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Hilbert-substructure of Real Measurable Spaces on Reductive Groups, *I*; Basic Theory

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Abstract

This paper reconsiders the age-long problem of normed linear spaces which do not admit inner product and shows that, for some subspaces, $\mathfrak{F}_n(G)$, of real $L^p(G)$−spaces (when G is a reductive group in the Harish-Chandra class and $p=2n$), the situation may be rectified, via an outlook which generalizes the fine structure of the Hilbert space, $L^2(G)$. This success opens the door for harmonic analysis of unitary representations, $G\to End(\mathfrak{F}_n(G))$, of G on the Hilbert-substructure $\mathfrak{F}_n(G)$, which has hitherto been considered impossible.

Keywords: Reductive groups; Hilbert spaces; Orthogonal polynomials

Introduction

Let G be a reductive group in the Harish-Chandra class and denote $L^p(G, \mathbb{K})$ as the Lebesque spaces of $\mathbb{K}$−valued functions on G, $0<p<\infty$. The harmonic analysis of $L^2(G, \mathbb{C})$ is complete with the successful decomposition of unitary representations of G in [1]. However not much, as known for $L^2(G, \mathbb{C})$, could be said about other members of the Lebesque spaces on G. This is largely due to the absence of a manageable 'Hilbert space theory' (which made the discussion of unitary representations of G on $L^{p>2}(G, \mathbb{C})$ a forbidden concept) and has led to the employment of indirect techniques to extract important results out of them. The most celebrated of these successful indirect techniques at harmonic analysis of $L^p(G, \mathbb{C})$ is the Trombi-Varadarajan theory [2] which entails refining the decay estimates of $L^2(G, \mathbb{C})$ for $L^p(G, \mathbb{C})$ – Schwartz-like functions (for only $0<p\leq 2$) in order to match the asymptotic estimates of the corresponding Fourier transform on G.

These refinements could however hardly hold nor match for other values of p, especially $p>2$. Had it been that (any of) the $L^{p>2}$ – spaces are Hilbert spaces, with respect to which unitary representations could be discussed, a direct approach to such analysis would have been possible, general and more satisfying than that of only $L^2(G, \mathbb{C})$ and would have subsumed the Trombi-Varadarajan theory as well.

The modalities for conducting harmonic analysis on $L^p(G, \mathbb{K})$ have been roughly and immaturely spelt out in [3]. where (as it will be shown in the course of this paper) an inner product which was proved to be consistent with the norm-convergence in $L^{2n}([a,b], \mathbb{R})$, $n\in\mathbb{N}$, $a,b\in\mathbb{R}$, and which led to a general Cauchy-Schwartz inequality and construction of higher orthogonal polynomials, was made available. In this paper we shall employ the techniques of [3] to initiate discussion on harmonic analysis of $L^{2n}(G, \mathbb{R})$ by showing explicitly that each of these real measurable spaces on G contains a Hilbert space substructure (thus correcting the outlook in [3] where the substructure was wrongly placed on all of $L^{2n}([a,b], \mathbb{R})$, thereby making the techniques of this earlier paper of the author available to a wider audience) and that this substructure is rich enough to allow L^p –harmonic analysis on G. The results of this paper lay a foundation for the successful treatments of unitary representations of G on $\mathfrak{F}_n(G)$.

Hilbert-Substructure of $L^p(G, \mathbb{R})$

Let G denote a reductive group in the Harish-Chandra class and let $C_c^\infty(G,\mathbb{R})$ represent the space of smooth real-valued functions on G, [4]. Let $L^p(G, \mathbb{R})$ denote the Lebesque space of real-valued functions on G, where $0 < p < \infty$. We shall write $\langle\cdot,\cdot\rangle_2$ for the inner product on the Hilbert space $L^2(G, \mathbb{R})$ It is well-known that each member of $L^p(G, \mathbb{R})$ is a completion of $C_c^\infty(G,\mathbb{R})$ in an appropriate norm, $\|\cdot\|_p$. This means that $C_c^\infty(G,\mathbb{R})$ is appropriately dense in each of $C_c^\infty(G,\mathbb{R})$, in particular in $L^2(G, \mathbb{R})$. Thus it should be possible to construct subspaces of $C_c^\infty(G,\mathbb{R})$ via requirement(s) given by the inner product $\langle\cdot,\cdot\rangle_2$. One of such subspaces is defined below.

Definition 2.1: Let $n\in\mathbb{N}$. The set $\mathfrak{F}_n^*(G)$ is given as

$$\mathfrak{F}_n^*(G):=\{f_i\in C^\infty(G,\mathbb{R}):\langle f_i,f_j^{2n-1}\rangle_2<\infty,\forall\, i\in\mathbb{N}\}.$$

The defining requirement, $\langle f_i,f_j^{2n-1}\rangle_2<\infty$, of members of $\mathfrak{F}_n^*(G)$ is appropriate, since $L^2(G, \mathbb{R})$ (the completeion of $C^\infty(G, \mathbb{R})$ in the norm $\|\cdot\|_2=\sqrt{\langle\cdot,\cdot\rangle_2}$) and $C^\infty(G, \mathbb{R})$ are multiplicative algebras (in that if $f\in C^\infty(G, \mathbb{R})$ (respectively, $L^2(G, \mathbb{R})$) then $f^n\in C^\infty(G, \mathbb{R})$ (respectively, $L^2(G, \mathbb{R})$), for all $n\in\mathbb{N}$). The denseness of $C^\infty(G, \mathbb{R})$ in $L^2(G, \mathbb{R})$ implies that $\mathfrak{F}_n^*(G)\neq\phi$ and that (when endowed with the sup-norm) $\mathfrak{F}_n^*(G)$ is an incomplete normed linear space over $\mathbb{R}$. We shall denote the completion of $\mathfrak{F}_n^*(G)$ under the L^p– norm, $\|\cdot\|_{p=2n}$, simply by $\mathrm{F}_n(G)$. A first property of $\mathrm{F}_n(G)$ giving its relationship with $L^{p=2n}(G, \mathbb{R})$ is proved as follows.

Lemma 2.2: $\mathrm{F}_n(G)\subseteq L^{2n}(G, \mathbb{R})$, for all $n\in\mathbb{N}$, with equality when, and only when, $n=1$.

Proof: Let $f\in\mathrm{F}_n(G)$, then $\langle f_i,f_j^{2n-1}\rangle_2<\infty$. Hence,

$$\|f\|_{p=2n}^{2n}=\int_G(f(x))^{2n}dx=\int_G(f(x))(f(x))^{2n-1}dx=\langle f,f^{2n-1}\rangle_2<\infty,$$

showing that $f\in L^{2n}(G, \mathbb{R})$.

It is clear that $\mathrm{F}_1(G)=L^2$– completion of $\mathfrak{F}_1^*(G)$ which, when combined with the fact that

$$\mathfrak{F}_1^*(G):=\{f\in C_c^\infty(G,\mathbb{R}):\|f\|_2^2=\langle f,f\rangle_2=\langle f,f^{2(1)-1}\rangle_2<\infty\},$$

establishes the inclusion $L^2(G, \mathbb{R})$ $\mathrm{F}_1(G)$ and hence $\mathrm{F}_1(G)=L^2(G, \mathbb{R})$.

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The details of the proof above show that each $\mathfrak{F}_n(G)$ is an $\|\cdot\|_{2n}$–normed linear subspace of $L^{2n}(G, \mathbb{R})$ and that $\mathfrak{F}_1(G)$ is a real Hilbert space. That each of $L^{2n}(G, \mathbb{R})$, for $n>1$ is not an inner product space does not preclude this possibility for each of $\mathfrak{F}_n(G)$ In fact we may convert each $\mathfrak{F}_n^*(G)$ into an inner product space in the following defined manner.

Definition 2.3: For any $f, g \in \mathfrak{F}_n^*(G)$, we set the pairing $\langle f, g\rangle_{2n}$ as

$$\langle f, g\rangle_{2n} := \langle f, g^{2n-1}\rangle_2.$$

It is clear, for any $f, g \in \mathfrak{F}_n^*(G)$, that $\langle f, g\rangle_{2n} < \infty$ since

$$\mathfrak{F}_n^*(G) \subset C^\infty(G,\mathbb{R}) \subset L^2(G,\mathbb{R}).$$

Theorem 2.4: The pair $(\mathfrak{F}_n(G), \langle\cdot,\cdot\rangle_{2n})$ is a real Hilbert space.

Proof: We note, for $f, g, h \in \mathfrak{F}_n^*(G)$ and $\alpha\in\mathbb{R}$, that $\langle f+g, h\rangle_{2n} = \langle f,h\rangle_{2n} + \langle g,h\rangle_{2n}$, $\langle \alpha f, g\rangle_{2n} = \alpha\langle f,g\rangle_{2n}$, $\langle f,f\rangle_{2n} \geq 0$, $\langle f,f\rangle_{2n} = 0$ if $f=0$ and $\langle f,g\rangle_{2n} = \langle g, f\rangle_{(2n)^*}$. (Here $(2n)^*$ denote that the power, $(2n)-1$ in the L^2–inner product now goes to the first entry). Since $\mathfrak{F}_n(G)$ is the completion of $\mathfrak{F}_n^*(G)$ in the $\|\cdot\|_{2n}$ – norm the result follows.

We shall refer to the pair $(\mathfrak{F}_n(G), \langle\cdot,\cdot\rangle 2n)$ as a Hilbert-substructure of $L^{2n}(G, \mathbb{R})$. It is our modest aim in this paper to use the present general outlook (afforded by $\mathfrak{F}_n(G)$) on Hilbert (function) spaces to prove some results about $\mathfrak{F}_n(G)$ in order to convince the mathematical public of the necessity of doing analysis on $(\mathfrak{F}_n(G), \langle\cdot,\cdot\rangle 2n)$, as against the consideration of only $(\mathfrak{F}_1(G), \langle\cdot,\cdot\rangle 2) = (L^2(G,\mathbb{R}), \langle\cdot,\cdot\rangle 2)$. We shall establish the foundation on which each of $\mathfrak{F}_n(G)$, $n\in\mathbb{N}$, would be seen to possess a generalization of the fine structure of $\mathfrak{F}_1(G) = L^2(G,\mathbb{R})$. A first among the fine structure well-known for $L^2(G,\mathbb{R})$ is the contribution of its inner product, $\langle\cdot,\cdot\rangle_2$, in the proof of the triangle-inequality axiom of an $\|\cdot\|_{2n}$ – norm. We are here referring to the direct contribution of Cauchy-Schwartz inequality in the proof of the triangle inequality

$$\|f+g\|_2 \leq \|f\|_2 + \|g\|_2,$$

for all $f,g\in L^2(G,\mathbb{R})$. Even though we already know that the Minkowski inequality

$$\|f+g\|_{2n} \leq \|f\|_{2n} + \|g\|_{2n},$$

holds for all $f,g\in L^2(G,\mathbb{R})$, via the truth of Holder's inequality and that hence $\|f+g\|_{2n} \leq \|f\|_{2n} + \|g\|_{2n}$ holds for all $f,g\in\mathfrak{F}_n(G)$ (by obvious restriction), it would be necessary (as usually performed for the $\|\cdot\|_{2n}$ – norm via the Cauchy-Schwartz inequality) to have what we may call an inner product proof of the Minkowski inequality for all the Hilbert-substructures, $\mathfrak{F}_n(G)$, $n \geq 1$ (and not just for $\mathfrak{F}_1(G) = L^2(G,\mathbb{R})$ only). Indeed, if accomplished, this will give credence to the independence of each of $\mathfrak{F}_n(G)$ from (the normed linear space) $L^{2n}(G,\mathbb{R})$, a reminiscence of the importance of Cauchy-Schwartz inequality and the independence of $L^2(G,\mathbb{R})$ from all $L^p(G,\mathbb{R})$ – spaces.

In order to achieve the feat outlined above we need a (general) Cauchy-Schwartz inequality for members of $\mathfrak{F}_n(G)$, $n \geq 1$. It happens that the much we need in order to achieve our aim is contained in the following.

Lemma 2.5: (A Cauchy-Schwartz inequality for $\mathfrak{F}_n(G)$).

Given that $f,g\in\mathfrak{F}_n(G)$, then

$$|\langle f^{p-k}, g^k\rangle_2| \leq \|f\|_p^{p-k}\,\|g\|_p^k,$$

for all $k\in\mathbb{N}$, $p\in 2\mathbb{N}$, with $p=2n$.

Proof: The classical Cauchy-Schwartz inequality implies that

$$|\langle f^{p-k}, g^k\rangle_2| \leq \|f^{p-k}\|_2 \|g^k\|_2,$$

for all $f,g\in\mathfrak{F}_1(G)$. We only need to show that

$$\|f^{p-k}\|_2\,\|g^k\|_2 = \|f\|_p^{p-k}\,\|g\|_p^k.$$

To this end we see that

$$\|f^{p-k}\|_2\,\|g^k\|_2 = \left(\int_G |f(x)|^{2(p-k)}\,dx\right)^{\frac{1}{2}} \cdot \left(\int_G |g(x)|^{2k}\,dx\right)^{\frac{1}{2}} =$$

$$\left[\left(\int_G |f(x)|^{2(p-k)}\,dx\right)^{\frac{1}{2(p-k)}}\right]^{(p-k)} \cdot \left[\left(\int_G |g(x)|^{2k}\,dx\right)^{\frac{1}{2k}}\right]^{k} = \|f\|_{2(p-k)}^{p-k} \cdot \|g\|_{2k}^{k}.$$

Since $k\in\mathbb{N}$ we may set $p=2k\in\mathbb{N}$, so that $2(p-k)=2(2k-k)=2k=p$. Hence $\|f^{p-k}\|_2\,\|g^k\|_2 = \|f\|_{2(p-k)}^{p-k} \cdot \|g\|_{2k}^{k} = \|f\|_p^{p-k} \cdot \|g\|_p^k$, as required.

On setting $p=2$ and $k=1$ in Lemma 2.5 we see that $n=1$ and we arrive at the classical Cauchy-Schwartz inequality for $\mathfrak{F}_1(G)$.

It may have been expected that a generalization of the classical Cauchy-Schwartz inequality for $\mathfrak{F}_1(G)$ to all of $\mathfrak{F}_n(G)$, $n\geq 1$, would be that of finding a bound for $|\langle f,g\rangle_{2n}|$, for $f,g\in\mathfrak{F}_n(G)$. We are however not motivated by blind generalization but by seeking an inequality that would serve the L^{2n}–norm on $\mathfrak{F}_n(G)$ (in exactly the same way the classical Cauchy-Schwartz inequality serves the L^2–norm on $\mathfrak{F}_1(G)$) in the proof of Minkowski inequality. We shall advise this cautionary measure in generalizing other inequalities (like the Bessel's inequality) of inner product spaces to all of $\mathfrak{F}_n(G)$. This use for Lemma 2.5 is, in this wise, contained in the following.

Theorem 2.6: (An inner-product proof of Minkowski inequality on $\mathfrak{F}_n(G)$).

Given that $f,g\in\mathfrak{F}_n(G)$, then

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p,$$

for all $p=2n$.

Proof: $\|f+g\|_p^p = \langle f+g, f+g\rangle_p = \langle f, f+g\rangle_p + \langle g, f+g\rangle_p$

$$= \sum_{k=0}^{p-1} {}^{(p-1)}C_k \langle f, f^{p-k-1}g^k\rangle_2 + \sum_{k=0}^{p-1} {}^{(p-1)}C_k \langle g, f^{p-k-1}g^k\rangle_2$$

$$= \sum_{k=0}^{p-1} {}^{(p-1)}C_k \langle f^{p-k}, g^k\rangle_2 + \sum_{k=0}^{p-1} {}^{(p-1)}C_k \langle f^{p-k-1}, g^{k+1}\rangle_2$$

$$= \|f\|_p^p + \sum_{k=1}^{p-1} {}^{(p-1)}C_k \langle f^{p-k}, g^k\rangle_2 + \sum_{k=0}^{p-2} {}^{(p-1)}C_k \langle f^{p-k-1}, g^{k+1}\rangle_2 + \|g\|_p^p$$

$$= \|f\|_p^p + \sum_{k=1}^{p-1} {}^{(p-1)}C_k \langle f^{p-k}, g^k\rangle_2 + \sum_{k=1}^{p-1} {}^{(p-1)}C_{k-1} \langle f^{p-k}, g^k\rangle_2 + \|g\|_p^p$$

$$= \|f\|_p^p + \sum_{k=1}^{p-1} \left({}^{(p-1)}C_k + {}^{(p-1)}C_{k-1}\right)\langle f^{p-k}, g^k\rangle_2 + \|g\|_p^p$$

$$\leq \|f\|_p^p + \sum_{k=1}^{p-1} \left({}^{(p-1)}C_k + {}^{(p-1)}C_{k-1}\right)|\langle f^{p-k}, g^k\rangle_2| + \|g\|_p^p$$

$$= \|f\|_p^p + \sum_{k=1}^{p-1} {}^{p}C_k\, |\langle f^{p-k}, g^k\rangle_2| + \|g\|_p^p$$

$$\leq \|f\|_p^p + \sum_{k=1}^{p-1} {}^{p}C_k\, \|f\|_p^{p-k} \cdot \|g\|_p^k + \|g\|_p^p = (\|f\|_p + \|g\|_p)^p.$$

The reader may check that the above computations go through even when $n=1$ (*i.e.*, $p=2$). This shows the universality of Lemma 2.5 and Theorem 2.6.

One of the cornerstones of inner product spaces, in particular of the space $L^2(G,\mathbb{R}) = \mathfrak{F}_1(G)$, is the parallelogram equality;

$$\|f+g\|_2^2 + \|f-g\|_2^2 - 2(\|f\|_2^2 + \|g\|_2^2) = 0,$$

so named because of its geometric contents. It is customary, in the

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general theory of inner product spaces, to verify this equality for any given norm in order to ascertain if the corresponding normed linear space could also be an inner product space. It was on this basis that each of the $L^{p>2}-$ spaces was rightly knocked out of the race for the possession of an inner product. However Theorem 2.4 has now shown that it is unfair to force all the $L^{p>2}-$ spaces to be induced by the inner product, $\langle\cdot,\cdot\rangle_2$, of the L^2- space.

In the light of Theorem 2.4 it would therefore be necessary to re-consider the properties of the polynomial map,

$\mathfrak{F}_n(G)\times\mathfrak{F}_n(G)\to\mathbb{R}:(f,g)\mapsto C_p(f,g)$

given as

$C_p(f,g):=\|f+g\|_p^p+\|f-g\|_p^p-2(\|f\|_p^p+\|g\|_p^p),$

For $p=2n$, $n\geq 1$.

As just remarked above, it has been classically known that $C_2(f,g)\equiv 0$ and by implication, $C_2(f,f)\equiv 0$ (a proof of which may be deduced from Lemma 2.7 (*iv.*) below), for all $f,g\in\mathfrak{F}_1(G)$. We should however not expect these nullities any more for $C_p(f,g)$, when $f,g\in\mathfrak{F}_n(G)$, for all $p=2n$ and it is in our expectations that the geometric contents of $C_p(f,g)$ would reveal the degree of inertial curvature of each of $\mathfrak{F}_n(G)$. In this wise the Hilbert space $\mathfrak{F}_1(G)$ has (just like the Euclidean space $\mathbb{R}^n$) a zero-degree of inertial curvature. It should also be of tremendous interest to consider this polynomial, $(f,g)\mapsto C_p(f,g)$, for all $f,g\in L^p(G,\mathbb{K})$, $p\in(0,\infty)$ and to investigate how solutions of its differential equations

$$\Psi(f,g,C_p(f,g),\frac{\partial C_p}{\partial f},\frac{\partial C_p}{\partial g})=0$$

(possibly for $p=4$) could be of use to the geometry of general relativity.

In the meantime we shall state some of the basic properties of the inertial curvature polynomial map, $(f,g)\mapsto C_p(f,g)$.

Lemma 2.7: Let $f,g\in\mathfrak{F}_n(G)$, $\alpha\in\mathbb{R}$ and $p=2n$ with $n\geq 1$. Then

(i.) $C_p(f,g)=C_p(g,f)$;

(ii.) $C_p(-f,g)=C_p(f,g)$;

(iii.) $C_p(\alpha f,\alpha g)=|\alpha|^p\cdot C_p(f,g)$;

(iv.) $C_p(f,f)=(2^p-4)\cdot\|f\|_p^p$. Hence $C_p(f,g)\neq 0$ *for all* $n>1$.

The results of the above Lemma follow from the earlier given expression for $C_p(f,g)$. It is therefore clear from (*iv.*) of Lemma 2.7 why

$C_2(f,f)\equiv 0$

for all $f\in\mathfrak{F}_n(G)$, $n>1$.We may in fact give an explicit formula for $C_p(f,g)$ in terms of $\langle\cdot,\cdot\rangle_2$ as follows,

Lemma 2.8: Let $f,g\in\mathfrak{F}_n(G)$ and $p=2n$, with $n\geq 1$. Then

$$C_p(f,g)=2\cdot\sum_{k\in 2\mathbb{N}}^{p-2}{}^pC_k\langle f^{p-k},g^k\rangle_2.$$

In particular, $C_2(f,g)\equiv 0$.

The above expression for $C_p(f,g)$ may be seen from explicit expansion of $\|f+g\|_p^p$ (respectively, of $\|f-g\|_p^p$) starting as

$\|f+g\|_p^p=\langle f+g,f+g\rangle_p=\langle f+g,(f+g)^{p-1}\rangle_2$

(respectively, as $\|f-g\|_p^p=\langle f-g,f-g\rangle_p=\langle f-g,(f-g)^{p-1}\rangle_2$) and using the fact that ${}^pC_k+{}^pC_k(-1)^k=\begin{cases}0 & ,\text{if } k \text{ is odd},\\ 2\cdot{}^pC_k & ,\text{if } k\in 2\mathbb{N}.\end{cases}$

We now introduce the notion of orthogonality in $\mathfrak{F}_n(G)$.

Definition 2.9: Let $f,g\in\mathfrak{F}_n(G)$ and $p=2n$. We shall say that f is $p-$ orthogonal to g if either $\langle f,g\rangle_p=0$ or $\langle g,f\rangle p=0$ (or both)

The usual relationships between orthogonality and linear independence hold (*cf.*[2.],*p*.21–23). The notions of Fourier transform and Fourier expansion of functions may also be given in terms of a $p-$ orthogonal set $\mathfrak{F}_n(G)$. Indeed if $\{e_{j,p}\}_{j=1}^\infty$ is a $p-$ orthogonal set in $\mathfrak{F}_n(G)$ and $f\in\{e_{1,p},\ldots,e_{m,p},\ldots\}$, then

$$f=\sum_{j=1}^\infty\alpha_j\cdot e_{j,p},$$

for some $\alpha_j\in\mathbb{R}$. The Fourier coefficients of f with respect to this $p-$ orthogonal set is then

$$\langle f,e_{k,p}\rangle_p=\sum_{j=1}^\infty\alpha_j\cdot\langle e_{j,p},e_{k,p}\rangle_p=\alpha_k\cdot\langle e_{k,p},e_{k,p}\rangle_p=\alpha_k\cdot\|e_{k,p}\|_p^p=\alpha_k;$$

so that

$$f=\sum_{j=1}^\infty\langle f,e_{j,p}\rangle_p\cdot e_{j,p},$$

for any $p\in 2\mathbb{N}$, is the Fourier expansion of the function f.

In order to then generate a $p-$orthogonal set from any given linearly independent set in $\mathfrak{F}_n(G)$ it will be necessary to have the so-called Gram-Schmidt procedure in place. We give this procedure in its generality and thereafter consider a well-known concrete example to show its universality.

Theorem 2.10 (Gram-Schmidt procedure in $\mathfrak{F}_n(G)$): Let $H\subseteq G$ for which $\mathfrak{F}_n(H)$ is a Hilbert subspace of $\mathfrak{F}_n(G)$ (under the obvious restriction of $\langle\cdot,\cdot\rangle_p$, $p=2n$) and let $\{f_i\}_{i=0}^\infty$ denote a linearly independent set in $\mathfrak{F}_n(H)$. Choose

$$e_{0,p}=\frac{f_0}{\|f_0\|_p};$$

$$e_{1,p}=\frac{v_1}{\|v_1\|_p},\ where\ v_1=f_1-\langle f_1,e_{0,p}\rangle_p\cdot e_{0,p};$$

$$e_{m,p}=\frac{v_m}{\|v_m\|_p},\ where\ v_m=f_m-\sum_{k=1}^{m-1}\langle f_m,e_{k,p}\rangle_p\cdot e_{k,p},\ m=2,3,4,\cdots.$$

Then the set $\{e_{j,p}\}_{j=0}^\infty$ is a $p-$orthonormal set in $\mathfrak{F}_n(H)$.

Proof. It is clear that

$\|e_{j,p}\|=1.$

We only need to show that (2.9) holds for members of $\{e_{j,p}\}$ and we may establish this by showing that each v_m is $p-$orthogonal to any of $\{e_{j,p}\}_{j=1}^{m-1}$ as follows: $\langle v_m,e_{j,p}\rangle_p=$

$$\langle f_m-\sum_{k=1}^{m-1}\langle f_m,e_{k,p}\rangle_p\cdot e_{k,p},e_{j,p}\rangle_p=\langle f_m,e_{j,p}\rangle_p-\sum_{k=1}^{m-1}\langle f_m,e_{j,p}\rangle_p\cdot\langle e_{k,p},e_{j,p}\rangle=$$
$$\langle f_m,e_{j,p}\rangle_p-\langle f_m,e_{j,p}\rangle_p\cdot\langle e_{j,p},e_{j,p}\rangle=\langle f_m,e_{j,p}\rangle_p-\langle f_m,e_{j,p}\rangle_p\cdot\|e_{j,p}\|_p^p=0.$$

This procedure may be explicitly exhibited in an example as follows.

Example 2.11: (Legendre and Higher Legendre polynomials in $\mathfrak{F}_n(G)$).

Let G=$\mathbb{R}$ and $H=[-1,1]$.Then $\mathfrak{F}_n([-1,1])$ is a Hilbert subspace of $\mathfrak{F}_n(\mathbb{R})$ for $n\in\mathbb{N}$. Now consider the linearly independent set $\{1,t,t^2,\ldots,t^k,\ldots\}$ in $\mathfrak{F}_n([-1,1])$. Following the procedure in Theorem 2.10 above (*cf.*[2.]), we arrive at the following $p=2n-$ orthogonal polynomials in $\mathfrak{F}_n([-1,1])$:

$$e_{0,p}(t)=\left(\frac{1}{2}\right)^{1/p},$$

$$e_{1,p}(t)=\left(\frac{p+1}{2}\right)^{1/p}\cdot t,$$

$$e_{2,p}(t)=\left(\frac{1}{2\cdot\sum_{k=0}^{p}(-1)^k\frac{3^{p-k}\cdot{}^pC_k}{(2p-2k+1)}}\right)^{1/p}\cdot(3t^2-1),$$

$$e_{3,p}(t)=\left(\frac{1}{2\cdot\sum_{k=0}^{p}(-1)^k\frac{(p+3)^{p-k}\cdot(p+1)^k\cdot{}^pC_k}{(3p-2k+1)}}\right)^{1/p}\cdot[(p+3)t^3-(p+1)t],\cdots$$

The reader may observe that when n=1 (and hence p=2) the above polynomials reduce to $e_{0,2}(t)=\sqrt{\frac{1}{2}}$, $e_{1,2}(t)=\sqrt{\frac{3}{2}}t$, $e_{2,2}(t)=\sqrt{\frac{5}{8}}(3t^2-1)$, $e_{3,2}(t)=\sqrt{\frac{7}{8}}(5t^3-3t),\cdots$ which are the well-known Legendre polynomials in $\mathfrak{F}_n([-1,1])$, up to factors of non-zero positive constants.

Looking a step further than n=1, we may equally see that when n=2 (and hence p=4) we have the polynomials $e_{0,4}(t)=\sqrt[4]{\frac{1}{2}}$, $e_{1,4}(t)=\sqrt[4]{\frac{5}{2}}t$, $e_{2,4}(t)=\sqrt[4]{\frac{35}{96}}(3t^2-1)$, $e_{3,4}(t)=\sqrt[4]{\frac{1287}{7008}}(7t^3-5t),\cdots$ which may be seen as the higher Legendre polynomials in $\mathfrak{F}_n([-1,1])$ (up to factors of non-zero positive constants). Other values of $n\in\mathbb{N}$ may also be considered.

A successful derivation of the differential equations satisfied by the higher Legendre polynomials above (or of any other higher orthogonal polynomials in $\mathfrak{F}_n(H)$) would lead to getting the corresponding higher Legendre functions. We already know that the zonal spherical functions on $G=SL(2,\mathbb{R})$ are exactly these Legendre functions. However we do not yet know the reductive group G, whether $G=SL(2,\mathbb{R})$ or its covering group, that would have these higher orthogonal functions as its zonal spherical functions and the explicit differential equations satisfied by the higher Legendre polynomials in $\mathfrak{F}_n(G)$.

In order to accomplish these feats we need to know (all) the series of representations of G on $\mathfrak{F}_n(G)$ which will then be used to define the corresponding zonal spherical functions, whose differential equations may now be sought in the manner of ([1]). We shall however embark on these in another paper.

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