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Hopf algebras, random walks and quantum master equations [1]

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Abstract

Hopf algebras of functions and operators are utilized to develop a mathematical construction scheme for building *algebraic random walks*. The main construction treats *systems of covariance* formed by translation operator and its associated operator valued measures on e.g. the circle and the line, and derives an algebraic quantum random walk by means of completely positive trace preserving maps. Asymptotic limit of the action of such maps is shown to lead to quantum master equations of Lindblad type.

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1 Introduction

Sets of objects such as functions or operators endowed with the structure of a Hopf algebra have extensively been utilized to construct algebraically motivated random walks and associated stochastic equations of various types [13]–[16]. It is a fruitful interface of Hopf algebras (in the form of e.g. discrete groups, Lie groups, quantum groups), with the so called *open systems*, namely systems that are not *closed systems*, and so they involve no Hamiltonian (conservative) structure. Cases of quantum random walks e.g. on the canonical algebra of quantum mechanics, or on braided algebras (smash line algebras), and their associated diffusion limit evolution equations have been constructed, and their solutions have been investigated c.f [3]–[6], and also [7].

In the present work we take up the problem of constructing evolution equations or *quantum master equations*; a problem of central important in the theory of open quantum systems[1]. Our approach is to start with a *system of covariance* [2], on a measurable space Ω, made of a one parameter group in e.g. circle or real line, and a covariant to it positive operator valued measure (POVM), for a chosen Hilbert space (see below). Then we consider the POVM as a function of Ω, which is endowed with the structure of a Hopf algebra of functions, and define on it an appropriate positive functional, in order to formulate an algebraic classical random walk. At the level of POVM one step of this classical walk manifests itself with the action of a completely positive trace preserving map i.e. as a quantum walk [3]–[7]. Namely we obtain a classically induced quantum random walk. The asymptotic diffusion limit is shown to lead to a quantum master equation for the density operator of the underlying quantum system. This equation is of the Kossakowski-Lindblad (KL) form [9, 10].

To pave the way for the classically induced quantum walk (section 4), in the next two chapters we recall two exemplary constructions of walks with classical (section 2), and quantum Hopf algebra (section 3).

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2 Classical algebraic random walks and diffusion equations

Denote by $H \equiv \mathbb{R}[[X]](\mu, \Delta, u, \epsilon)$ the real line Hopf algebra with the multiplication map $\mu(X^m \otimes X^n) = X^{m+n}$, comultiplication map $\Delta(X) = X \otimes \mathbf{1} + \mathbf{1} \otimes X \equiv X_1 + X_2$, unit map $c \in \mathbb{R}$, $u(c) = c\mathbf{1}$, and co-unit map $\epsilon(X) = 0$, $\epsilon(\mathbf{1}) = 1$. Its elements are the coordinate functions $X^m(x) = x^m, m = 0, 1, 2, \ldots$ their comultiplication of which is the sum of two one-variable functions

$$\Delta(X)(x, y) = (X \otimes \mathbf{1} + \mathbf{1} \otimes X)(x, y) = x + y$$

Similarly the n-fold comultiplication $\Delta^n(X) = X_1 + ... + X_n$ leads to a sum of n one-variable coordinate functions which in the context of algebraic random walks are identified as sum of n independent identically distributed random variables. Also we introduce the star map $* : H \to H$, and the state map $\phi : H \to \mathbf{C}$ on H, which is positive definite i.e. $\phi(XX^*) \geq 0$ for any $X \in H$, and normalized $\phi(\mathbf{1}) = 1$. On any formal power series $f(X) \in H$ we act with the state ϕ to obtain

$$\phi(f) \equiv \langle f \rangle_\phi = \langle \phi, f \rangle = \int \rho f$$

where the result is re-expressed by means of formal integral involving the probability density function $\rho \in H$, with properties $\rho > 0, \int \rho = 1$. Further the convolution product between two states is defined by $(\phi * \psi)(f) = (\phi \otimes \psi) \circ \Delta(f)$. The Markov transition operator $T_\phi : H \to H$, is defined from state ϕ, by $T_\phi = (\phi \otimes id) \circ \Delta$,and leads to that state by $\epsilon \circ T_\phi = \phi$. Also the state convolution product leads to product of transition operators by $\psi * \phi = \epsilon \circ T_\psi T_\phi$.

As example we obtain from this set up the random walk on the line and its diffusion equation limit [13]. Really, choices $\phi \equiv \delta_1$ and $\delta_1(X^n) = \delta_{n1}$ lead to Markov operator $T_{\delta_1} \equiv \frac{d}{dx}$, and in order to deal with random walk with step $a \in \mathbb{R}^+$, and stepping probability p on the line, we choose the state $\phi(f) = pf(a) + (1 - p)f(-a)$, which invokes the density $\rho(X) = p\delta(X - a) + (1 - p)\delta(X - a)$. To derive the continuous limit for the state $\phi_t(f) \equiv \lim_{n\to\infty} \phi^{*n}(f) \equiv \int \rho_t f$, or dually for the transition operator $T_t(f) \equiv \lim_{n\to\infty} T^n(f)$, we need to introduce continuous time t, as well as the drift c and diffusion γ coefficients, by the respective definitions $2a(p - 1/2) = \frac{ct}{n}$, and $a^2/2 = \frac{\gamma t}{n}$. Then the resulting density function ρ_t, obeys the diffusion equation

$$\partial_t \rho_t = \left(-c\partial_x + \gamma \partial_x^2\right) \rho_t \tag{2.1}$$

Similar constructions for other Hopf and bialgebras such as the anyonic (or braided) line Hopf algebra $H \equiv \mathbb{R}[\xi]$, and the so called smash line Hopf algebra (real and anyonic braided line) $H \equiv \mathbb{R}[[X, \xi]]$, have shown to lead to generalized diffusion equations for the corresponding density functions [4, 5].

3 Quantum algebraic random walks and master equations

Let the canonical or Heisenberg-Weyl algebra $\mathbf{hw} \equiv \{a, a^\dagger, \mathbf{1}\}$, generated by creation, annihilation and unit operator respectively. Let $N = a^\dagger a$ be the number operator. We aim to construct an algebraic quantum random walk along the lines of the previous classical walk, but now starting with an operator Hopf algebra. To this end we proceed by introducing appropriate Hopf algebra structure in the enveloping algebra of $\mathbf{hw}$, denoted by $\mathcal{U}(\mathbf{hw})$ [13, 3]. Choose a state ϕ which acts on $\mathcal{U}(\mathbf{hw})$, and is determined by eigenvectors of annihilation operator, and is also parameterized by the probability p, as a measure of asymmetry of the walk, and by the random step size a. In the diffusion limit these parameters lead to drift and diffusion parameters, as in the case of classical walk discussed above. The diffusion limit of this quantum walk is expressed by its associated master equation that is satisfied by density operator ρ_t of the underlying quantum system. It reads [3]

$$\dot{\rho_t} = [ca^\dagger - c^* a, \rho_t] + \gamma(a^2\rho_t + \rho_t a^2 - 2a\rho_t a) - \gamma^*(a^{\dagger 2}\rho_t + \rho_t a^{\dagger 2} - 2a^\dagger \rho_t a^\dagger)$$
$$- |\gamma|^2((2N+1)\rho_t + \rho_t(2N+1) - 2a^\dagger\rho_t a - 2a\rho_t a^\dagger)$$

This equation leads to conservative (Hamiltonian) dynamics (see first commutation in rhs), and to dissipative dynamics, determined by the strength of diffusion coefficient γ. The solution is given in terms of operator valued Appell polynomials (see for further discussion [3].

4 Quantum algebraic random walks, POVMs and master equations

Let Ω be an non empty set, $\mathcal{F}$ the σ-algebra of subsets of Ω, $(\Omega, \mathcal{F})$ a measurable space. Also let a quantum system with Hilbert space $\mathcal{H}$, and $\mathcal{L}(\mathcal{H})$ its set of bounded operators. Further let the set of density operators or states $\mathcal{D}(\mathcal{H})= \{\rho \in \mathcal{L}(\mathcal{H}) | \rho \geq 0, \mathrm{tr}\,(\rho) = 1$. As special cases of density operators we have the *pure states* $\rho = |\Psi\rangle\langle\Psi|$, for any normalized vector $|\Psi\rangle \in \mathcal{H}$, with the projectivity property $\rho^2 = \rho$. Then $\mathcal{D}(\mathcal{H})$ is the convex hull of pure states. Dually we consider quantum observables (including vonNeumann projectors and generalizations) in the form of positive operator valued probability measure (POVM) $M : \mathcal{F} \to \mathcal{L}(\mathcal{H})$, with properties $M(X) \geq 0$, $M(\Omega) = \mathbf{1}$, $M(\cup_i X_i) = \sum_i M(X_i)$, valid for $X^/s$ taken as disjoint sets in $\mathcal{F}$. Special case of these generalized observables are the rank-1 projectors $M(X)^2 = M(X)$. Given a quantum system in state ρ, the probability measure corresponding to an interval $X \in \mathcal{F}$ in association with a POVM $M(X)$ is defined (c.f [2]) as follows: $p^M_\rho : \mathcal{F} \to [0,1]$, $X \mapsto p^M_\rho(X) = \mathrm{tr}\,(\rho M(X))$ We next need to introduce an operator $N \in \mathcal{L}(\mathcal{H})$ with the property to form with POVM $M(X)$ a *covariant pair* i.e $e^{iaN}M(X)e^{-iaN} = M(X + a)$. Construction of such a covariant pair for some given Ω and Hilbert space $\mathcal{H}$ requires a mathematical investigation that has been carried out for some interesting cases elsewhere c.f [11, 12, 15]. Here we only recall e.g the case of $\mathcal{H} = l_2(\mathbb{N})$; the POVM $M(X)$, named *covariant phase observable* , has the general form

$$M(X) = \sum_{mn=0}^{\infty} c_{nm}\, i_{nm}(X)|n\rangle\langle m|$$

where $i_{nm}(X) = \int_X e^{i(n-m)\varphi} d\varphi$, and $c = (c_{nm})$ is the *phase matrix*, i.e a positive definite (complex) matrix with elements $c_{nn} = 1$, $n \in \mathbb{N}$. In the following we choose $c = \mathbf{1}$. Also for concreteness we consider examples of the definitions given above: $(\Omega, \mathcal{F}) \equiv (\mathbb{R}^k, \mathrm{Borel}(\mathbb{R}^k))$ e.g $k = 1$, with POVMs associated to canonical observables of position and momentum; $k = 2$, associated to phase space POVMs; and $(\Omega, \mathcal{F}) \equiv ([0, 2\pi), \mathrm{Borel}([0, 2\pi))$ associated to POVMs related to angular momentum and spin observables. Let us also give some concrete examples of Hilbert spaces that will be involved in the general construction. In terms of the groups $SU(2)$, $ISO(2)$, and HW, we adopt the corresponding index set $J := \{-j, ..., j\}$ (j is integer and half integer) $\mathbb{Z}$, $\mathbb{N}$, and the three respective Hilbert spaces $\mathcal{H} = l_2(J) = \mathrm{span}(\{|n\rangle\}, n \in J)$, that become the modules of their irreducible representations. Also we introduce the generic *number operator* $N = \sum_{n\in J} n|n\rangle\langle n|$ and the *phase state vector* $|\varphi\rangle = \sum_{n\in J} e^{in\varphi}|n\rangle$, $0 \leq \varphi < 2\pi$, and proceed to construct an algebraic random walk over the circle $\Omega \equiv [0, 2\pi)$. To this end we introduce the *unsharp covariant phase observable*, which is a POVM defined on any interval X of the circle,

$$M(X) = \sum_{mn\in J} \int_X e^{i(n-m)\varphi} d\varphi |n\rangle\langle m| = \int_X d\varphi |\varphi\rangle\langle\varphi|$$

On this POVM seen as an operator valued function of variable φ, we now introduce the functional

$$\phi(f) = pf(\varphi = a) + (1-p)f(\varphi = -a)$$

with parameters $p \in [0, 2\pi]$, and $a \in \mathbb{R}^+$. The associated Markov operator acts on the POVM as

$$T_\phi(M(X)) = (\phi \otimes id) \circ \Delta(M(X)) = pM(X+a) + (1-p)M(X-a)$$

Above $X + a = \{a' + a \ (mod 2\pi) \ | a' \in X\}$ is a rigid translation by a of the interval X. By means of the covariance property shared by (N, M) operators and the definition of group adjoint action $Ad(F)(\beta) = F\beta F^*$ and of algebra adjoint action $ad(F) = [F, \beta]$, the action of Markov operator on the POVM is expressed as $T_\phi(M(X)) = \left[pAd(e^{iaN}) + (1-p)Ad(e^{-iaN})\right] M(X)$. From this we deduce the form of Markov transition operator

$$T_\phi \equiv pe^{ia \ ad(N)} + (1-p)e^{-ia \ ad(N)}$$

Next we seek the asymptotic form of transition operator defined as

$$T_t(M(X)) \equiv \lim_{n\to\infty} T_\phi^n(M(X))$$

and its associated generator $\lim_{t\to 0} \frac{dT_t}{dt} = L$. To derive the (weak) limit we work in the eigenbasis of number operator in which T_ϕ is a diagonal matrix, for the matrix elements of which the numerical limit is evaluated by means of the time parameter and the drift and diffusion coefficients as defined above equation (2.1). The resulting generator reads

$$\lim_{t\to 0} \frac{dT_t}{dt} = L \equiv c \ ad(N) + \gamma \ ad(N)ad(N)$$

The time dependent expectation value of POVM is evaluated according to quantum mechanical standards. The cyclic property of trace allows to either have a time evolved (state) density operator (Schroedinger picture), or a time evolved (observable) POVM i.e.

$$\langle M(X)\rangle_t \equiv Tr(\rho T_t(M(X))) = Tr(T_{-t}(\rho)M(X))$$

Then the time evolution equation for an initial density operator under the Markov operator flow $\rho_t \equiv T_{-t}(\rho)$, is

$$\frac{d\rho_t}{dt} = L(\rho_t) = c[N, \rho_t] + \gamma[N[N, \rho_t]]$$

This in its more familiar form is the quantum master equation

$$\frac{d\rho_t}{dt} = c[N, \rho_t] + \gamma(N^2\rho_t + \rho_t N^2 - 2N\rho_t N) \tag{4.1}$$

Remarks 4.1. 1) Interpretation of quantum random walk: POVM $M(X)$ provides a probability measure for obtaining a measurement result in interval X, for a system in state ρ. A noisy jittering of the position of window X might be due to imperfections of measuring device, and so this classical noise, formulated as classical algebraic walk, we have shown to *induce* a quantum walk in POVMs and states. 2) The commutator part of quantum master equation is related to the drift coefficient and can be attributed to the Hamiltonian component of the evolution. The term proportional to diffusion coefficient represents the irreversible part of dynamics. A density matrix initially chosen diagonal in the number operator eigenbasis develops no off-diagonal elements in its evolution as described by the quantum master equation, so its solution is easily obtained. 3) The quantum master equation is the evolution equation of a quantum system immersed in Markovian bath, and it generates a dynamical semigroup, which is a completely positive trace preserving one parameter linear map acting on the density matrix. The generator L of this semigroup is of the KL form[1]. 4) If we work with $\Omega = \mathbb{R}$, and a POVM constructed from the spectral measures of position or momentum canonical operators i.e. $M(X) = \int_X ds|s\rangle\langle s|$, for $s = q, p$, then a similar construction of algebraic walk leads to the respective quantum master equations; both these equations are of the KL form,

$$\dot{\rho}_t = c[P, \rho_t] + \gamma(P^2\rho_t + \rho_t P^2 - 2P\rho_t P), \quad \dot{\rho}_t = c[Q, \rho_t] + \gamma(Q^2\rho_t + \rho_t Q^2 - 2Q\rho_t Q)$$

Special case: let the group $SU(2)$ and its fundamental irrep $j = 1/2$, then the master equation (4.1) becomes an evolution equation for the density matrix of a quantum spin in contact with a heat bath and reads in terms of Pauli sigma matrices as $\frac{d\rho_t}{dt} = c[\sigma_3, \rho_t] + 2\gamma(\rho_t - \sigma_3\rho_t\sigma_3)$.

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