How to Prove the Riemann Hypothesis

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Abstract

The aim of this paper is to prove the celebrated Riemann Hypothesis. I have already discovered a simple proof of the Riemann Hypothesis. The hypothesis states that the nontrivial zeros of the Riemann zeta function have real part equal to 0.5. I assume that any such zero is \( s = a + bi \). I use integral calculus in the first part of the proof. In the second part I employ variational calculus. Through equations (50) to (59) I consider \( a \) as a fixed exponent, and verify that \( a = 0.5 \). From equation (60) onward I view \( a \) as a parameter (\( a < 0.5 \)) and arrive at a contradiction. At the end of the proof (from equation (73)) and through the assumption that \( a \) is a parameter, I verify again that \( a = 0.5 \).

Keywords: Definite integral; Indefinite integral; Variational calculus

Introduction

The Riemann zeta function is the function of the complex variable \( s = a + bi \) (\( i = \sqrt{-1} \)), defined in the half plane \( a > 1 \) by the absolute convergent series.

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \]  
(1)

and in the whole complex plane by analytic continuation.

The function \( \zeta(S) \) has zeros at the negative even integers -2, -4, \( \ldots \) and one refers to them as the trivial zeros. The Riemann hypothesis states that the nontrivial zeros of \( \zeta(S) \) have real part equal to 0.5 [1].

Proof of the Hypothesis

We begin with the equation,

\[ \zeta(S) = 0 \]  
(2)

And with,

\[ s = a + bi \]  
(3)

\[ \zeta(a+bi) = 0 \]  
(4)

It is known that the nontrivial zeros of \( \zeta(S) \) are all complex. Their real parts lie between zero and one.

If \( 0 < a < 1 \) then,

\[ \zeta(x) = \sum_{n=1}^{\infty} \frac{1}{x^n} \]  
(5)

\( x \) is the integer function. Hence,

\[ \int_{0}^{\infty} \frac{x-x}{x^s+1} \, dx = 0 \]  
(6)

Therefore,

\[ \int_{0}^{\infty} \frac{\left[ x \right] - x}{x^s+1} \, dx = 0 \]  
(7)

\[ \int_{0}^{\infty} \left[ x \right] - x \, dx = 0 \]  
(8)

\[ \int_{0}^{\infty} \left[ x \right] - x \, \cos(b \log x) \, dx = 0 \]  
(9)

\[ \int_{0}^{\infty} x^{-\alpha} \, a \, b \, x^{i} \, dx = 0 \]  
(10)

\[ \int_{0}^{\infty} x^{-\alpha} \, a \, b \, x^{i} \, \sin(b \log x) \, dx = 0 \]  
(11)

According to the functional equation, if \( \zeta(S) = 0 \) then \( \zeta(1-S) = 0 \). Hence we get besides equation (11)

\[ \int_{0}^{\infty} y^{-1-a} \, (y)^{1} \, \sin(b \log y) \, dy = 0 \]  
(12)

In equation (11) replace the dummy variable \( x \) by the dummy variable \( y \).

\[ \int_{0}^{\infty} y^{-1-a} \, (y)^{1} \, \sin(b \log y) \, dy = 0 \]  
(13)

We form the product of the integrals (12) and (13). This is justified by the fact that both integrals (12) and (13) are absolutely convergent. As to integral (12) we notice that [2]

\[ \int_{0}^{\infty} y^{-2} \, (y)^{1} \, \sin(b \log y) \, dy \leq \int_{0}^{\infty} y^{-2} \, (y) \, dy \leq \int_{0}^{\infty} x^{-2} \, (x) \, dx \]

\[ \left( ((z)) \right) \text{ the fractional part of } z, \, 0 \leq (\left( ((z)) \right) < 1 \]

\[ = \lim_{t \to 0} \int_{t}^{\infty} x^{-2} \, (x) \, dx \]

\[ = \frac{1}{a} \, \int_{t}^{\infty} x^{-2} \, (x) \, dx \]

\[ = \frac{1}{a} \, \lim_{t \to 0} \int_{t}^{\infty} x^{-2} \, (x) \, dx \]

\[ \frac{1}{a} \, \lim_{t \to 0} \int_{t}^{\infty} x^{-2} \, (x) \, dx = \frac{1}{a} + \frac{1}{a - 1} \]

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And as to integral (13) \[ \int_0^\infty y^{1-a}(y-y)\sin(b\log y) dy \]
\[ \leq \int_0^\infty y^{1-a}((y) - y)\sin(b\log y) dy \]
\[ \leq \int_0^\infty y^{1-a}((y)) dy \]
\[ = \lim(t \to 0) \int_t^\infty y^{1-a}((y)) dy + \lim(t \to 0) \int_0^t y^{1-a}((y)) dy \]
\[ = (t) \text{ is a very small positive number} \]
\[ \leq 1 - \frac{1}{a} + (t) \]
\[ \leq \frac{1}{1-a} + \int_{t}^{\infty} y^{1-a}((y)) dy \]
\[ \leq \frac{1}{1-a} + \frac{1}{1-a} \]

Since the limits of integration do not involve \( x \) or \( y \), the product can be expressed as the double integral,
\[ \int_{0}^{\infty} \int_{0}^{\infty} x^{2-a} y^{1-a}(x-y)\sin(b\log xy) \sin(b\log xy) dx \, dy = 0 \]

Thus,\[ \int_{0}^{\infty} x^{2-a} y^{1-a}((x)-(y)) dx = 0 \]

And as to integral (13) \[ \int_0^\infty x^{2-a} y^{1-a}((x)-(y)) dx \]
\[ \leq \int_0^\infty x^{2-a} y^{1-a}((x)-y) dx \]
\[ \leq \int_0^\infty x^{2-a} y^{1-a}((x)) dx \]
\[ = \lim(t \to 0) \int_t^\infty x^{2-a} y^{1-a}((x)) dx + \lim(t \to 0) \int_0^t x^{2-a} y^{1-a}((x)) dx \]
\[ = x \rightarrow \infty \]
\[ \leq \frac{1}{2} + \lim(t \to 0) \int_0^t x^{2-a} y^{1-a}((x)) dx \]

Thus equation 17 becomes,
\[ \int_0^\infty x^{2-a} y^{1-a}((x)-(y)) \cos(b\log xy) dx \, dy = \]

Write the last equation in the form,
\[ \int_0^\infty x^{2-a} y^{1-a}((x)-(y)) \cos(b\log xy) \int_{([x]-x)} \int_{([y]-y)} dx \, dy = 0 \]

Let \( p \) be an arbitrary small positive number. We consider the following regions in the \( x-y \) plane [4].

The region of integration \( I=([0,\infty) \times(0,\infty) \)

The large region \( I_1=[p,\infty) \times(0,\infty) \)

The narrow strip \( I_2=[p,p \times(0,p] \)

The narrow strip \( I_3=[0,p) \times(0,\infty) \)

Note that,
\[ I=I_1 \cup I_2 \cup I_3 \]

Denote the integer and in the left hand side of equation (25) by:
\[ F(x,y) = x^{2-a} y^{1-a}((x)-(y)) \cos(b\log xy) \]

Let us find the limit of \( F(x,y) \) as \( x \rightarrow \infty \) and \( y \rightarrow \infty \). This limit is given by:
\[ \lim_{x \rightarrow \infty} x^{2-a} y^{1-a}((x)-(y)) \cos(b\log xy) \left( x^{2-a} y^{2-a} \right) \]

\( (x) \) is the fractional part of the number \( z \), \( 0 \leq (x) < 1 \)

The above limit vanishes, since all the functions \( \left( (x) \right) \) remain bounded as \( x \rightarrow \infty \) and \( y \rightarrow \infty \). Note that we can find the limit of \( F(x,y) \) as \( x \rightarrow \infty \) and \( y \rightarrow \infty \) for all \( (x) \) remain bounded as \( x \rightarrow \infty \) and \( y \rightarrow \infty \). We can prove that the integral,
\[ \int_0^\infty \int_0^\infty F(x,y) dx \, dy \]

is bounded as follows
\[ \leq \int_0^1 \left( \int_0^\infty x^{2-a} y^{1-a}((x)-(y)) \cos(b\log xy) \left( x^{2-a} y^{2-a} \right) \right) dx \, dy \]

\[ \leq \int_0^1 \left( \int_0^\infty x^{2-a} \cos(b\log xy) \left( x^{2-a} y^{2-a} \right) dx \right) \]

If we replace the dummy variable \( z \) by the dummy variable \( x \), the integral takes the form [3]
\[ \int_0^\infty \int_0^\infty x^{2-a} y^{1-a}((x)-(y)) \cos(b\log xy) \, dy \]

Rewrite this integral in the equivalent form,
\[ \int_0^\infty \int_0^\infty x^{2-a} y^{1-a}((x)-(y)) \cos(b\log xy) \, dy \]

\[ \leq \int_0^\infty \left( \int_0^\infty x^{2-a} \cos(b\log xy) \left( x^{2-a} y^{2-a} \right) dx \right) \]
\[< \int_{\frac{1}{x}} x^{-a} \left( \left( \frac{1}{x} \right) + \left( x \right) \right)^{2a-2} \ dx \left| y \right|^{-1-a} \]

\[- \frac{1}{ap} \int_{\frac{1}{x}} x^{-a} \left( \left( \frac{1}{x} \right) + \left( x \right) \right)^{2a-2} \ dx \]  \hspace{1cm} (34)

\[- \frac{1}{ap} \int_{\lim(t \to 0)} x^{-a} \left( \left( \frac{1}{x} \right) + \left( x \right) \right)^{2a-2} \ dx \]

Where \( t \) is a very small arbitrary positive number. Since the integral:

\[ \lim(t \to 0) \int_{\frac{1}{x}} x^{-a} \left( \left( \frac{1}{x} \right) + \left( x \right) \right)^{2a-2} \ dx \]

is bounded, it remains to show that \( \lim(t \to 0) \int_{\frac{1}{x}} x^{-a} \left( \left( \frac{1}{x} \right) + \left( x \right) \right)^{2a-2} \ dx \) is bounded.

Since \( x > 1 \), then \( \left( \frac{1}{x} \right) \), and we have:\n
\[ \int_{\frac{1}{x}} x^{-a} \left( \left( \frac{1}{x} \right) + \left( x \right) \right)^{2a-2} \ dx \]

\[ = \lim(t \to 0) \int_{\frac{1}{x}} x^{-a} \left( \left( \frac{1}{x} \right) + \left( x \right) \right)^{2a-2} \ dx \]

\[ = \lim(t \to 0) \int_{\frac{1}{x}} x^{-a-1} + \left( x \right)^{2a-2} \ dx \]

\[ = \frac{1}{a(1-a)} \]

Hence the boundedness of the integral \( \int_{\frac{1}{x}} F(x, y) \ dy \) is proved.

Consider the region:

\[ I_1 = \{ (x, y) \mid x > 1, x > 1 \} \]

We know that,

\[ 0 = \int_{\frac{1}{x}} F(x, y) \ dy = \int_{\frac{1}{x}} F(x, y) \ dy + \int_{I_1} F(x, y) \ dy \]  \hspace{1cm} (35)

and that,

\[ \int_{I_1} F(x, y) \ dy \]  \hspace{1cm} (37)

From which we deduce that the integral \[ \int_{I_1} F(x, y) \ dy \] is bounded.

(38)

Remember that,

\[ \int_{I_1} F(x, y) \ dy = \int_{\frac{1}{x}} F(x, y) \ dy + \int_{I_1} F(x, y) \ dy \]  \hspace{1cm} (39)

Consider the integral,

\[ \int_{\frac{1}{x}} F(x, y) \ dy \leq \int_{\frac{1}{x}} F(x, y) \ dy \]  \hspace{1cm} (40)

\[ \left| \int_{0}^{\frac{1}{x}} x^{-a} \left( \left( \frac{1}{x} \right) - \left( x \right) \right)^{2a-2} \cos(b \log y) \ dx \right| \frac{1}{y} \ dy \]

\[ \leq \left| \int_{0}^{\frac{1}{x}} (x^{-a} \left( \frac{1}{x} \right) - \left( x \right))^{2a-2} \right| \left| \cos(b \log y) \ dx \right| \frac{1}{y} \ dy \]

\[ \leq \left| \int_{0}^{\frac{1}{x}} (x^{-a} \left( \frac{1}{x} \right) - \left( x \right))^{2a-2} \right| \left| \cos(b \log y) \ dx \right| \frac{1}{y} \ dy \]

\[ \leq \left| \int_{0}^{\frac{1}{x}} x^{-a} \left( \left( \frac{1}{x} \right) - \left( x \right) \right)^{2a-2} \right| \left| \cos(b \log y) \ dx \right| \frac{1}{y} \ dy \]

\[ \leq \int_{0}^{\frac{1}{x}} x^{-a} \left( \left( \frac{1}{x} \right) - \left( x \right) \right)^{2a-2} \ dx \times \int_{0}^{\frac{1}{y}} \frac{1}{y} \ dy \]

\[ \int_{0}^{\frac{1}{x}} \frac{1}{y} \ dy \]

is bounded. Thus we deduce that the integral \[ \int_{0}^{\frac{1}{x}} F(x, y) \ dy \] is bounded.

Hence, according to equation (39), the integral \[ \int_{I_1} F(x, y) \ dy \] is bounded.

(41)

Now consider the integral,

\[ \int_{I_1} F(x, y) \ dy \]  \hspace{1cm} (42)

(43)

We write it in the form:

\[ \int_{I_1} F(x, y) \ dy = \int_{0}^{\frac{1}{x}} y^{-a} \left( \left( y \right) \cos(b \log y) \ dy \right) \left| \left( \frac{1}{x} \right) - \left( x \right) \right| \ dx \]

\[ \leq \int_{0}^{\frac{1}{x}} y^{-a} \left( \left( y \right) \cos(b \log y) \ dy \right) \left| \left( \frac{1}{x} \right) - \left( x \right) \right| \ dx \]

\[ \leq \int_{0}^{\frac{1}{x}} y^{-a} \left( \left( y \right) \cos(b \log y) \ dy \right) \left| \left( \frac{1}{x} \right) - \left( x \right) \right| \ dx \]

Now we consider the integral with respect to \( y \),

\[ \int_{0}^{\frac{1}{x}} y^{-a} \left( \left( y \right) \right) \ dy \]  \hspace{1cm} (43)

\[ = \left( \lim t \to 0 \right) \int_{0}^{\frac{1}{t}} y^{-a} \times y \ dy + \left( \lim t \to 0 \right) \int_{\frac{1}{t}}^{\frac{1}{x}} y^{-a} \left( \left( y \right) \right) \ dy \]

(44)

(45)

(46)

(47)

(48)

(49)

(50)

(51)

(52)
Now equation (44) gives us,
\[ -K \leq \int y^{\alpha-x} (y) \cos (b \log y) dy \leq K \] (45)
According to equation (42) we have,
\[ \int_{\delta}^{\hat{\delta}} F(x,y) dy = \int_{\delta}^{\hat{\delta}} y^{x-a} (y) \cos (b \log y) dy \frac{((1/y) - x^{2a-1})}{x^a} dx \] (46)
Hence we have,
\[ \int_{\delta}^{\hat{\delta}} F(x,y) dy \text{ is bounded. Therefore the integral,} \]
\[ \int_{\delta}^{\hat{\delta}} \frac{((1/x) - x^{2a-1})}{x^a} dx \text{ is also bounded. Therefore the integral,} \]
\[ G = \frac{((1/x) - x^{2a-1})}{x^a} \text{ is bounded} \] (47)
We denote the integer and of (47) by:
\[ F = \frac{((1/x) - x^{2a-1})}{x^a} \] (48)
Let \( \delta G[F] \) be the variation of the integral \( G \) due to the variation of the integrand \( F \).
Since,
\[ G[F] = \int F dx \text{ (the integral (49) is indefinite)} \] (49)
(here we do not consider \( a \) as a parameter, rather we consider it as a given exponent)
We deduce that
\[ \frac{\delta G[F]}{\delta F(x)} = 1 \]
that is,
\[ \delta G[F] = \delta F(x) \] (50)
But we have,
\[ \delta G[F] = \int \delta F(x) \delta F(x) \text{ (the integral (51) is indefinite)} \] (51)
Using equation (50) we deduce that,
\[ \delta G[F] = \delta F(x) \text{ (the integral (52) is indefinite)} \] (52)
Since \( G[F] \) is bounded across the elementary interval \([0,p]\), we must have that,
\[ \delta G[F] \text{ is bounded across this interval} \] (53)
From (52) we conclude that,
\[ \delta G = \int_{0}^{c} dx \delta F(x) = \int_{0}^{c} dx \delta F(x) \text{ (at } x = 0) - \frac{d}{dx} \left( F(x) \right) \text{ (at } x = 0) \] (54)
Since the value of \( F(x) \text{ (at } x = 0) \) is bounded, we deduce from equation (54) that,
\[ \lim (x \to 0) \delta G \text{ must remain bounded.} \] (55)
Thus we must have that
\[ \lim (x \to 0) \frac{d}{dx} \left( \frac{((1/x) - x^{2a-1})}{x^a} \right) \text{ is bounded.} \] (56)
First we compute,
\[ \lim (x \to 0) \frac{d}{dx} \left( \frac{((1/x) - x^{2a-1})}{x^a} \right) \] (57)
Applying L’Hospital’ rule we get,
\[ \lim x \to 0 \frac{d}{dx} \left( \frac{((1/x) - x^{2a-1})}{x^a} \right) = \lim x \to 0 \frac{1}{x^a} \times x^{1-a} \times \frac{d}{dx} \left( \frac{1}{x} \right) = 0 \] (58)
We conclude from (56) that the product, \[ 0 \times \lim x \to 0 \left( \frac{((1/x) - x^{2a-1})}{x^a} \right) \text{ must remain bounded.} \] (59)
Assume that \( a = 0.5 \). (Remember that we considered \( a \) as a given exponent) This value \( a = 0.5 \) will guarantee that the quantity \[ \left( \frac{1}{x} \right) - x^{2a-1} \]
will remain bounded in the limit as \( x \to 0 \). Therefore, in this case \( a = 0.5 \) (56) will approach zero as \( x \to 0 \) and hence remain bounded.
Now suppose that \( a < 0.5 \). In this case we consider \( a \) as a parameter. Hence we have,
\[ G_{\delta} = \int_{0}^{\infty} \frac{F(x,a)}{x} dx \text{ (the integral (60) is indefinite)} \] (60)
Thus,
\[ \frac{\partial G}{\partial x} = \frac{F(x,a)}{x} \] (61)
But we have that,
\[ \delta G_{\delta} = \int_{0}^{\infty} \frac{F(x,a)}{x} \delta x \text{ (the integral (62) is indefinite)} \] (62)
Substituting from (61) we get,
\[ \delta G_{\delta} = \int_{0}^{\infty} \frac{F(x,a)}{x} \delta x \] (63)
We return to equation (49) and write,
\[ G = \lim (x \to 0) \int F dx \text{ (is a very small positive number } 0 < t < p \) \] (64)
\[ = \{ F(x(t)) - \lim (x \to 0) F(x) \} \] (65)
Let us compute,
\[ \lim (x \to 0) F(x) \] (66)
Thus equation (64) reduces to,
\[ G = \lim (x \to 0) \int F dx \] (67)
Note that the left – hand side of equation (66) is bounded. Equation (63) gives us,
\[ \delta G_{\delta} = \lim (x \to 0) \int_{0}^{\infty} \frac{F(x)}{x} \delta x \] (68)
\[ \text{ (is the same small positive number } 0 < t < p \) \]
We can easily prove that the two integrals \[ \int_{0}^{\infty} dx \text{ and } \int_{0}^{\infty} dx \delta x \]
are absolutely convergent. Since the limits of integration do not involve any variable, we form the product of (66) and (67)
\[ K = \lim (t \to 0) \int_{0}^{\infty} \frac{F(x)}{x} \delta x = \lim (t \to 0) \int_{0}^{\infty} F dx \times \int_{0}^{\infty} \delta x \] (69)
\[ \text{(K is a bounded quantity)} \]
That is,
\[ K = \lim (t \to 0) \int_{0}^{\infty} \frac{F(x)}{x} \delta x \] (70)
We conclude from this equation that,
Therefore, 
\[
\lim (x \to 0) - \frac{1}{x} \text{ must remain bounded. (71)}
\]

But, 
\[
\lim (x \to 0) - \frac{1}{x} \text{ remains bounded as } (x \to 0).
\]

\[
\text{Since } \lim (x \to 0) \delta = 0, \text{ which is the same thing as } \lim (x \to 0) \delta x = 0.
\]

\[
\text{We have that:}
\]
\[
\lim (x \to 0) - \frac{1}{x} \text{ must remain bounded in the limit as } (x \to 0).
\]

\[
\text{Therefore:}
\]
\[
\lim (x \to 0) - \frac{1}{x} \text{ remains bounded as } (x \to 0).
\]

\[
\text{Hence we get:}
\]
\[
\lim (x \to 0) - \frac{1}{x} = \lim (x \to 0) x^{\alpha - 1} = \lim (x \to 0) x^{\alpha - 1} = 1
\]

\[
(\text{Since } \lim (x \to 0) x^{\alpha - 1} = 1)
\]

\[
\text{Thus we must apply I. 'Hospital' rule with respect to } x \text{ in the limiting process (75)}
\]
\[
\lim (x \to 0) - \frac{1}{x} = \lim (x \to 0) - \frac{2(x - 1)x^{2 - 1}}{x}
\]

\[
\lim (x \to 0) - \frac{1}{x} = \lim (x \to 0) - \frac{2(x - 1)x^{2 - 1}}{x}
\]

\[
\text{Now we write again,}
\]
\[
a = \lim (x \to 0) (0.5 + x)
\]

\[
\text{Thus the limit (78) becomes:}
\]
\[
\lim (x \to 0) - \frac{(x - 2)}{x} = \lim (x \to 0) \frac{(0.5 + x)^{1 - 2}}{x^2} = \lim (x \to 0) \frac{(0.5 + x)^{1 - 2}}{x^2} - \lim (x \to 0) \frac{(0.5 + x)^{1 - 2}}{x^2}
\]

\[
(\text{Since } \lim (x \to 0) x^{-1} = 1)
\]

\[
(80)
\]

\[
\text{We must apply I. 'Hospital' rule,}
\]
\[
\lim (x \to 0) \frac{(0.5 + x)^{1 - 2}}{x^2} = \lim (x \to 0) \frac{(0.5 + x)^{1 - 2}}{x^2} - \lim (x \to 0) \frac{(0.5 + x)^{1 - 2}}{x^2}
\]

\[
(81)
\]

\[
\text{Thus we have verified here that, for } a = 0.5 \text{ (71) approaches zero as } (x \to 0) \text{ and hence remains bounded.}
\]

\[
\text{We consider the case } a > 0.5. \text{ This case is also rejected, since according to the functional equation, if } (\zeta (s)) = 0 \text{ has a root with } a > 0.5, \text{ then it must have another root with another value of } a < 0.5.
\]

\[
\text{But we have already rejected this last case with } a < 0.5.
\]

\[
\text{Thus we are left with the only possible value of } a \text{ which is } a = 0.5
\]

\[
\text{Therefore } a = 0.5
\]

\[
\text{This proves the Riemann Hypothesis.}
\]

\[\text{Conclusion}\]

\[\text{The Riemann Hypothesis is now proved. The hypothesis states that the nontrivial zeros of the Riemann zeta function have real part equal to 0.5. I assume that any such zero is } z = a + bi. \text{ I use integral calculus in the first part of the proof. In the second part I employ variational calculus. Through equations (50) to (59) I consider } (a) \text{ as a fixed exponent, and verify that } a = 0.5. \text{ From equation (60) onward I view } (a) \text{ as a parameter } (a < 0.5) \text{ and arrive at a contradiction. At the end of the proof (from equation (73)) and through the assumption that } (a) \text{ is a parameter, I verify again that } a = 0.5.\]

\[\text{References}\]


