Journal of
Applied & Computational Mathematics
ISSN: 2168-9679

Tai, J Appl Computat Math 2013, 2:2
DOI: 10.4172/2168-9679.1000126

Research Article Open Access

# Global Asymptotic Stability of a Neutral Stochastic Lotka-Volterra System

**Zhixin Tai***

*School of Astronautics and Aeronautics, Tsinghua University, Beijing-100084, P.R. China*

**Abstract**

The work addresses a stochastic Lotka-Volterra system with delays of neutral type for which global asymptotic stability criteria are established.

**Keywords:** Neutral delays; Lotka-Volterra system; Stochastic stability

## Introduction

More recently, Liu and Wang [1] have discussed the following stochastic Lotka-Volterra system with delays of *retarded* type

$$dx_i(t) = x_i(t)\left[r_i + \sum_{j=1}^n a_{ij}x_j(t) + \sum_{j=1}^n b_{ij}x_j(t-\tau_{ij}) + \sum_{j=1}^n c_{ij}\int_{-\infty}^0 x_j(t+\theta)d\mu_{ij}(\theta)\right] \quad (1)$$
$$dt + \sigma_i x_i(t)(x_i(t)-x_i^*)dB_i(t), 1\le i\le n$$

where $\sigma_i^2$ denotes the intensity of the noise, $B_i(t)$ is a standard Brownian motion defined on a complete probability space (Ω,F,P) with a filtration $\{F_t\}_{t\in R_+}$ satisfying the usual conditions, and $x_i^*$ ($i$=1,…, $n$) is positive equilibrium of system (1).

However, derivatives of delayed states appear often in the natural ecosystem models. Motivated by the works mentioned above, we consider a more general Lotka-Volterra system with delays of *neutral* type

$$dx_i(t) = x_i(t)\left[r_i + \sum_{j=1}^n a_{ij}x_j(t) + \sum_{j=1}^n b_{ij}x_j(t-\tau_{ij}) + \sum_{j=1}^n c_{ij}\int_{-\infty}^0 x_j(t+\theta)d\mu_{ij}(\theta) + \sum_{j=1}^n \eta_{ij}\dot{x}_j(t-\tau_{ij})\right] \quad (2)$$
$$dt + \sigma_i x_i(t)(x_i(t)-x_i^*)dB_i(t), 1\le i\le n$$

For more biological significance of the above model, we refer the readers to [2,3].

## Results

Motivated by the biological background, assume that population size $x_i(t)>0$, parameter $a_{ii}<0$ ($i$=1, …, $n$): Using the thought of similar proof in [1]; some conditions under which system (2) has a global positive solution are given in a straightforward way as follows.

**Lemma 4.1.** If $\sigma_i>0$ ($i$=1, …, $n$), then for any given initial value $\xi(\theta)\in BC((-\infty,0),R_+^n)$, there exists a unique global positive solution $x(t)$ to system (2), where $BC((-\infty,0),R_+^n)$ denotes the family of bounded and continuous functions from $(-\infty,0)$ to $R_+^n$ with the norm $\|\xi\| = \sup_{\theta\le 0}\|\xi(\theta)\|$.

**Theorem 4.1.** If there exist positive numbers $d_1, d_2,\ldots, d_n$ such that $\begin{bmatrix}\bar{D}A + A^T\bar{D} & 0\\ 0 & B\end{bmatrix}$ is negative definite, then positive equilibrium state $x^*$ in Eq.(2) is globally asymptotically stable almost surely (a.s.), i.e.,

$\lim_{t\to\infty} x_i(t) = x_i^*$ a.s. $1\le i\le n$ where $\bar{D} = diag(d_1, d_2,\ldots,d_n)$, $A=(\alpha_{ij})_{n\times n}$, $B = diag(\beta_1,\beta_2,\ldots,\beta_n)$, $\beta_i = \sum_{j=1}^n d_j|\eta_{ji}|, (i=1,\ldots,n)$

$$a_{ij} = \begin{cases} a_{ij}, i\ne j \\ a_{ii} + 0.5x_i^*\sigma_i^2 + 0.5\sum_{j=1}^n\left(|b_{ij}| + |c_{ij}| + |\eta_{ij}| + \dfrac{d_j}{d_i}|c_{ji}|\right), i=j\end{cases}$$

**Proof:** Applying Itô's formula yields that

$$LV_1 \le (x) = \sum_{i=1}^n d_i(x_i - x_i^*)\left[r_i + \sum_{i=1}^n a_{ij}x_j(t) + \sum_{i=1}^n b_{ij}x_j(t-\tau_{ij}) + \sum_{i=1}^n c_{ij}\int_{-\infty}^0 x_j(t+\theta)d\mu_{ij}(\theta) + \sum_{i=1}^n \eta_{ij}\dot{x}_j(t-\tau_{ij})\right]$$
$$+0.5\sum_{i=1}^n d_i x_i^*\sigma_i^2(x_i - x_i^*)^2$$
$$= \sum_{i=1}^n d_i(x_i - x_i^*)\left[\sum_{j=1}^n a_{ij}(x_j x_j^*) + \sum_{j=1}^n b_{ij}(x_j(t-\tau_{ij}) - x_j^*) + \sum_{j=1}^n c_{ij}\int_{-\infty}^0 (x_j(t+\theta) - x_j^*)d\mu_{ij}(\theta) + \sum_{j=1}^n \eta_{ij}\dot{x}_j(t-\tau_{ij})\right]$$
$$+0.5\sum_{i=1}^n d_i x_i^*\sigma_i^2(x_i - x_i^*)^2$$
$$= \sum_{i=1}^n d_i\left[\sum_{j=1}^n a_{ij}(x_i - x_i^*)(x_j - x_j^*) + \sum_{j=1}^n b_{ij}(x_i - x_i^*)(x_j(t-\tau_{ij}) - x_j^*)\right.$$
$$\left.+\sum_{j=1}^n c_{ij}\int_{-\infty}^0 (x_i - x_i^*)(x_j(t+\theta) - x_j^*)d\mu_{ij}(\theta) + \sum_{j=1}^n \eta_{ij}(x_i - x_i^*)\dot{x}_j(t-\tau_{ij})\right] + 0.5\sum_{i=1}^n d_i x_i^*\sigma_i^2(x_i - x_i^*)^2$$
$$\le \sum_{i=1}^n d_i\left[\sum_{j=1}^n a_{ij}(x_i - x_i^*)(x_j - x_j^*) + 0.5x_i^*\sigma_i^2(x_i^*)^2\right.$$
$$+0.5\sum_{j=1}^n |b_{ij}|(x_i - x_i^*)^2 + 0.5\sum_{j=1}^n |b_{ij}|((x_j(t-\tau_{ij}) - x_j^*)^2$$
$$+0.5\sum_{j=1}^n |c_{ij}|(x_i - x_i^*)^2 + 0.5\sum_{j=1}^n |c_{ij}|\left(\int_{-\infty}^0 (x_j(t+\theta) - x_j^*)d\mu_{ij}(\theta)\right)^2$$
$$\left.+0.5\sum_{j=1}^n |\eta_{ij}|(x_i - x_i^*)^2 + 0.5\sum_{j=1}^n |\eta_{ij}|\dot{x}_j^2(t-\tau_{ij})\right]$$
$$LV_2(t) = 0.5\sum_{i=1}^n\sum_{j=1}^n d_i|b_{ij}|(x_j(t) - x_j^*)^2 - 0.5\sum_{i=1}^n\sum_{j=1}^n d_i|b_{ij}|(x_j(t-\tau_{ij}) - x_j^*)^2$$
$$0.5\sum_{i=1}^n\sum_{j=1}^n d_j|b_{ji}|(x_i(t) - x_i^*)^2 - 0.5\sum_{i=1}^n\sum_{j=1}^n d_i|b_{ij}|(x_j(t-\tau_{ij}) - x_j^*)^2$$
$$LV_3(t) = 0.5\sum_{i=1}^n\sum_{j=1}^n d_i|c_{ij}|(x_j(t) - x_j^*)^2 - 0.5\sum_{i=1}^n\sum_{j=1}^n d_i|c_{ij}|\int_{-\infty}^0 (x_j(t+\theta) - x_j^*)^2 d\mu_{ij}(\theta)$$
$$= 0.5\sum_{i=1}^n\sum_{j=1}^n d_j|c_{ji}|(x_i(t) - x_i^*)^2 - 0.5\sum_{i=1}^n\sum_{j=1}^n d_i|c_{ij}|\int_{-\infty}^0 (x_j(t+\theta) - x_j^*)^2 d\mu_{ij}(\theta)$$
$$LV_4(t) = 0.5\sum_{i=1}^n\sum_{j=1}^n d_i|\eta_{ij}|(\dot{x}_j^2(t) - 0.5\sum_{i=1}^n\sum_{j=1}^n d_i|\eta_{ij}|(\dot{x}_j^2(t-\tau_{ij})$$

***Corresponding author:** Zhixin Tai, School of Astronautics and Aeronautics, Tsinghua University, Beijing-100084, P.R. China, E-mail: taizhixin01@163.com

**Received** April 05, 2013; **Accepted** April 10, 2013; **Published** April 25, 2013

**Citation:** Tai Z (2013) Global Asymptotic Stability of a Neutral Stochastic Lotka-Volterra System. J Appl Computat Math 2: 126. doi:10.4172/2168-9679.1000126

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$$=0.5\sum_{i=1}^{n}\sum_{j=1}^{n} d_j \left|\eta_{ji}\right| (\dot{x}_j^2(t) - 0.5\sum_{i=1}^{n}\sum_{j=1}^{n} d_i \left|\eta_{ij}\right| (\dot{x}_j^2(t-\tau_{ij})$$

Using Hölder inequality, it follows that

$$\sum_{j=1, j\neq i}^{n} a_{ij}(x_i - x_i^*)(x_j - x_j^*) + 0.5\sum_{j=1}^{n}\left|b_{ij}\right|(x_j(t-\tau_{ij}) - x_j^*)^2$$

$$+0.5\sum_{j=1}^{n}\left|c_{ij}\right|\int_{-\infty}^{0}(x_j(t-\theta) - x_j^*)^2 d\mu_{ij}(\theta) + 0.5\sum_{j=1}^{n}\left|\eta_{ij}\right|\dot{x}_j^2(t-\tau_{ij})\Bigg]$$

from which it can be concluded that

$$LV(x,t) \leq \sum_{i=1}^{n} d_i \left[\left(a_{ii} + 0.5x_i^*\sigma_i^2 + 0.5\sum_{j=1}^{n}\left(\left|b_{ij}\right| + \left|c_{ij}\right| + \left|\eta_{ij}\right| + \frac{d_j}{d_i}\left|b_{ji}\right| + \frac{d_j}{d_i}\left|c_{ji}\right|\right)\right).(x_i - x_i^*)^2\right.$$

$$\left. + \sum_{j=1, j\neq i}^{n} a_{ij}(x_i - x_i^*)(x_j - x_j^*)\right] + 0.5\sum_{i=1, j=1}^{n} d_j \left|\eta_{ji}\right| \dot{x}_i^2(t) = 0.5\begin{bmatrix} x - x^* \\ \dot{x} \end{bmatrix}^T \begin{bmatrix} \bar{D}A + A^T\bar{D} & 0 \\ 0 & B \end{bmatrix}\begin{bmatrix} x - x^* \\ \dot{x} \end{bmatrix}$$

From the assumption of $\begin{bmatrix} \bar{D}A + A^T\bar{D} & 0 \\ 0 & B \end{bmatrix}$ being negative definite, we have that $LV<0$ along all trajectories in $R^+$ except $x^*$ the proof is completed.


## Acknowledgment

This work was partially supported by the National Natural Science Foundation of China (60976071, 61174058, 60974052, 61134001, 60974071), the National Key Basic Research Program, China (2012CB215202), and the 111 Project (B12018).


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