Global Classical Solutions to the Mixed Initial-boundary Value Problem for a Class of Quasilinear Hyperbolic Systems of Balance Laws

Zhi-Qiang Shao

Department of Mathematics, Fuzhou University, Fuzhou 350002, China

Abstract

It is proven that the mixed initial-boundary value problem for a class of quasilinear hyperbolic systems of balance laws with general nonlinear boundary conditions in the half space \(|\{(t, x) | t \geq 0, x \geq 0\}\) admits a unique global C1 solution \(u = u(t, x)\) with small C1 norm, provided that each characteristic with positive velocity is weakly linearly degenerate. This result is also applied to the flow equations of a model class of fluids with viscosity induced by fading memory.

MSC: 35L45; 35L50; 35Q72.

Keywords: Mixed initial-boundary value problem; Global classical solution; Quasilinear hyperbolic; systems of balance laws; Weakly linearly degenerate characteristics

Introduction and Main Result

Consider the following quasilinear hyperbolic system of balance laws in one space dimension:

\[
\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} + Lu = 0
\]

(1.1)

where \(L > 0\) is a constant; \(u = (u_1, \ldots, u_n)^T\) is the unknown vector function of \((t, x)\), \(f(u)\) is a given \(C^3\) vector function of \(u\).

It is assumed that system (1.1) is strictly hyperbolic, i.e., for any given \(u\) on the domain under consideration, the Jacobian \(A(u) = \nabla f(u)\) has \(n\) real distinct eigenvalues.

\[
\lambda_1(u) < \lambda_2(u) < \ldots < \lambda_n(u)
\]

(1.2)

Let \(l_i(u) = (l_{i1}(u), \ldots, l_{in}(u))\) (resp. \(r_i(u) = (r_{i1}(u), \ldots, r_{in}(u))\)) be a left (resp. right) eigenvector corresponding to \(\lambda_i(u)\) (resp. \(\lambda_i(u)\)).

\[
l_i(u)A(u) = \lambda_i(u)l_i(u)
\]

(1.3)

Then we have

\[
det \begin{vmatrix} l_i(u) \end{vmatrix} \neq 0 \quad (\text{Equivalently, } det \begin{vmatrix} r_i(u) \end{vmatrix} \neq 0).
\]

(1.4)

Without loss of generality, we may assume that on the domain under consideration

\[
l_i(u)r_j(u) = \delta_{ij} (i, j = 1, \ldots, n)
\]

(1.5)

And

\[
r_i^T(u)r_j(u) = 1 \quad (i = 1, \ldots, n)
\]

(1.6)

Where \(\delta_{ij}\) stands for the Kronecker's symbol.

Clearly, all \(\lambda_i(u), l_i(u)\) and \(r_j(u)\) (i, j = 1, ..., n) have the same regularity as \(A(u)\), i.e., \(C^2\) regularity.

We assume that on the domain under consideration, each characteristic with positive velocity is weakly linearly degenerate and the eigenvalues of \(A(u) = \nabla f(u)\) satisfy the non-characteristic condition.

\[
\lambda r(u) < 0 < \lambda s(u)
\]

(1.9)

(r = 1, ..., m; s = m + 1, ..., n)

(1.10)

We are concerned with the existence and uniqueness of global C1 solutions to the mixed initial-boundary value problem for system (1.1) in the half space

\[
D = \{(t, x) | t \geq 0, x \geq 0\}
\]

(1.11)

with the initial condition:

\[
t = 0 : u = \varphi(x)(x \geq 0)
\]

(1.12)

and the nonlinear boundary condition:

\[
x = 0 : v_e = G_i(\alpha(t), v_1,..., v_m) h_i(t), s = m + 1, ..., n(t \geq 0)
\]

(1.13)

Where

\[
v_i = l_i(u)u(i = 1, ..., n)
\]

(1.14)

And

\[
\alpha(t) = (\alpha_1(t), \ldots, \alpha_n(t))
\]

Then, we have

\[
x_0 t_0 \geq \max\{\sup_{|x| \leq 1, \|h(t)\| + \|h'(t)\| < +\infty} h(t)\}
\]

(1.15)

in which

\[
h(t) = (h_{i1}(t), \ldots, h_{in}(t))
\]

*Corresponding author: Zhi-Qiang Shao, Department of Mathematics, Fuzhou University, Fuzhou 350002, China, Tel: 86-0591-83852790; E-mail: zqshao@fzu.edu.cn

Received November 24, 2014; Accepted November 25, 2014; Published December 10, 2014


Copyright: © 2014 Shao ZQ. This is an open-access article distributed under the terms of the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original author and source are credited.
Without loss of generality, we assume that
\[ G_i(\alpha(t),...,0) = 0 (s = m+1,...,n) \]  
(1.16)

For the special case where (1.1) is a quasilinear hyperbolic system of conservation laws, i.e., L=0, such kinds of problems have been extensively studied (for instance, see [1-8] and the references therein). In particular, Li and Wang proved the existence and uniqueness of global C^1 solutions to the mixed initial boundary value problem for first order quasilinear hyperbolic systems with general nonlinear boundary conditions in the half space \{ (t,x | t \geq 0, x \geq 0) \}. On the other hand, for quasilinear hyperbolic systems of balance laws, many results on the existence of global solutions have also been obtained by Liu, et al., (for instance, see [8-14] and the references therein), and some methods have been established. So the following question arises naturally: when can we obtain the existence and uniqueness of semi-global C1 solutions for quasilinear hyperbolic systems of balance laws? It is well known that for first-order quasilinear hyperbolic systems of balance laws, generically speaking, the classical solution exists only locally in time and the singularity will appear in a finite time even if the data are sufficiently smooth and small [15-20]. However, in some cases global existence in time of classical solutions can be obtained. In this paper, we will generalize the results in [21] to a nonhomogeneous quasilinear hyperbolic system, the analysis relies on a careful study of the interaction of the nonhomogeneous term. Our main results can be stated as follows:

**Theorem 1.1.** Suppose that the non-characteristic condition (1.10) holds and system (1.1) is strictly hyperbolic. Suppose furthermore that for \( j = m+1,..., n \); each j-characteristic field with positive velocity is weakly linearly degenerate. Suppose finally that \( \varphi, \alpha, G_i, h_i \) are all C^1 functions with respect to their arguments, satisfying (1.15)-(1.16) and the conditions of C^1 compatibility at the point \( (0, 0) \). Then there exists a sufficiently small \( \theta_0 > 0 \) such that for any given \( 0 \leq \theta < \theta_0 \), the ith characteristic trajectory passing through \( \hat{u} = 0 \) coincides with the \( \hat{u}_i \)-axis at least for \( | \hat{u}_i | \) small, namely,
\[
\tilde{\gamma}(\hat{u}_i, c_i) = c_i, \forall | \hat{u}_i | \text{ small} \ (i = 1, ..., n);
\]
(2.1)

Where
\[
e_i = (0, ..., 0, 1, 0, ..., 0)^T
\]

This transformation is called the normalized transformation, and the corresponding unknown variables \( \hat{u} = (\hat{u}_1, ..., \hat{u}_n)^T \) are called the normalized variables or normalized coordinates [24-28].

Let
\[ w_i = li(u)(\hat{u}) = (l_i(u); : : : ; l_n(u)) \]

where
\[ li(u) = (l_1(u); : : : ; l_n(u)) \]
denotes the ith left eigenvector.

By (1.5), it follows from (1.14) and (2.2) that
\[ u = \sum_{k=1}^{n} V_i \tilde{\gamma}_i(u) \]
and
\[ u_i = \sum_{k=1}^{n} w_i \tilde{\gamma}_i(u) \]

Let

\[ \frac{d}{dt} \tilde{\gamma}_i(u) = \tilde{\gamma}_i(u) \frac{\partial}{\partial \hat{x}} \]  
(2.5)

be the directional derivative along the ith characteristic. Our aim in this section is to prove several formulas on the decomposition of waves for system (1.1), which will play an important role in our discussion.

**Lemma 2.1.**
\[
\frac{d(e_i \cdot w_i)}{dt} = \sum_{j,k=1}^{n} \gamma_{ijk} (u) w_j w_k + \sum_{j,k=1}^{n} \gamma_{ijk} (u) v_j w_k + \sum_{j,k=1}^{n} \gamma_{ijk} (u) v_j v_k (i = 1, ..., n) \]  
(2.6)

Where
\[ \gamma_{ijk} (u) = (\lambda_i (u) - \lambda_j (u)) r_{j}^i (u) \]  
\[ \gamma_{ijk} (u) = -L_{r_{j}^i} (u) V_{r_{j}^i} (u) \]  
(2.7)

Hence, we have
\[ \tilde{\gamma}_i(u) = 0, \forall j \neq i (i = 1, ..., n) \]
(2.9)

Moreover, in the normalized coordinates,
\[ \tilde{\gamma}_i(u; e_i) = 0, \forall | u_i | \text{ small, } \forall i, j \]
(2.11)

while, when the ith characteristic \( \lambda_i (u) \) is weakly linearly degenerate, in the normalized coordinates,
\[ \gamma_{ijk} (u; e_i) = 0, \forall | u_i | \text{ small, } \forall i, j \]
(2.12)

**Lemma 2.2.**
\[
\frac{d(e_i \cdot v_i)}{dt} = \sum_{j,k=1}^{n} \beta_{ijk} (u) v_j w_k + \sum_{j,k=1}^{n} \beta_{ijk} (u) v_j v_k (i = 1, ..., n) \]  
(2.13)

Where
\[ \beta_{ijk} (u) = (\lambda_i (u)) r_{j}^i (u) V_{l}(u) \]  
\[ \beta_{ijk} (u) = -L_{r_{j}^i} (u) V_{l}(u) \]  
(2.14)

Thus, we have
\[ \tilde{\beta}_{i}(u) = 0, \forall i, j (i, j = 1, ..., n) \]
(2.16)

Moreover, by (2.1), in the normalized coordinates we have
\[ \beta_{a(j)}(u, e_j) = 0, \quad \forall \text{ } |u_j| \text{ small}, \forall \text{ } i, j \quad (2.17) \]

And

\[ \tilde{\beta}_{a(j)}(u, e_j) = 0, \quad \forall \text{ } |u_j| \text{ small}, \forall \text{ } i, j \quad (2.18) \]

The proofs of Lemmas 2.1-2.2 can be found in [29]. Similarly, we have

Lemma 2.3. In the normalized coordinates, it follows that

\[ \frac{d(e^i u_i)}{d t} = \sum_{j=1}^{N} e^i u_j w_j (i = 1, \ldots, n) \quad (2.19) \]

Where

\[ p_{a(i)}(u) = 0, \quad \forall \text{ } i, j \quad (2.20) \]

And

\[ p_{a(i)}(u) = \left( \lambda_a(u) \right) \int_0^1 \frac{e^i u_j}{\xi_j} (Tu_1, \ldots, Tu_{i-1}, u_{i+1}, \ldots, Tu_n) \, dT \quad (2.21) \]

Hence, the characteristic passing through point \((u(t, x), y(t, x))\) gives

\[ \Gamma_{a(i)}(u) = \left( \lambda_a(u) - \lambda_i(u) \right) r^i_j (u) \nabla l_i(u) r_i(u) \quad (2.29) \]

Hence, \( \Gamma_{a(i)}(u) = 0 \quad \forall i, j \quad (2.30) \)

Proof. Differentiating the first equation of (2.27) with respect to \(y\) gives

\[ \frac{d}{d t} \frac{\partial \tilde{\xi}_i(t, y)}{\partial y} = \nabla \lambda_i(u(t, \hat{x}(t, y))) \frac{\partial u}{\partial x} (t, \hat{x}(t, y)) \frac{\partial \hat{x}_i(t, y)}{\partial y} \quad (2.31) \]

Then, noting (2.6), it follows from (2.31) that

\[ \frac{d(e^i q_i)}{d t} = \frac{d(e^i w_i)}{d t} \frac{\partial \tilde{\xi}_i(t, y)}{\partial y} + e^i w_i \frac{d}{d t} \frac{\partial \tilde{\xi}_i(t, y)}{\partial y} = \sum_{i=1}^{N} e^i \left[ \gamma_{a(i)}(u) w_{i-1} + \tilde{\gamma}_{a(i)}(u) w_i \right] + e^i w_i [\nabla \lambda_i(u)] w_i \quad (2.32) \]

Thus, from (2.4), (2.7) and (2.32), we immediately get (2.28)-(2.30). The proof of Lemma 2.4 is finished.

Similarly, noting (2.4), by (2.13) and (2.31), we have

\[ \text{Lemma 2.5. Let } p(t; x) \text{ be defined by } p(t; x) = v(t; \hat{x}(t, y)) \frac{\partial \tilde{\xi}_i(t, y)}{\partial y} \quad (2.33) \]

Then along the characteristic \(x = \hat{x}(t, y)\) we have

\[ \frac{d(e^i \beta_{p(i)})}{d t} = \sum_{i=1}^{N} e^i \Gamma_{a(i)}(u) \frac{\partial \tilde{\xi}_i(t, y)}{\partial y} w_i + \sum_{i=1}^{N} e^i \tilde{\gamma}_{a(i)}(u) \frac{\partial \tilde{\xi}_i(t, y)}{\partial y} w_i \quad (2.34) \]

Where \( \tilde{\beta}_{a(i)}(u) \) is given by (2.15) and

\[ \tilde{\beta}_{a(i)}(u) = \tilde{\beta}_{a(i)}(u) + \nabla \lambda_i(u) r_i(u) \delta_{i(i)} \quad (2.35) \]

By (2.16), it is easy to see that

\[ B_{a(i)}(u) = 0 \quad \forall i \neq j, i = 1, \ldots, n \quad (2.36) \]

Moreover, by (2.17), in the normalized coordinates we have

\[ B_{a(i)}(u) = 0 \quad \forall \text{ } |u_j| \text{ small}, \forall \text{ } j \neq i \quad (2.37) \]

while, when the \(i\)th characteristic \(\lambda_i(u)\) is weakly linearly degenerate, in the normalized coordinates,

\[ B_{a(i)}(u) = 0 \quad \forall \text{ } |u_i| \text{ small} \quad \forall i \quad (2.38) \]

Lemma 2.6. Let \(q(t; x)\) be defined by \( q(t; x) = u(t; \hat{x}(t, y)) \frac{\partial \tilde{\xi}_i(t, y)}{\partial y} \)

Then along the i-th characteristic \(x = \hat{x}(t, y)\) we have

\[ \frac{d(e^i Z_{a(i)})}{d t} = \sum_{i=1}^{N} e^i \Gamma_{a(i)}(u) \frac{\partial \tilde{\xi}_i(t, y)}{\partial y} w_i \quad (2.39) \]

Where

\[ F_{a(i)}(u) = p_{a(i)}(u) + \nabla \lambda_i(u) r_i(u) \delta_{i(i)} \quad (2.40) \]

By (2.20) and (2.22), it is easy to see that

\[ F_{a(i)}(u) = 0, \quad \forall i \neq j, i = 1, \ldots, n \quad (2.41) \]

\[ F_{a(i)}(u) = 0, \quad \forall i \neq j, i = 1, \ldots, n \quad (2.42) \]
And
\[ F_p(u) = \nabla \lambda_i(u) \nabla (u) \quad \forall i = 1, \ldots, n \] (2.43)

Proof. Differentiating the first equation of (2.27) with respect to \( y \) gives
\[ \frac{d}{dt} \left( \delta \lambda_i(t, y) \right) = \nabla \lambda_i(u(t, \delta \lambda_i(t, y))) \frac{\partial u}{\partial x} (t, \delta \lambda_i(t, y)) \frac{\partial \delta \lambda_i(t, y)}{\partial y} \] (2.44)
Then, noting (2.19), it follows from (2.44) that
\[ \frac{d(e^{\lambda_i Z_i})}{dt} = \frac{d(e^{\lambda_i Z_i})}{dt} \frac{\partial \delta \lambda_i(t, y)}{\partial y} + e^{\lambda_i Z_i} \frac{d}{dt} \left( \delta \lambda_i(t, y) \right) = \left( \sum_{i=1}^{n} e^{\lambda_i Z_i} u_i u_i + e^{\lambda_i Z_i} \nabla \lambda_i(u) u_i \right) \frac{\partial \delta \lambda_i(t, y)}{\partial y} \] (2.45)

Thus, from (2.4), (2.20)-(2.22) and (2.45), we immediately get (2.39)-(2.43). The proof of Lemma 2.6 is finished.

Proof of Theorem 1.1

By the existence and uniqueness of a local \( C^1 \) solution for quasilinear hyperbolic systems [22], there exists \( T_0 > 0 \) such that the mixed initial-boundary value problem (1.1) and (1.12)-(1.13) admits a unique \( C^1 \) solution \( u = u(t, x) \) on the domain
\[ D(T_0) = \{ (t, x) | 0 \leq t \leq T_0, x \geq 0 \} \] (3.1)
Thus, in order to prove Theorem 1.1 it suffices to establish a uniform a priori estimate for the \( C^1 \) norm of \( u \) and \( u_x \) on any given domain of existence of the \( C^1 \) solution \( u = u(t, x) \).

Noting (1.2) and (1.10), we have
\[ \lambda_i(0) < \ldots < \lambda_n(0) < \lambda_{m+1}(0) < \ldots < \lambda_n(0) \] (3.2)
Thus, there exist sufficiently small positive constants \( \delta \) and \( \delta_0 \) such that
\[ \lambda_i(u) - \lambda_i(0) \geq 4\delta_0 \quad \forall |u|_\infty \leq \delta |i-1, \ldots, n-1| \] (3.3)
\[ \lambda_i(u) - \lambda_i(0) \leq \delta_0 \quad \forall |u|_\infty \leq \delta |i-1, \ldots, n| \] (3.4)
And
\[ |\lambda_i(0)| \geq \delta_0 \quad (i = 1, \ldots, n) \] (3.5)

For the time being it is supposed that on the domain of existence of the \( C^1 \) solution \( u = u(t, x) \) to the mixed initial-boundary value problem (1.1) and (1.12)-(1.13), we have
\[ |u(t, x)| \leq \delta \] (3.6)

At the end of the proof of Lemma 3.3, we will explain that this hypothesis is reasonable. Thus, in order to prove Theorem 1.1, we only need to establish a uniform a priori estimate for the piecewise \( C^1 \) norm of \( v \) and \( w \) defined by (1.14) and (2.1) on the domain of existence of the \( C^1 \) solution \( u = u(t, x) \).

For any fixed \( T > 0 \), let
\[ D_T^+ = \{ (t, x) | 0 \leq t \leq T, x \geq \lambda_i(0) + \delta_0 |t| \} \] (3.7)
\[ D_T^- = \{ (t, x) | 0 \leq t \leq T, 0 \leq x \leq \lambda_{m+1}(0) + \delta_0 |t| \} \] (3.8)
\[ D_T^0 = \{ (t, x) | 0 \leq t \leq T, (\lambda_i(0) + \delta_0 |t|) \leq x \leq (\lambda_n(0) + \delta_0 |t|) \} \] (3.9)
and for \( i = m+1, \ldots, n \) let
\[ D_T^+ = \{ (t, x) | 0 \leq t \leq T, -[\delta_0 + n(\lambda_i(0) - \lambda_{m+1}(0))] |t| \leq x - \lambda_i(0) |t| \leq [\delta_0 + n(\lambda_n(0))] |t| \} \] (3.10)
where \( n > 0 \) is suitably small (Figure 1).

Noting that \( n > 0 \) is small, by (3.3), it is easy to see that
\[ D_T^+ \cap D_T^- = \emptyset, \quad \forall i \neq j \] (3.11)
And
\[ \bigcup_{i=m+1}^{n} D_T^i \subset D_T^1 \] (3.12)

By the definitions of \( D_T^+ \) and \( D_T^- \), it is easy to get the following lemma. Lemma 3.1. For each \( i = m+1, \ldots, n \), on the domain \( D_T^i / D_T^0 \) we have
\[ c |t| \leq x - \lambda_i(0) |t| \leq C t, \quad c |t| \leq x - \lambda_i(0) |t| \leq C x \] (3.13)
where \( c \) and \( C \) are positive constants independent of \( T \).

Let
\[ V(D_T^i) = \max_{i=m+1}^{n} |1 + t|x|^{|t|} u_i(t, x) \|_{L^1(D_T^i)} \] (3.14)
\[ W(D_T^i) = \max_{i=m+1}^{n} |1 + t|x|^{|t|} u_i(t, x) \|_{L^\infty(D_T^i)} \] (3.15)
\[ V(D_T^i) = \max_{i=m+1}^{n} |1 + t|x|^{|t|} u_i(t, x) \|_{L^1(D_T^i)} \] (3.16)
\[ W(D_T^i) = \max_{i=m+1}^{n} |1 + t|x|^{|t|} u_i(t, x) \|_{L^\infty(D_T^i)} \] (3.17)
\[ V_i(T) = \max_{i=m+1}^{n} \max_{a \in D_T^0} (1 + T)|x|^{|t|} u_i(t, x) \|_{L^1(D_T^0)} \] (3.18)
\[ W_i(T) = \max_{i=m+1}^{n} \max_{a \in D_T^0} (1 + T)|x|^{|t|} u_i(t, x) \|_{L^\infty(D_T^0)} \] (3.19)
\[ U_i(T) = \max_{i=m+1}^{n} \max_{a \in D_T^0} (1 + T)|x|^{|t|} u_i(t, x) \|_{L^1(D_T^0)} \] (3.20)
\[ \tilde{V}_i(T) = \max_{i=m+1}^{n} \max_{a \in D_T^0} \int_{t}^{T} \tilde{V}_i(t, x) \|_{L^1(D_T^0)} \] (3.21)
\[ \hat{W}_i(T) = \max_{i=1,...,n} \max_{j=1} \int_{C_j} |w_i(t,x)| \, dt \]  
(3.22)

\[ \hat{U}_i(T) = \max_{i=1,...,n} \max_{j=1} \int_{C_j} |U_i(t,x)| \, dt \]  
(3.23)

Where \( C_j \) denotes any given jth characteristic in \( D_j^i \) (j ≠ i, i = m + 1, ..., n).

\[ V(T) = \max_{i=1,...,n} V_i(t,x) \]  
(3.24)

\[ W_i(T) = \max_{i=1,...,n} \int_{i=0}^{T} |w_i(t,x)| \, dx \]  
(3.25)

\[ U_i(T) = \max_{i=1,...,n} \int_{i=0}^{T} |u_i(t,x)| \, dx \]  
(3.26)

Where \( D_j^i(t) \) (t ≥ 0) denotes the t-section of \( D_j^i \).

\[ D_j^i(t) = \{(t,x) \mid T = t, (t,x) \in D_j^i \} \]  
(3.27)

\[ V_n(T) = \max_{i=1,...,n} \int_{i=0}^{T} |v_n(t,x)| \, dx \]  
(3.28)

And

\[ V_n(T) = \max_{i=1,...,n} \int_{i=0}^{T} |w_i(t,x)| \, dx \]  
(3.29)

Clearly, \( V_n(T) \) is equivalent to \( U_n(T) \).

\[ U_n(T) = \max_{i=1,...,n} \int_{i=0}^{T} |U_i(t,x)| \, dx \]  
(3.30)

In the present situation, similar to the corresponding result in [24,30-33], we have

**Lemma 3.2.** Suppose that in a neighborhood of \( \mu = 0 \): \( A(u) \in C^2 \) system (1.1) is strictly hyperbolic and (1.10) holds. Suppose furthermore that \( \phi(x) \) satisfies (1.15). Then there exists a sufficiently small \( \theta \) > 0 such that for any fixed \( \theta \) ∈ [0, \theta_0] on any given existence domain \( \{t,x\} \mid 0 \leq t \leq T \times x \geq 0 \} \) of the \( C^2 \) solution \( u = u(t,x) \) to the mixed initial-boundary value problem (1.1) and (1.12)-(1.13), we have the following uniform a priori estimates:

\[ V(D_n^i), W(D_n^i) \leq k, \theta \]  
(3.31)

where here and henceforth, \( k(i=1; 2,...) \) are positive constants independent of \( \theta \) and \( T \).

**Proof.** We first estimate \( W(D_n^i) \)

(i) For \( i = 1, ..., m \), let \( \xi = x_i(s,y) \) be the ith characteristic passing through any fixed point \( (t,x) \in D_n^i \) and intersecting the x-axis at a point \( (0,y) \). Noting (3.6), by (3.3)-(3.4), it is easy to see that the whole characteristic \( \xi = x_i(s,y) \) is included in \( D_n^i \).

Noting (3.6), by (3.4) we get

\[ y + (\lambda_i(0) - \delta_0 / 2)s \leq x_i(s,y) \leq y \]  
(3.32)

By (3.4), it is easy to see that

\[ s \leq t \leq t_0 \]  
(3.33)

where \( t_0 \) denotes the t-coordinate of the intersection point of the straight line \( x = \lambda_i(0) + \delta_0 \) with the straight line \( x = y + (\lambda_i(0) + \delta_0 / 2)t \) passing through the point \( (0,y) \).

\[ t_0 = \frac{y}{\lambda_i(0) - \lambda_i(0) + \delta_0 / 2} \]  
(3.34)

Therefore it follows from (3.32)-(3.34) that

\[ \lambda_i(0) - \lambda_i(0) + \delta_0 / 2 \leq y \leq x_i(s,y) \leq y \]  
(3.35)

By integrating (2.6) along this ith characteristic, we have

\[ W_i(t,x) = e^{-\Theta} W_i(0,y) + \int_0^t e^{-\Theta(s-y)} \sum_{j=1}^n \gamma_{ij}(u) w_j w_k \]  
(3.36)

Thus, noting (3.6) and the fact that \( L > 0 \), using (3.33)-(3.35), it follows from (3.36) that

\[ (1 + x)^{-\Theta} |w_i(t,x)| \leq C_i(1 + y)^{-\Theta} |w_i(0,y)| + (1 + y)^{-\Theta} \sum_{j=1}^n (|W(D_j^i)| + W(D_j^i)) V(D_j^i) \]  
(3.37)

where here and henceforth, \( C_i(i=1; 2,...) \) denote positive constants independent of \( \theta \) and \( T \).

(ii) For \( m + 1; \ldots; n \); let \( \xi = x_i(s,y) \) be the ith characteristic passing through any fixed point \( (t,x) \in D_i \) and intersecting the x-axis at a point \( (0,y) \). Noting (3.6), by (3.3)-(3.4), it is easy to see that the whole characteristic \( \xi = x_i(s,y) \) is included in \( D_n^i \).

Noting (3.6), by (3.4) we get

\[ y \leq x_i(s,y) \leq y + \lambda_i(0) + \delta_0 / 2) s \]  
(3.38)

By (3.4), it is easy to see that

\[ s \leq t \leq t_0 \]  
(3.39)

where \( t_0 \) denotes the t-coordinate of the intersection point of the straight line \( x = \lambda_i(0) + \delta_0/2 \) with the straight line \( x = y + (\lambda_i(0) + \delta_0/2)t \) passing through the point \( (0,y) \). Clearly,

\[ t_0 = \frac{y}{\lambda_i(0) - \lambda_i(0) + \delta_0 / 2} \]  
(3.40)

Therefore it follows from (3.38)-(3.40) that

\[ y \leq x_i(s,y) \leq y + \lambda_i(0) + \delta_0 / 2) s \]  
(3.41)

Then, similar to (3.37), we have

\[ (1 + x)^{-\Theta} |w_i(t,x)| \leq C_i(1 + y)^{-\Theta} \sum_{j=1}^n \gamma_{ij}(u) w_j w_k \]  
(3.42)

Combining (3.37) and (3.42), we obtain

\[ W(D_i^i) \leq C_i \sum_{j=1}^n \gamma_{ij}(u) \sum_{j=1}^n \gamma_{ij}(u) \]  
(3.43)

Similarly, we have

\[ W(D_i^i) \leq C_i \sum_{j=1}^n \gamma_{ij}(u) \sum_{j=1}^n \gamma_{ij}(u) \]  
(3.44)

By (3.43) and (3.44), it is easy to prove that for \( \mu > 0 \) suitably small, there exists a positive constant \( k1 \) independent of \( \theta \) and \( T \), such that for any fixed \( T_0(0 < T_0 < T) \) if
\[ W(D^2) \leq k_2 \theta \]  
(3.45)

Then
\[ W(D^1) \leq k_2 \theta \]  
(3.46)

Hence, noting (1.15), by continuity we immediately get (3.31). The proof of Lemma 3.2 is finished. 2 Lemma 3.3. Under the assumptions of Lemma 3.2, suppose furthermore that system (1.1) is weakly linearly degenerate. Then in the normalized coordinates there exists a sufficiently small \( \psi_0 > 0 \) such that for any fixed \( \theta \in [0, \psi_0] \) on any given existence domain \( \{(t,x) | 0 \leq t \leq T, x \geq x_0 \} \) of the C1 solution \( u = u(t; x) \) to the mixed initial-boundary value problem (1.1) and (1.12)-(1.13), we have the following uniform a priori estimates:

\[ W(x) \leq k \theta \]  
(3.53)

Proof. We first estimate \( W(D^1) \).

For \( j = 1, \ldots, m \), passing through any fixed point \( (t,x) \in D^1 \) we draw the \( j \)-th characteristic \( C_j : \xi = \xi_j(s,t,x) \) which must intersect the boundary \( x = (\lambda_j(0) + \delta_j) t \) of \( D^1 \) at a point \( (0,y) \).

Proposition 3.1. On this \( j \)-th characteristic \( C_j : \xi = \xi_j(s,t,x) \) it follows that
\[ t \geq t_0 \geq \frac{-\lambda_j(0) - \delta_j}{2} t \]
(3.54)

Proof. Noting (3.4), it is easy to see that
\[ x = (\lambda_j(0) + \delta_j) t \leq y - (\lambda_j(0) + \delta_j) t_0 \]
(3.55)

On the other hand, from (3.8), we have
\[ x \geq 0 \]
(3.56)

Since
\[ y = (\lambda_j(0) + \delta_j) t_0 \]
(3.57)

we conclude from (3.55)-(3.57) that
\[ t \geq \frac{-\lambda_j(0) - \delta_j}{2} t \]
(3.58)

Noting the fact that \( t \geq t_0 \) we immediately get (3.54).

By integrating (2.6) along \( \xi = \xi_j(s,t,x) \) and noting (2.9) and (2.11), we have
\[ w_j(t,x) = e^{-\int t} w_j(t_0,y) \]
(3.59)

Thus, noting the fact that \( L \geq 0 \), by (3.35) and (3.36), we obtain from (3.59) that

\[ x = 0 : \left\{ \begin{array}{l}
\frac{\partial V}{\partial t} + \frac{\partial}{\partial s} (\alpha(t), V(t), v_{m+1}(t), y_{m+1}(t)) = 0 \\
- \lambda_j(0) V_j(t) + h_j(t) \end{array} \right\} 
(3.60)

Therefore it follows from (3.63)-(3.65) that
\[ x = 0 : (I_{n-m} - B(1)) \left\{ \begin{array}{l}
w_{m+1} \\
\vdots \\
w_n
\end{array} \right\} = B_2 \left\{ \begin{array}{l}
w_1 \\
\vdots \\
w_n \end{array} \right\} - B_3 
(3.66)

where \( B(1) \) is a matrix whose elements are all C1 functions of \( u \), which satisfy

\[ In-m - B(1) \] is invertible; for sufficiently small \( \psi \)

\[ B2 \text{ is an } (n-m) \times m \text{ matrix independent of } w(t) \]
(3.67)

\[ B_3 = \left( \begin{array}{c}
\sum_{i=1}^{m} \frac{\partial G_i}{\partial u_i} (\alpha(t), V(t), v_{m+1}(t), y_{m+1}(t)) \\
\lambda_j(u)
\end{array} \right)_{j=1}^{n-m} 
(3.68)

in which \( F(s = m + 1, \ldots, n) \) are continuous functions of \( t \) and \( u \).

Thus, noting (3.6), for \( \delta > 0 \) small enough, by (3.66)-(3.68) we easily get
\[ x = 0 : ws = \sum_{j=1}^{ld} f_j(t,u)w_j + \sum_{i=1}^{ld} f_i(t,u)a_i'(t) \]
\[ + \sum_{i=1}^{ld} \mathcal{T}_i(u,h(t,u)) + f_i(t,u)u(s = m + 1, \ldots, n) \]  
(3.69)

Where \( f_j, f_i \) and \( \mathcal{T}_i \) are continuous functions of \( t \) and \( u \).

For \( j = m+1; \ldots; n \), passing through any fixed point \((t,x) \in D^T \)
we draw the \( j \)th characteristic \( c_j : \xi = \xi_j(s; t,x) \) which must intersect the \( t \)-axis at a point \((t_0,0)\). Then, we have

Proposition 3.2. On this \( j \)th characteristic \( c_j : \xi = \xi_j(s; t,x) \) it follows that

\[ t \geq t_0 \geq \frac{\lambda_j(0) - \lambda_{m+1}(0) + \frac{\delta_1}{2} t}{\lambda_j(0) - \frac{\delta_1}{2}} \]  
(3.70)

Noting the fact that \( t \geq t_0 \), we immediately get (3.70).

By integrating (2.6) along \( c_j : \xi = \xi_j(s; t,x) \) we have

\[ w_j(t,x) = e^{-\lambda_j(t_0) \int_0^t} wj(t_0,0)^{d0} + \int_0^t e^{-\lambda_j(t_0) \int_0^t} \sum_{k=1}^{m} [j_{jk}(u)w_k w_i] + \mathcal{T}_{jk}(u)w_j[s, \xi_j(s; t,x)] ds \]  
(3.74)

By (3.69), we have

\[ w_j(t_0,0) = \sum_{i=1}^{ld} f_i(t_0,u(t_0))w_i(t_0,0) + \sum_{i=1}^{ld} \mathcal{T}_i(t_0,u(t_0))a'_i(t_0,0) \]
\[ + \sum_{i=1}^{ld} f_i(t_0,u(t_0)a_i(t_0,0))u_i(t_0,0) \]  
(3.75)

By employing the same arguments as in (i), we can obtain

\[ (1 + t_0^\eta) \begin{array}{l} |w_i(t_0,0)| \leq c_i(t_0 + (WD'(y))^2 + WD'(y) V(D')) \\ + W_j(T)W_i(T) + W_j(T) + V_j(T) + C_j(T)V_j(T) + U_j(T)V_j(T) + W_j(T)V_j(T) \end{array} \]  
(3.76)

Thus, noting (1.15), (3.6) and (3.70), it follows from (3.75) and (3.76) that

\[ (1 + t_0^\eta) \begin{array}{l} |w_i(t,0)| \leq c_i(t_0 + (WD'(y))^2 + WD'(y) V(D')) \\ + W_j(T)W_i(T) + W_j(T) + V_j(T) + C_j(T)V_j(T) + U_j(T)V_j(T) + W_j(T)V_j(T) \end{array} \]  
(3.77)

Hence, noting the fact that \( L > 0 \), we obtain from (3.74) that

\[ (1 + t_0^\eta) \begin{array}{l} |w_i(t,0)| \leq c_i(t_0 + (WD'(y))^2 + WD'(y) V(D')) \\ + W_j(T)W_i(T) + W_j(T) + V_j(T) + C_j(T)V_j(T) + U_j(T)V_j(T) + W_j(T)V_j(T) \end{array} \]  
(3.78)

Combining (3.62) with (3.78), we get

\[ W(D') \leq \begin{array}{l} C_i(t_0 + (WD'(y))^2 + WD'(y) V(D')) + W_j(T) \\ + W_j(T)V_j(T) + U_j(T)V_j(T) \end{array} \]  
(3.79)

We next estimate \( \mathcal{W}_j(T) \)

Let \( \mathcal{C}_j : x = x_j(t), t \leq t_j \) be any given \( j \)th characteristic in \( D^T(j \neq i, i = m + 1, \ldots, n) \) By (3.4), the whole \( i \)th characteristic \( x = x_i(t) \) passing through \( O(0,0) \) is included in \( D^T \). Let \( (t_0, x_j(t_0)) \) be the intersection point of this characteristic with \( \mathcal{C}_j \). Passing through any given point \((t_0, x_j(t_0)) \) on \( \mathcal{C}_j \), we draw the \( i \)th characteristic \( \xi = \xi_i(s; x_0) \) which intersects one of the boundaries of \( DT \) at \( x = (\lambda_i(t_0), (\lambda_i(t_0) + \delta_i) t) \) at a point \( A_i(x_i(t_0) = \xi_i(t_0), y) \) \( (\xi_i(t_0) = \xi_i(t_0) + \delta_i) t) \) if \( t_0 \leq t_1 \), \( (\xi_i(t_0) = \xi_i(t_0) + \delta_i) t) \) Clearly, we have

\[ \tilde{x}(t_0, x_j(t_0)) = x_j(t_0) \]  
(3.80)

which gives a one-to-one correspondence \( t = t(y) \) between the segment \( \overline{A_i} \) and \( \tilde{x}(t_0, x_j(t_0)) \) resp. \( \tilde{x}(t_0, x_j(t_0)) \) Thus, the integral on \( \mathcal{C}_j \) with respect to \( t \) can be reduced to the integral with respect to \( y \). Differentiating (3.80) with respect to \( t \) gives

\[ dt = \frac{1}{\lambda_j(u(t, \tilde{x}(t,y))) - \lambda_i(u(t, \tilde{x}(t,y)))} \frac{\partial \xi_j(t,y)}{\partial y} dy \]  
(3.81)

in which \( t = t(y) \). Then, noting (3.3) and (3.6), it is easy to see that in order to estimate

\[ \int_{t_0}^{t_j} \left| \begin{array}{l} w_i(t_0, x(t,y)) \left( \right) \right| \right| dt \]  
\[ \int_{t_0}^{t_j} \left| \begin{array}{l} w_i(t_0, x(t,y)) \left( \right) \right| \right| dt \]  
(3.82)

it suffices to estimate

\[ \int_{t_0}^{t_j} \left| q_i(t, \tilde{x}(t,y)) \left( \right) \right| dy \]  
\[ \int_{t_0}^{t_j} \left| q_i(t, \tilde{x}(t,y)) \left( \right) \right| dy \]  
(3.83)

We now estimate

\[ \int_{t_0}^{t_j} \left| q_i(t, \tilde{x}(t,y)) \left( \right) \right| dy \]  
(3.84)

By integrating (2.28) along \( \xi = \xi(s,y) \) and noting (2.30) and the fact that \( \tilde{x}(y/((\lambda_i(t_0) + \delta_i) t)) = y \) we obtain

\[ q_i(t, x(t,y)) = e^{\lambda_i(t_0) \int_0^t} \begin{array}{l} \frac{u_j(t_0,y)}{\lambda_i(u(t, \tilde{x}(t,y))) - \lambda_i(u(t, \tilde{x}(t,y)))} \right| \\ + \frac{\delta_i}{\lambda_i(u(t, \tilde{x}(t,y))) - \lambda_i(u(t, \tilde{x}(t,y)))} \right| \\ + \frac{\delta_i}{\lambda_i(u(t, \tilde{x}(t,y))) - \lambda_i(u(t, \tilde{x}(t,y)))} \right| \end{array} \]  
(3.85)

By Hadamard's formula and (2.11), we have
\[ \gamma_i(u) = \gamma_i(u) - \gamma_i(u) = \int_0^1 \frac{\partial}{\partial t} \partial_t \gamma_i(u) \, dt \] (3.85)

Noting (3.6), (3.11), (3.13) and the fact that \( L > 0 \) and \( \partial_t \gamma_i(u) \) obtain from (3.84) and (3.85) that

\[ \begin{align*}
\int_{D(t)} (1 - s)^{-(n-1)} | V_i(s, \tilde{x}(s,y)) | \, ds \\
+ \int_{D(t)} (1 - s)^{-(n-1)} | u_i(s, \tilde{x}(s,y)) | \, ds
\end{align*} \] (3.86)

Noting that the transformation \( \int_{x=\tilde{x}(s,y)} \) gives the area element

\[ dx = \frac{\partial x(s,y)}{\partial y} \, ds \, dy \] (3.87)

by Lemma 3.2, it easily follows from (3.86) that

\[ \int_{D(t)} (1 - s)^{-(n-1)} | V_i(s, \tilde{x}(s,y)) | \, ds \\
+ \int_{D(t)} (1 - s)^{-(n-1)} | u_i(s, \tilde{x}(s,y)) | \, ds
\] (3.88)

Similarly, we have

\[ \int_{D(t)} (1 - s)^{-(n-1)} | V_i(s, \tilde{x}(s,y)) | \, ds \\
+ \int_{D(t)} (1 - s)^{-(n-1)} | u_i(s, \tilde{x}(s,y)) | \, ds
\] (3.89)

Thus, we obtain

\[ \left( \begin{array}{c}
\int_{D(t)} (1 - s)^{-(n-1)} | V_i(s, \tilde{x}(s,y)) | \\
+ \int_{D(t)} (1 - s)^{-(n-1)} | u_i(s, \tilde{x}(s,y)) |
\end{array} \right)
\] (3.90)

Next, estimate \( W(T) \).

For \( i = m+1, \ldots, n \), passing through any given point \( (t, x) \in D_i^r(t) \) we draw the \( r \)-th characteristic \( C_i : \xi = \xi_i(s,t,x) \) which intersects one of the boundaries of \( DT \), say \( x = (\lambda_i(0) - \delta_i(0)) t \) (resp. \( x = (\lambda_i(0) + \delta_i(0)) t \)) at a point \( (\psi(y, (\lambda_i(0) - \delta_i(0)) t, y)) \) (resp. \( B_i(y, (\lambda_i(0) + \delta_i(0)) t, y)) \). Clearly, we have \( x = \tilde{x}(s,y) \) Therefore we obtain

\[ \int_{D_i^r(t)} w_i(t, x) \, dx = \int_0^1 \eta_i(t, \tilde{x}(t, y)) \, dy \] (3.91)

where \( y_1 \) and \( y_2 \) are shown in Figure 2.

**Figure 2**: Where \( y_1 \) and \( y_2 \) are shown in figure.

Similar to (3.90), it follows from (3.91) that

\[ \left( \begin{array}{c}
\int_{D_i^r(t)} (1 - s)^{-(n-1)} | V_i(s, \tilde{x}(s,y)) | \\
+ \int_{D_i^r(t)} (1 - s)^{-(n-1)} | u_i(s, \tilde{x}(s,y)) |
\end{array} \right)
\] (3.92)

We next estimate \( W(T) \).

For \( r = 1 ; \ldots , m \), passing through any fixed point \( (t, x) \in D_i^r \) we draw the \( r \)-th characteristic \( C_i : \xi = \xi_i(s,t,x) \) which must intersect the boundary \( x = (\lambda_i(0) - \delta_i(0)) t \) at a point \((0, y) \). Then, we have

Proposition 3.3. On this characteristic \( C_i : \xi = \xi_i(s,t,x) \) it follows that

\[ t \geq t_0 \geq \frac{\lambda_{i-1}(0) - \lambda_i(0) - \delta_i(0)}{2} \] (3.93)

**Proof.** By (3.4), it is easy to see that

\[ x - \left( \lambda_i(0) - \frac{\delta_i(0)}{2} t \right) \leq y - \left( \lambda_i(0) + \frac{\delta_i(0)}{2} t \right) \] (3.94)

On the other hand, from (3.9), we have

\[ x \geq (\lambda_{i-1}(0) - \delta_i(0)) t \] (3.95)

Since

\[ y = (\lambda_i(0) + \delta_i) t \] (3.96)

we conclude from (3.94)-(3.96) that

\[ t_0 \geq \frac{\lambda_{i-1}(0) - \lambda_i(0) - \delta_i(0)}{2} \] (3.97)

Noting the fact that \( t \geq t_0 \) we immediately get (3.93).

By integrating (2.6) along \( \xi = \xi_i(s,t,x) \) and noting (2.9) and (2.11), we have

\[ w_i(t, x) = e^{\lambda_i(0) t} \eta_i(w_i, (0, y)) \] (3.98)

\[ + \int_k e^{\lambda_i(0) t} \sum_{j=1}^m \sum_{1 \leq l \leq n} \gamma_{ij}(u) \eta_i(w_i, (\xi_i(s, t, x))) \, ds \] (3.99)

By Hadamard's formula, we have

\[ \tilde{\gamma}_i(u) - \tilde{\gamma}_i(u, e) = \int_{D_i^r(t)} \frac{\partial}{\partial t} \partial_t \tilde{\gamma}_i(u, e) \, dt \] (3.100)

Thus, noting (3.93) and the fact that \( L > 0 \), we obtain from (3.98) that

\[ (1 + \psi)^{-1} | w_i(t, x) \leq C_{\lambda_i(0)} |(1 + \psi)^{-1} | e^{\lambda_i(0) t} | (1 + \psi)^{-1} | \eta_i(w_i, (\xi_i(s, t, x))) | ds \] (3.101)

We fix the idea we may assume that
which implies \( i < n \). Let \( \xi = \xi(s; t, x) \) be the \( i \)th characteristic passing through \((t, x)\), which intersects the boundary \( x = (\lambda_{\gamma}(0) + \delta_{\gamma}) t \) of \( D^T \) at a point \((0; y)\) (Figure 3).

Recalling (3.4), it is easy to see that

\[
x - \lambda_{\gamma}(0)t > \left[ \delta_{\gamma} + \eta(\lambda(0) - \lambda_{\gamma}(0)) \right] t
\]

(3.102)

which implies \( i < n \). Let \( \xi = \xi(s; t, x) \) be the \( i \)th characteristic passing through \((t, x)\), which intersects the boundary \( x = (\lambda_{\gamma}(0) + \delta_{\gamma}) t \) of \( D^T \) at a point \((0; y)\) (Figure 3).

Recalling (3.4), it is easy to see that

\[
x - \lambda_{\gamma}(0)t \geq y - (\lambda_{\gamma}(0) + \delta_{\gamma}) t_{0}
\]

(3.103)

Since

\[
y = (\lambda_{\gamma}(0) + \delta_{\gamma}) t_{0},
\]

(3.104)

recalling (3.102) and the fact that \( t \geq t_{0} \) it follows from (3.103) that

\[
t \geq t_{0} \geq y t.
\]

(3.105)

By integrating (2.6) along \( \xi = \xi(s; t, x) \) and noting (2.9) and (2.11), we have

\[
w_{t}(t, x) = e^{-\lambda_{\gamma}(0) t}w_{0}(t, y)
\]

\[
+ \int_{t_{0}}^{t} e^{\lambda_{\gamma}(0) t_{0}} \sum_{i=1}^{n} \gamma_{i}(u) w_{i}(s; \xi(s; t, x))ds
\]

\[
+ \int_{t_{0}}^{t} e^{\lambda_{\gamma}(0) t_{0}} \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{m} \gamma_{i j}(u) w_{i j}(s; \xi(s; t, x))ds
\]

(3.106)

By Lemma 3.2 and observing (3.104)-(3.105), it is easy to see that

\[
| w_{t}(t, y) | \leq k b(1 + y)^{-(i+\alpha)} \leq C_{m} b(1 + t_{0})^{-(i+\alpha)} \leq C_{m} b(1 + t)^{-(i+\alpha)}
\]

(3.107)

By Hadamard’s formula, we have

\[
\tilde{\gamma}_{i}(u) - \gamma_{i}(u, \xi) = \int_{t_{0}}^{t} \frac{\partial \gamma_{i}}{\partial u}(Tu_{1}, \ldots, Tu_{i-1}, u_{i}, Tu_{i+1}, \ldots, Tu_{n})u_{i} dt
\]

(3.108)

Thus, recalling (3.13) and (3.105), and noting the fact that \( L > 0 \), it follows from (3.106) that

\[
W^{(\alpha+\beta)}_{\gamma}(T) \leq C_{m} b(1 + t)^{(i+\alpha)} \leq C_{m} b(1 + t)^{(i+\alpha)}
\]

(3.109)

Next, we assume that

\[
x - \lambda_{\gamma}(0)t < -\left[ \delta_{\gamma} + \eta(\lambda(0) - \lambda_{\gamma}(0)) \right] t
\]

(3.110)

which implies \( i > m + 1 \). Let \( \xi = \xi(s; t, x) \) be the \( i \)th characteristic passing through \((t, x)\), which intersects the boundary \( x = (\lambda_{\gamma+1}(0) - \delta_{\gamma}) t \) of \( D^T \) at a point \((0; y)\) (Figure 3).

Recalling (3.4), it is easy to see that

\[
x - \lambda_{\gamma+1}(0) - \delta_{\gamma}) t \geq y - (\lambda_{\gamma+1}(0) - \delta_{\gamma}) t_{0}
\]

(3.111)

Since

\[
y = (\lambda_{\gamma+1}(0) - \delta_{\gamma}) t_{0},
\]

(3.112)

noting (3.110) and the fact that \( t \geq t_{0} \) it follows from (3.111) that

\[
t \geq t_{0} \geq y t.
\]

(3.113)

By integrating (2.6) along \( \xi = \xi(s; t, x) \) and noting (2.9) and (2.11), we have

\[
w_{t}(t, x) = e^{\lambda_{\gamma}(0) t}w_{0}(t, y)
\]

\[
+ \int_{t_{0}}^{t} e^{\lambda_{\gamma}(0) t_{0}} \sum_{i=1}^{n} \gamma_{i}(u) w_{i}(s; \xi(s; t, x))ds
\]

\[
+ \int_{t_{0}}^{t} e^{\lambda_{\gamma}(0) t_{0}} \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{m} \gamma_{i j}(u) w_{i j}(s; \xi(s; t, x))ds
\]

(3.114)

By (3.113), it is easy to see that

\[
| w_{t}(t, y) | \leq C_{m} W(D^T)(1 + t_{0})^{-(i+\alpha)} \leq C_{m} W(D^T)(1 + t)^{-(i+\alpha)}
\]

(3.115)

Thus, using (3.13), (3.108) and (3.113), and noting the fact that \( L > 0 \), it follows from (3.114) that

\[
l t_{0} - x - \lambda_{\gamma}(0)t \geq 0 \Rightarrow w_{t}(t, y) \leq C_{m} b(1 + t)^{(i+\alpha)} \leq C_{m} b(1 + t)^{(i+\alpha)}
\]

(3.116)

Combining (3.101) and (3.109), (3.116), we obtain

\[
W^{\alpha}_{\gamma}(T) \leq C_{m} W(D^T) \leq C_{m} W(D^T) \leq C_{m} W(D^T)
\]

(3.117)

We next estimate \( V(D^T) \). For \( j = 1, \ldots, m \), for any fixed point \((t, x) \in D^T\) similar to (3.59), by integrating (2.13) along \( \xi = \xi(s; t, x) \) and noting (2.17)-(2.18), we have

\[
v_{j}(t, x) = e^{\lambda_{\gamma+1}(0) t}v_{j}(t, y)
\]

\[
+ \int_{t_{0}}^{t} e^{\lambda_{\gamma+1}(0) t_{0}} \sum_{i=1}^{n} \gamma_{i}(u) w_{i}(s; \xi(s; t, x))ds
\]

\[
+ \int_{t_{0}}^{t} e^{\lambda_{\gamma+1}(0) t_{0}} \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{m} \gamma_{i j}(u) w_{i j}(s; \xi(s; t, x))ds
\]

(3.118)

By using Lemma 3.2 and noting (3.54) and (3.57), it is easy to see that
Combining (3.122) and (3.128), we get

\[ \frac{1}{\xi_1} \int_0^1 |p_j(t, \bar{x}(t, y))|_{(T, y)} dy + \frac{1}{\xi_2} \int_0^1 |p_j(t, \bar{x}(t, y))|_{(T, y)} dy \]  

Thus, noting the fact that \( L > 0 \), and using (3.13) and (3.54), we obtain from (3.118) that

\[ (1+\xi_1)^{1/2} |v_j(t, x)| \leq C_0 |\xi_1|^{1/2} + W(D') \eta(D', \xi') |v_j(t, x)| + W(D') \eta(D', \xi') |v_j(t, x)| \]  

for any fixed point \((t, x) \in D^T\), similar to (7.4), we have

\[ v_j(t, x) = e^{-L(t-\xi)} v_j(t_0, 0) + \int_0^t e^{-L(s-\xi)} \sum_{k=1}^m |\beta_{jk}(u)| v_j(t_0, 0) ds \]  

Noting (1.16), by (1.13), it is easy to see that

\[ v_j(t_0, 0) = \sum_{i=1}^n g_p(t_0) v_j(t_0) + h_j(t_0) \]  

Where

\[ g_p(t_0) = \int_0^1 \frac{\partial^2}{\partial y^2} \left( \alpha(t, x), T v_j(t_0), \ldots, T w_j(t_0) \right) dT \]  

By employing the same arguments as in (1), we can obtain

\[ (1+\xi_1)^{1/2} |v_j(t, x)| \leq C_0 |\xi_1|^{1/2} + W(D') \eta(D', \xi') |v_j(t, x)| + W(D') \eta(D', \xi') |v_j(t, x)| \]  

Thus, noting (1.15), (3.6) and (3.70), it follows from (3.124)-(3.126) that

\[ (1+\xi_1)^{1/2} |v_j(t, x)| \leq C_0 |\xi_1|^{1/2} + W(D') \eta(D', \xi') |v_j(t, x)| + W(D') \eta(D', \xi') |v_j(t, x)| \]  

Hence, noting the fact that \( L > 0 \), we obtain from (3.123) that

\[ \left| v_j(t, x) \right| \leq C_0 |\xi_1|^{1/2} + W(D') \eta(D', \xi') |v_j(t, x)| + W(D') \eta(D', \xi') |v_j(t, x)| \]  

Combining (3.122) and (3.128), we get

\[ v_j(t, x) = e^{-L(t-\xi)} v_j(t_0, 0) + \int_0^t e^{-L(s-\xi)} \sum_{k=1}^m |\beta_{jk}(u)| v_j(t_0, 0) ds \]  

We next estimate \( \bar{V}(T) \) and \( V(T) \).

For \( i = m + 1, \ldots, n \), for any given \( j \)-characteristic \( C_j \), in \( D^T_j \) \((j = i)\) as in the proof of (3.90), in order to estimate \( \bar{V}(T) \) it suffices to estimate

\[ \left| V_j(t_1, x) \right| \leq k_0(1+\xi)^{1/2} \leq C_0(1+\xi)^{1/2} \leq C_0(1+\xi)^{1/2} \]  

By Hadamard's formula, we have

\[ \beta_{jk}(u) - \beta_{jk}(u_0) = \int_0^1 \frac{\partial^2}{\partial y^2} \left( \alpha(t, x), T v_j(t_0), \ldots, T w_j(t_0) \right) dT \]  

And

\[ \beta_{jk}(u) - \beta_{jk}(u_0) = \int_0^1 \frac{\partial^2}{\partial y^2} \left( \alpha(t, x), T v_j(t_0), \ldots, T w_j(t_0) \right) dT \]  

Noting that \( \lambda_j(u) \)\( (i = m + 1; \ldots; n) \) are weakly linearly degenerate, by (2.37) and (2.38), we have

\[ B(u, c) \leq 0, \forall j \]  

By Hadamard's formula, and noting (2.18) and (3.122), we have

\[ B_j(u, c) = B_j(u) \]  

Similarly, we have

\[ \left| v_j(t, x) \right| \leq C_0 |\xi_1|^{1/2} + W(D') \eta(D', \xi') |v_j(t, x)| + W(D') \eta(D', \xi') |v_j(t, x)| \]  

Thus, we obtain

\[ V_j(t, x) \leq C_0 |\xi_1|^{1/2} + W(D') \eta(D', \xi') |v_j(t, x)| + W(D') \eta(D', \xi') |v_j(t, x)| \]  

Similarly, we have

\[ V_j(t, x) \leq C_0 |\xi_1|^{1/2} + W(D') \eta(D', \xi') |v_j(t, x)| + W(D') \eta(D', \xi') |v_j(t, x)| \]  

We next estimate \( V(T) \).

For \( r = 1; \ldots; m \), for any fixed point \((t, x) \in D^T\), noting (2.17) and (2.18), similar to (3.98), we have
By using Lemma 3.2 and noting (3.93) and (3.96), it is easy to see that

\[ |V(t,y)| \leq K|\theta(1+y)e^{\alpha y}| \leq C_0 \theta(1+y)e^{\alpha y} \quad (3.140) \]

By Hadamard’s formula, we have

\[ \beta_{\Omega}(u) - \beta_{\Omega}(u_k) = \int_0^\infty \sum_{j=1}^m \beta_j(u_k(t,x), u_k(t,x)) \, dt \quad (3.141) \]

And

\[ \beta_{\Omega}(u) - \beta_{\Omega}(u_k) = \int_0^\infty \sum_{j=1}^m \beta_j(u_k(t,x), u_k(t,x)) \, dt \quad (3.142) \]

Thus, noting (3.93) and the fact that \( L > 0 \), we obtain from (3.139) that

\[ (t,x) \in \Omega \quad (3.143) \] and

\[ L \leq C_\Omega u_k \quad (3.144) \]

Then, it follows from (3.143) and (3.144) that

\[ \frac{\partial u_k}{\partial t}(t,x) \leq C_\Omega u_k |i(t,x)| \quad (3.145) \]

We next estimate \( U \), \( \int_0^T \frac{d}{dt} \int_{\Omega} u_k^2 \, dx \, dy \)

We write \( u_k^2 \) in the form of (3.90), in order to estimate \( \int_0^T \frac{d}{dt} \int_{\Omega} u_k^2 \, dx \, dy \)

By integrating (2.39) along \( \xi = \xi(s,y) \) (2.41), similar to (3.84), we have

\[ u_k^2 = \int_0^T Z(t,x \xi(t,y)) \, dy \quad (3.146) \]

Since \( A(u) \) is weakly linearly degenerate and \( u = (u_1; \ldots; u_m)T \) are normalized coordinates, by (2.43),

we have

\[ F_{\Omega}(u) = F_{\Omega}(u) - F_{\Omega}(u_k) \quad (3.147) \]

Hence, noting (3.6), (3.11), (3.13) and the fact that

\[ L > 0 \quad \frac{\partial u_k}{\partial y} > 0 \]

we obtain from (3.147) and (3.149) that

\[ u_k \leq C_\Omega |i(s,y)| \quad (3.148) \]

Thus, we obtain

\[ U \leq C_\Omega |i(s,y)| \quad (3.149) \]

Similarly, we have

\[ \int_0^T \frac{d}{dt} \int_{\Omega} u_k^2 \, dx \, dy \leq C_\Omega |i(s,y)| \quad (3.150) \]

By Lemma 3.2, similar to (3.88), it follows from (3.150) that

\[ U_k \leq C_\Omega |i(s,y)| \quad (3.151) \]

Thus, we obtain

\[ \int_0^T \frac{d}{dt} \int_{\Omega} u_k^2 \, dx \, dy \leq C_\Omega |i(s,y)| \quad (3.152) \]

Similarly, we have

\[ U_k \leq C_\Omega |i(s,y)| \quad (3.153) \]

We next estimate \( U_k \), \( \int_0^T \frac{d}{dt} \int_{\Omega} u_k^2 \, dx \, dy \)

For \( r = 1; \ldots; m \), for any fixed point \( (t,x) \in \Omega \) noting (2.19) and (2.20), similar to (3.98), we have

\[ u_k(t,x) = e^{-\alpha y} u_k(0,0) \quad (3.154) \]

By using Lemma 3.2 and noting that (3.93) and (3.96), it is easy to see that

\[ |u_k(t,x)| \leq C_\Omega \theta(1+y)e^{\alpha y} \quad (3.155) \]

Then, it follows from (3.157) and (3.158) that

\[ u_k \leq C_\Omega |i(s,y)| \quad (3.159) \]

We now estimate \( V \), \( \int_0^T \frac{d}{dt} \int_{\Omega} u_k^2 \, dx \, dy \)

For \( r = m+1; \ldots; n \), passing through any given point \( (t,x) \in \Omega \) we draw the \( i \)th characteristic \( \xi = \xi(s,t,x) \) which intersects one of the boundaries of \( \Omega \) at one point. For fixing the idea, suppose that this characteristic intersects \( s = \lambda_0(0) + \lambda_1 t \) at a point \( (y^* \lambda_0(0) + \lambda_1, y) \)

\[ (3.159) \]

\[ (3.157) \]

\[ \int_0^T \frac{d}{dt} \int_{\Omega} u_k^2 \, dx \, dy \leq C_\Omega |i(s,y)| \quad (3.158) \]

We now estimate \( V \), \( \int_0^T \frac{d}{dt} \int_{\Omega} u_k^2 \, dx \, dy \)

For \( r = m+1; \ldots; n \), passing through any given point \( (t,x) \in \Omega \) we draw the \( i \)th characteristic \( \xi = \xi(s,t,x) \) which intersects one of the boundaries of \( \Omega \) at one point. For fixing the idea, suppose that this characteristic intersects \( s = \lambda_0(0) + \lambda_1 t \) at a point \( (y^* \lambda_0(0) + \lambda_1, y) \)

\[ (3.156) \]

\[ (3.155) \]
By integrating (2.13) along this characteristic and noting (2.16)-(2.18), we have
\[ V_r(x,t) = e^{-\int^t_0 \frac{\partial W_i}{\partial x}(r(s),x(s),s) ds} \]
\[ + \int^t_0 e^{-\int^s_0 \frac{\partial W_i}{\partial x}(r(r),x(r),r) dr} \frac{\partial W_i}{\partial x}(r(s),x(s),s) ds \]
\[ + \int^t_0 e^{-\int^s_0 \frac{\partial W_i}{\partial x}(r(r),x(r),r) dr} \frac{\partial W_i}{\partial x}(r(s),x(s),s) ds \]
\[ + \int^t_0 e^{-\int^s_0 \frac{\partial W_i}{\partial x}(r(r),x(r),r) dr} \frac{\partial W_i}{\partial x}(r(s),x(s),s) ds \]
\[ + \int^t_0 e^{-\int^s_0 \frac{\partial W_i}{\partial x}(r(r),x(r),r) dr} \frac{\partial W_i}{\partial x}(r(s),x(s),s) ds \]
\[ (3.160) \]

Noting Lemma 3.1 and Lemma 3.2, and using Hadamard’s formula, it follows from (3.160) that
\[ |V_r(x,t)| \leq C_{10}(\theta + V(D^2) + W(U) U'(T) + U(T) W(T) + W(U) U(T)) \]
\[ + |W(U) U'(T) + U(T) W(T) + W(U) U(T)|_V(T) \]
\[ (3.161) \]

On the other hand, for \( m = 1, \ldots, n \), for any fixed point \( (x,t) \in D(T) = \{(x,t) | 0 \leq x \leq T, x > 0 \} \), we have
\[ |V_r(x,t)| \leq C_{10}(\theta + V(D^2) + W(U) U'(T) + U(T) W(T) + W(U) U(T)) \]
\[ + |W(U) U'(T) + U(T) W(T) + W(U) U(T)|_V(T) \]
\[ (3.162) \]

We finally estimate \( W_{0}^s(T) \)

For \( i = 1, \ldots, n \), passing through any given point \( (t,x) \in D(T) \), similar to (3.160), noting (2.9), (2.11)-(2.12) and the fact that \( \lambda_i(u) \) is weakly linearly degenerate, we have
\[ \eta_i(t,x) = e^{-\int^t_0 \frac{\partial W_i}{\partial x}(r(r),x(r),r) dr} \]
\[ + \int^t_0 e^{-\int^s_0 \frac{\partial W_i}{\partial x}(r(r),x(r),r) dr} \frac{\partial W_i}{\partial x}(r(s),x(s),s) ds \]
\[ + \int^t_0 e^{-\int^s_0 \frac{\partial W_i}{\partial x}(r(r),x(r),r) dr} \frac{\partial W_i}{\partial x}(r(s),x(s),s) ds \]
\[ + \int^t_0 e^{-\int^s_0 \frac{\partial W_i}{\partial x}(r(r),x(r),r) dr} \frac{\partial W_i}{\partial x}(r(s),x(s),s) ds \]
\[ + \int^t_0 e^{-\int^s_0 \frac{\partial W_i}{\partial x}(r(r),x(r),r) dr} \frac{\partial W_i}{\partial x}(r(s),x(s),s) ds \]
\[ (3.163) \]

Noting Lemma 3.1 and Lemma 3.2, and using Hadamard’s formula, it follows from (3.163) that
\[ |\eta_i(t,x)| \leq C_{10}(\theta + W(U) U'(T) + U(T) W(T) + W(U) U(T)) \]
\[ + |W(U) U'(T) + U(T) W(T) + W(U) U(T)|_V(T) \]
\[ (3.164) \]

Thus, by using the definitions of \( W_{0}^s(T) \), \( W(D^2) \) and \( W(D^2) \), and using Lemma 3.2, we have
\[ W_{0}^s(T) \leq C_{10}(\theta + W(D^2) + W(U) U'(T) + U(T) W(T) + W(U) U(T)) \]
\[ + |W(U) U'(T) + U(T) W(T) + W(U) U(T)|_V(T) \]
\[ (3.165) \]

We now prove (3.47)-(3.53).

Noting (1.15), evidently we have
\[ W_{0}^s(0), V_{0}^s(0), U_{0}^s(0) \leq C_{10} \theta \]
\[ (3.166) \]
\[ W(0) = V(0) = U(0) = \bar{W}(0) = \bar{V}(0) = \bar{U}(0) = 0 \]
\[ (3.167) \]
Consider the following mixed initial-boundary value problem for the system of the flow equations of a model class of fluids with viscosity induced by fading memory (cf. [7]):

\[
\begin{align*}
& w' - vx + w = 0 \\
& v' - (\sigma(w)) x + v = 0
\end{align*}
\]

with the initial condition

\[
t = 0 : w = w_0(x), v = v_0(x), (x \geq 0)
\]

and the boundary condition

\[
x = 0 : v = h(t), (t \geq 0)
\]

Here, \( w \) is the displacement gradient and \( v \) the velocity of a model class of fluids, the stress-strain function \( \sigma(w) \) is a smoothly function of \( w \) such that

\[
\sigma'(0) > 0
\]

Moreover, the conditions of \( C1 \) compatibility are supposed to be satisfied at the point \((0, 0)\).

Let \( \sigma_0 = (W_0, V_0) \in C^1 \) and we assume that there exists a constant \( \mu > 0 \) such that

\[
\sup_{t \geq 0} (1 + t)^{\mu \gamma} (|w_0(x)| + |v_0(x)| + |V_0(x)| + |V_0'|(x) + \gamma |V_0'(x)|) < +\infty
\]  

In addition, we assume that \( h(t) \in C^1 \) such that

\[
\sup_{t \geq 0} (1 + t)^{\mu \gamma} (|h(t)| + |h'(t)|) < +\infty
\]

Finally, we observe that when \( \mu > 0 \) is suitably small, by (3.52) we have

\[
U_\infty (T) \leq k, \theta \leq k, \theta \leq \delta / 2
\]

This implies the validity of hypothesis (3.6). The proof of Lemma 3.3 is finished.

Proof of Theorem 1.1. It suffices to prove Theorem 1.1 in the normalized coordinates. Under the assumptions of Theorem 1.1, by (3.52) and (3.53), we know that there is a suitably small \( \delta, \theta > 0 \) such that for any fixed \( \theta \in (0, \theta_1] \) on any given domain of existence \( D(T) = \{(t,x) | 0 \leq t \leq T, x \geq 0\} \) of the C1 solution \( u = u(t,x) \) to the mixed initial-boundary value problem (1.1) and (1.12)-(1.13), we have the following uniform a priori estimate for the C1 norm of the solution

\[
\| u(t,\cdot) \| c^1 \leq \| u(t,\cdot) \| c^0 + \| u_1(t,\cdot) \| c^0 \leq k_0 T
\]  

Thus we immediately get the conclusion of Theorem 1.1. The proof of Theorem 1.1 is finished.