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Introduction to Geo*Arithmetic Series

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Abstract

In this manuscript, we introduce Geo*Arithmetic Series. Mainly, we use theories and theorems on series to find partial sum of Geo*Arithmetic Series. Moreover, we show a Geo*Arithmetic Series is convergent (divergent) series whenever the absolute value of the common ratio of its terms is less than one (greater than one). Furthermore, we find the sum of a Geo*Arithmetic Series whenever the absolute value of the common ratio of its terms is less than one.

Keywords: Terms; Partial sums; Convergence tests

Introduction

In this manuscript we introduce Geo*Arithmetic Series. There are convergent Geo*Arithmetic Series. Thus, we want to know the sum of Geo*Arithmetic Series if it converges. Moreover, we find the sum of Geo*Arithmetic Series if it converges. We define first sequence because the theory of series directly or indirectly depends on the theory of sequence. A sequence of real numbers is an ordered list of real numbers. Now, we define a series of real numbers as the sum of terms of real sequence. A given series has its partial sum, that is, the sum of the first in terms of a series is its partial sum. We want partial sum of a series because the sum of that series depends on the infinite limit of its partial sum. Moreover, we apply convergence tests for series to check whether the given series is convergent or not.

Geo*Arithmetic Series is a series for which the term is the product of the term of Geometric Series and Arithmetic Series. The main objective of this manuscript is to find the sum of Geo*Arithmetic Series if it converges. Mainly, we apply the generalized ratio test to test the convergence of Geo*Arithmetic Series. Finally, we find the sum of Geo*Arithmetic Series if it converges.

Sequences

Definition 1: A sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers is an ordered list of numbers $a_n \in R$, called the terms of the sequence [1], indexed by the natural numbers n $n \in N$.

Series

Definition 2: Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers, then the expression $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ which is denoted by $\sum_{i=1}^{\infty} a_i$, that is, $\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ is called an infinite series.

Definition 3: The Sequence of Partial Sums of $\{a_n\}_{n=1}^{\infty}$ [2].

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence and define a new sequence $\{s_n\}_{n=1}^{\infty}$ by the recursion relation $s_1 = a_1$, and $S_{n+1} = S_n + a_{n+1}$. The sequence $\{s_n\}_{n=1}^{\infty}$ is called the Sequence of partial sums of $\{a_n\}_{n=1}^{\infty}$.

Definition 4: Convergence of Series [3].

An infinite series $\sum_{n=1}^{\infty} a_n$ with sequence of partial sums $\{s_n\}_{n=1}^{\infty}$ is said to be convergent if and

only if the sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ converges, i.e., if $\lim_{n\to\infty} s_n$ exists, then we say that the series $\sum_{n=1}^{\infty} a_n$ is a convergent series and we write it as $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$ A series $\sum_{n=1}^{\infty} a_n$ is said to be divergent if it is not convergent.

Geo*Arithmetic Series

Definition 5: Geo*Arithmetic Series

The series of the form $\sum_{n=1}^{\infty} (ar^n)(b+nd)$ is called Geo*Arithmetic Series, where a, b, d and r are real numbers.

The main questions for a series

Question 1: Given a series does it converge or diverge?

Question 2: If it converges, what does it converge to?

Can we answer the above main questions for Geo*Arithmetic Series?

Answer: Yes we can answer the above main questions for Geo*Arithmetic Series.

There are several convergence tests for series. Now we apply the generalized ratio test for Geo*Arithmetic Series [4].

Theorem 1: Suppose $\{a_n\}_{n=1}^{\infty}$ is a real sequence. If the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then it is convergent [5].

Theorem 2: The Generalized Ratio Test.

Suppose that $a_n \neq 0$ for $n \geq 1$ and $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = r$ (possibly ∞)

If $r < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

If $r > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

If $r = 1$, we cannot draw any conclusions from this test alone about the convergence of the series [5].

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Convergence of Geo*Arithmetic Series

Let $\sum_{n=1}^{\infty}[(ar^n)(b+nd)]$ be Geo*Arithmetic Series. Then $a_n=(ar^n)(b+nd)$ and $a_{n+1}=(ar^{n+1})(b+(n+1)d)$, where a, b, d and r are real numbers. Suppose that a, b, d and r are real numbers such that $a_n=(ar^n)(b+nd)\neq 0$ for $n\geq 1$.

Let's consider the ratio of a_{n+1} and a_n.

Clearly

$$\frac{a_{n+1}}{a_n}=[\frac{(ar^{n+1})(b+(n+1)d)}{(ar^n)(b+nd)}]=r[\frac{(b+(n+1)d)}{(b+nd)}]=r[1+\frac{d}{b+nd}]$$

This implies that

$$\lim_{n\to\infty}|\frac{a_{n+1}}{a_n}|=\lim_{n\to\infty}|r[1+\frac{d}{b+nd}]|=|r|.$$

Therefore, the Geo*Arithmetic Series $\sum_{n=1}^{\infty}[(ar^n)(b+nd)]$ converges for $|r|<1$ and diverges for $|r|>1$ where a, b, d and r are real numbers such that $a_n=(ar^n)(b+nd)\neq 0$ for $n\geq 1$.

The Sequence of Partial Sums of the Geo*Arithmetic Series

Lemma 1:

$$\sum_{i=1}^{n}\frac{i}{r^i}=\frac{1}{r^n}\sum_{i=1}^{n}ir^{n-i}\quad\forall n\in N$$ and non-zero scalar r.

Proof

We apply mathematical induction method on n to show $\sum_{i=1}^{n}\frac{i}{r^i}=\frac{1}{r^n}\sum_{i=1}^{n}ir^{n-i}\quad\forall n\in N$ and non-zero scalar r.

Step 1

For $n=1$, $\frac{1}{r}\quad\frac{1}{r}$ This is always true for non-zero r.

Step 2

Suppose that $\sum_{i=1}^{n}\frac{i}{r^i}=\frac{1}{r^n}\sum_{i=1}^{n}ir^{n-i}$

Step 3

Consider $\sum_{i=1}^{n+1}\frac{i}{r^i}$

$$\sum_{i=1}^{n+1}\frac{i}{r^i}=\sum_{i=1}^{n+1}\frac{i}{r^i}+\frac{n+1}{r^{n+1}}=\left(\frac{1}{r^n}\sum_{i=1}^{n}ir^{n-i}\right)+\frac{n+1}{r^{n+1}}=\frac{(n+1)\sum_{i=1}^{n}ir^{n+1-i}}{r^{n+1}}$$
$$=\frac{1}{r^{n+1}}\sum_{i=1}^{n+1}ir^{n+1-i}$$

Hence proved

Theorem 3:

$$\sum_{i=1}^{n}\frac{i}{r^i}=\frac{r^{n+1}-1+(1-r)(n+1)}{(r-1)^2r^n}\forall n\in N$$ and scalar $r(r\neq 0,1)$

Proof

Let $x=\sum_{i=1}^{n}\frac{i}{r^i}$. Then by the above Lemma, $x=\frac{1}{r^n}\sum_{i=1}^{n}ir^{n-i}$.

It follows that

$$xr=\frac{1}{r^n}\sum_{i=1}^{n}ir^{n+1-i}.$$

This implies that

$$xr-x=\frac{1}{r^n}[(\sum_{i=1}^{n}ir^{n+1-i})-(\sum_{i=1}^{n}ir^{n-i})]=\frac{1}{r^n}[(\sum_{i=1}^{n}ir^{n+1-i})-(\sum_{i=1}^{n}ir^{n-i})]$$
$$=\frac{1}{r^n}[(\sum_{i=0}^{n-1}(i+1)r^{n-i})-(\sum_{i=1}^{n}ir^{n-i})]=\frac{1}{r^n}[(\sum_{i=0}^{n-1}ir^{n-i})-(\sum_{i=1}^{n}ir^{n-i})+(\sum_{i=0}^{n-1}r^{n-i})]$$
$$=\frac{1}{r^n}[(\sum_{i=0}^{n-1}r^{n-i})-n]=\frac{1}{r^n}[\frac{1-r^{n+1}}{1-r}-(1+n)]$$

Therefore, $\sum_{i=1}^{n}\frac{i}{r^i}=\frac{r^{n+1}-1+(1-r)(n+1)}{(r-1)^2r^n}\forall n\in N$ and scalar $r(r\neq 0,1)$

Here note that

$$\sum_{i=1}^{n}iu^i=\frac{(\frac{1}{u})^{n+1}-1+(1-\frac{1}{u})(n+1)}{(\frac{1}{u}-1)^2(\frac{1}{u})^n}=\frac{\frac{1-u^{n+1}}{u^{n+1}}+(\frac{u-1}{u})(n+1)}{(\frac{1-u}{u})^2(\frac{1}{u})^n}$$
$$=\frac{u(1-u^{n+1})}{(1-u)^2}+\frac{(n+1)u^{n+1}}{u-1}=\frac{nu^{n+1}}{u-1}+\frac{u-u^{n+1}}{(u-1)^2}\;for\;r=\frac{1}{u}.$$

Therefore, $\sum_{i=1}^{n}ir^i=\frac{nr^{n+1}}{1-r}+\frac{r-r^{n+1}}{(1-r)^2}\forall n\in N$ and scalar $r(r\neq 0,1)$

Let $\sum_{n=1}^{\infty}[(ar^n)(b+nd)]$ be Geo*Arithmetic Series. Then its n^{th} partial sum is given by

$$s_n=\sum_{i=1}^{n}[(ar^i)(b+id)]=\sum_{i=1}^{n}[(ar^i)(b+id)]=\sum_{i=1}^{n}(abr^i)+\sum_{i=1}^{n}(adir^i)=\sum_{i=1}^{n}r^i+ad\sum_{i=1}^{n}ir^i.$$

Let $s_n^*=\sum_{i=1}^{n}r^i$ and $s_n^{**}=\sum_{i=1}^{n}ir^i$. Then $s_n=ab\,s_n^*+ad\,s_n^{**}$.

Clearly

$$s_n^*=r[\frac{r^n-1}{r-1}]\text{ for } r\neq 1$$

$$s_n^{**}=\frac{nr^{n+1}}{1-r}+\frac{r-r^{n+1}}{(1-r)^2}\forall n\in N$$ and scalar $r(r\neq 0,1)$

Thus, $s_n=abr[\frac{r^n-1}{r-1}]+ad[\frac{nr^{n+1}}{1-r}+\frac{r-r^{n+1}}{(1-r)^2}]\forall n\in N$ and scalar $r(r\neq 0,1)$

Therefore, the n[th] partial sum of the Geo*Arithmetic Series $\sum_{n=1}^{\infty}[(ar^n)(b+nd)]$ is $s_n=abr[\frac{r^n-1}{r-1}]+ad[\frac{nr^{n+1}}{1-r}+\frac{r-r^{n+1}}{(1-r)^2}]\forall n\in N$ and scalar $r(r\neq 0,1)$

Convergence of the Sequence of Partial Sums of the Geo*Arithmetic Series

Clearly the Sequence of Partial Sums of the Geo*Arithmetic Series $\sum_{n=1}^{\infty}[(ar^n)(b+nd)]$ is $\{Sn\}_{n=1}^{\infty}$, where $s_n=abr[\frac{r^n-1}{r-1}]+ad[\frac{nr^{n+1}}{1-r}+\frac{r-r^{n+1}}{(1-r)^2}]\forall n\in N$ and $r(r\neq 0,1)$ is scalar.

We know that the Geo*Arithmetic Series $\sum_{n=1}^{\infty}[(ar^n)(b+nd)]$ converges for $|r|<1$, where a, b, d and r are real numbers such that $a_n=(ar^n)(b+nd)\neq 0$ for $n\geq 1$.

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Therefore, the Geo*Arithmetic Series $\sum_{n=1}^{\infty}[(ar^n)(b+nd)]$ converges to $\lim_{n\to\infty} s_n \; for |r|<1$ and $r \neq 0$ if this limit exists, where a, b, d and r are real numbers such that $a_n = (ar^n)(b+nd) \neq 0$ for $n \geq 1$ and $s_n = abr[\frac{r^n-1}{r-1}] + ad[\frac{nr^{n+1}}{1-r} + \frac{r-r^{n+1}}{(1-r)^2}]$.

Here we would like to raise basic question as follows. Does the above limit $\lim_{n\to\infty} s_n$ exist?

We apply the following theorems to find the sum of Geo*Arithmetic Series.

Theorem 4: (Test for Divergence) [5].

Consider the real series $\sum_{n=1}^{\infty} a_n$.

If $\lim_{n\to\infty} a_n \neq 0$, then the series does not converge. That is, if the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Theorem 5:

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series and c is a real number, [5] then

i. $\sum_{n=1}^{\infty}(a_n \pm b_n)$ converges and $\sum_{n=1}^{\infty}(a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$.

ii. $\sum_{n=1}^{\infty} ca_n$ converges and $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$.

Clearly $\sum_{n=1}^{\infty} r^n$ and $\sum_{n=1}^{\infty} nr^n$ are convergent series for $|\mathrm{r}|<1$ by generalized ratio test for series. Therefore, $\lim_{n\to\infty} r^n = 0$ and $\lim_{n\to\infty} nr^n = 0$ for $|\mathrm{r}|<1$

Let's consider $\lim_{n\to\infty} s_n$ for $|\mathrm{r}|<1$ and $r \neq 0$, where a, b, d and r are real numbers such that $a_n = (ar^n)(b+nd) \neq 0$ for $n \geq 1$ and $s_n = abr[\frac{r^n-1}{r-1}] + ad[\frac{nr^{n+1}}{1-r} + \frac{r-r^{n+1}}{(1-r)^2}]$.

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}[abr(\frac{r^n-1}{r-1}) + ad(\frac{nr^{n+1}}{1-r} + \frac{r-r^{n+1}}{(1-r)^2})]$$

$$= \lim_{n\to\infty}[abr(\frac{r^n-1}{r-1})] + \lim_{n\to\infty}[ad(\frac{nr^{n+1}}{1-r} + \frac{r-r^{n+1}}{(1-r)^2})]$$

$$= \left(\frac{abr}{r-1}\right)\lim_{n\to\infty}[(r^n-1)] + \lim_{n\to\infty}[ad(\frac{nr^{n+1}}{1-r})] + \lim_{n\to\infty}[ad(\frac{r-r^{n+1}}{(1-r)^2})]$$

$$= (\frac{abr}{r-1})\lim_{n\to\infty}[r^n-1] + (\frac{adr}{1-r})\lim_{n\to\infty}[nr^n] + \lim_{n\to\infty}[ad(\frac{r}{(1-r)^2})] + \lim_{n\to\infty}[-rad(\frac{r^n}{(1-r)^2})]$$

$$= (\frac{abr}{r-1})\lim_{n\to\infty} r^n - (\frac{abr}{r-1})\lim_{n\to\infty} 1 + (\frac{adr}{1-r})\lim_{n\to\infty}[nr^n] + \lim_{n\to\infty}[ad(\frac{r}{(1-r)^2})] - (\frac{adr}{(1-r)^2})\lim_{n\to\infty} r^n$$

$$= -(\frac{abr}{r-1}) + [ad(\frac{r}{(1-r)^2})] = ar[\frac{d}{(1-r)^2} - \frac{b}{r-1}] = ar[\frac{d+b(1-r)}{(1-r)^2}]$$

Thus, $\lim_{n\to\infty} s_n = ar[\frac{d+b(1-r)}{(1-r)^2}]$ where a, b, d and r are real numbers such that $a_n = (ar^n)(b+nd) \neq 0$ for $n \geq 1$ and $s_n = abr\left(\frac{r^{n-1}}{r-1}\right) + ad\left[\frac{nr^{n+1}}{r-1} + \frac{r-r^{n+1}}{(r-1)^2}\right]$.

Therefore, the Geo*Arithmetic Series $\sum_{n=1}^{\infty}[(ar^n)(b+nd)]$ converges to $s = ar[\frac{d+b(1-r)}{(1-r)^2}]$ for $|\mathrm{r}|<1$ and $r \neq 0$, where a, b, d and r are real numbers such that $a_n = (ar^n)(b+nd) \neq 0$ for $n \geq 1$.

Result and Discussion

We know that searching for sum of a series is difficult task. Thus, we try to find sum of some special series. In this manuscript we introduced the Geo*Arithmetic Series as one special series. We observed that there are some convergent Geo*Arithmetic Series. Moreover, we found the sum of convergent Geo*Arithmetic Series.

Conclusion

In this manuscript we introduced Geo*Arithmetic Series. Moreover, we found the sum of the Geo*Arithmetic Series $\sum^{\infty}[(ar \;\;)(b \;\;\; nd)]$ as $s = ar[\frac{d+b(1-r)}{(1-r)^2}]$ for $|\mathrm{r}|<1$ and $r \neq 0$, where a, b, d and r are real numbers such that $a_n = (ar^n)(b+nd) \neq 0$ for $n \geq 1$.

Recommendation

Author would like to recommend that researchers shall search for the sum of generalized Geo*Arithmetic Series $\sum_{n=1}^{\infty}[(ar^n)(b+nd)^k]$, where a, b and d are real numbers and k is natural number. Moreover, I will search for recursive formula of partial sum of $\sum_{n=1}^{\infty}[(ar^n)(b+nd)^k]$, in terms of the partial sum of $\sum_{n=1}^{\infty}[(ar^n)(b+nd)^{k-1}]$.

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