1)^{th}$ Integral of Differential Equations of Order *n*

We know that $(a_{1(n)} + (1-n)a_{1(n+1)}^{(1)})y + a_{1(n+1)}y^{(1)} = \sum_{s=2}^{n}\left(\frac{c_s x^{n-s}}{(n-s)!}\right) + h^{<n-1>}$

is the $(n-1)^{th}$ integral of multistage exact higher order differential equation $a_{11}y + a_{12}y^{(1)} + \cdots + a_{1(n)}y^{(n-1)} + a_{1(n+1)}y^{(n)} = h(x)$. This equation is first order linear ordinary differential equation. Thus, its general solution is given by

$$y = \frac{\int \frac{\mu(x)g(x)}{a_1}dx}{\mu(x)}, \qquad (5.1)$$

Where $\mu(x) = \exp(\int(\frac{a_0(x)}{a_1(x)})dx)$, $g(x) = \sum_{s=2}^{n}\left(\frac{c_s x^{n-s}}{(n-s)!}\right) + h^{<n-1>}$,

$a_0(x) = (a_{1(n)} + (1-n)a_{1(n+1)}^{(1)})$ and $a_1(x) = a_{1(n+1)}$.

Result and Discussion

We derived the reduced differential equation the so called *k – th* integral of multi-stage exact higher order differential equation of order n. Let's discuss about the general solution of multistage exact higher order differential equation. We reduced the *(n –1)th* stage multistage exact higher order differential equation of order n to first or-der linear differential equation. Note that an exact higher order differential equation may or may not be a multistage exact higher order differential equation. However, all multistage exact higher order differential equations are exact higher order differential equation.

Conclusion

In this manuscript, we introduced multistage exact higher order differential equation. In particular, we reduced the $(n-1)^{th}$ stage multistage exact higher order differential equation of order n to first order linear differential equation. Moreover, we found the general solution of the $(n-1)^{th}$ stage multistage exact higher order differential equation of order n. Therefore, the general solution of the $(n-1)^{th}$ multistage exact higher order differential equation $a_{11}y + a_{12}y^{(1)} + \cdots + a_{1(n)}y^{(n-1)} + a_{1(n+1)}y^{(n)} = h(x)$ is given by

$$y = \frac{\int \frac{\mu(x)g(x)}{a_1}dx}{\mu(x)}$$

where $\mu(x) = \exp(\int (\frac{a_0(x)}{a_1(x)})dx)$, $g(x) = \sum_{s=2}^{n} (\frac{c_s x^{n-s}}{(n-s)!}) + h^{<n-1>}$,

$a_0(x) = (a_{1(n)} + (1-n)a_{1(n+1)}^{(1)})$ and $a_1(x) = a_{1(n+1)}$.

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