Journal of Generalized Lie Theory and Applications
ISSN: 1736-4337

Zuevsky, J Generalized Lie Theory Appl 2018, 12:2
DOI: 10.4172/1736-4337.1000292

Research Article Open Access

Geometric Formula for Prime form on a Sewn Riemann Surface

Alexander Zuevsky*

Institute of Mathematics, Czech Academy of Sciences, Prague, Czech Republic

Abstract

We use the geometric representation for the Szegö kernel on genus g1+g2 Riemann surface obtained in the frames of the Yamada sewing construction of two Riemann surfaces of genus g1 and genus g2 in order to derive new formulas expressing prime forms. These formulas can be used in vertex operator algebra and conformal field theory computation as well as in algebraic geometry.

Keywords: Szegö kernel; Prime forms; Theta-functions

Mathematical classification: 17B69, 30F10, 11F03, 97F60

Differentials and Kernels on a Riemann Surface

Differentials on a Riemann surface

Consider a compact Riemann surface $\Sigma^{(g)}$ of genus g with canonical homology cycle basis $a_1,\ldots, a_g$, $b_1,\ldots, b_g$. There exists g holomorphic one-forms $\nu_i^{(g)}$, $i=1,\ldots, g$ which may be normalized [1,2] by:

$$\oint_{a_i} \nu_j^{(g)} = 2\pi i \delta_{ij}. \tag{1}$$

The genus g period matrix $\Omega^{(g)}$ is defined by:

$$\Omega_{ij}^{(g)} = \frac{1}{2\pi i}\oint_{b_i} \nu_j^{(g)}, \tag{2}$$

For $i, j=1,\ldots, g$. $\Omega^{(g)}$ is symmetric with positive imaginary part, i.e., $\Omega^{(g)}\in\mathbb{H}_g$, the Siegel upper half plane. Next we give the definition of the theta function with real characteristics [1,3].

$$\vartheta^{(g)}\begin{bmatrix}\alpha^{(g)} \\ \beta^{(g)}\end{bmatrix}(z|\Omega^{(g)}) = \sum_{m\in\mathbb{Z}^g} \exp\left(i\pi(m+\alpha^{(g)}).\Omega^{(g)}.(m+\alpha^{(g)}) + (m+\alpha^{(g)}).(z+2\pi i\beta^{(g)})\right),$$

for $\alpha^{(g)}=(_i)$, $\beta^{(g)}=(\beta_i)\in\mathbb{R}^g$, $z=(z_i)\in\mathbb{C}^g$, and $i=1,\ldots, g$. There exists [3,4] a non-singular odd character $\begin{bmatrix}\gamma^{(g)} \\ \delta^{(g)}\end{bmatrix}$ such that $\vartheta^{(g)}\begin{bmatrix}\gamma^{(g)} \\ \delta^{(g)}\end{bmatrix}\left(0\middle|\Omega^{(g)}\right) = 0, \partial_{z_i}\vartheta^{(g)}\begin{bmatrix}\gamma^{(g)} \\ \delta^{(g)}\end{bmatrix}\left(0\middle|\Omega^{(g)}\right) \neq 0$,

Then introduce:

$$\zeta^{(g)}(x) = \sum_{i=1}^{g} \partial_{z_i}\vartheta^{(g)}\begin{bmatrix}\gamma^{(g)} \\ \delta^{(g)}\end{bmatrix}\left(0\middle|\Omega^{(g)}\right)\nu_i^{(g)}(x), \tag{3}$$

a holomorphic one-form, and let $\left(\zeta^{(g)}(x)\right)^{\frac{1}{2}}$ denote the form of weight $\frac{1}{2}$ on the double cover $\widetilde{\Sigma^{(g)}}$ of $\Sigma^{(g)}$. We also refer to $\left(\zeta^{(g)}(x)\right)^{\frac{1}{2}}$ as a double-valued $\frac{1}{2}$-form on $\Sigma^{(g)}$.

Let us define the prime form $\mathcal{E}^{(g)}(x,y)$ by:

$$\mathcal{E}^{(g)}(x,y) = \vartheta^{(g)}\begin{bmatrix}\gamma^{(g)} \\ \delta^{(g)}\end{bmatrix}\left(\int_y^x \nu^{(g)}\,|\,\Omega^{(g)}\right)\left(\zeta^{(g)}(x)\right)^{-\frac{1}{2}}\left(\zeta^{(g)}(y)\right)^{-\frac{1}{2}}, \tag{4}$$

where $\int_y^x \nu^{(g)} = \left(\int_y^x \nu_i^{(g)}\right)\in\mathbb{C}^g$. The prime form $\mathcal{E}^{(g)}(x,y) = -\mathcal{E}^{(g)}(y,x)$ is a holomorphic differential form of weight $(-\frac{1}{2},-\frac{1}{2})$ on $\widetilde{\Sigma^{(g)}}\times\widetilde{\Sigma^{(g)}}$.

We define the Szegö kernel [4-6] for $\vartheta^{(g)}\begin{bmatrix}\alpha^{(g)} \\ \beta^{(g)}\end{bmatrix}(0|\Omega^{(g)})\neq 0$ by the formula:

$$S^{(g)}\begin{bmatrix}\theta^{(g)} \\ \phi^{(g)}\end{bmatrix}(x,y|\Omega^{(g)}) = \frac{\vartheta^{(g)}\begin{bmatrix}\alpha^{(g)} \\ \beta^{(g)}\end{bmatrix}\left(\int_y^x \nu^{(g)}\,|\,\Omega^{(g)}\right)}{\vartheta^{(g)}\begin{bmatrix}\alpha^{(g)} \\ \beta^{(g)}\end{bmatrix}(0|\Omega^{(g)})\,\mathcal{E}^{(g)}(x,y)}, \tag{5}$$

where $\theta^{(g)} = (\theta_i), \phi^{(g)} = (\phi_i)\in U(1)^n$ for $\theta_j = -e^{-2\pi i\beta_j}$, $\phi_j = -e^{2\pi i\alpha_j}$, $j=1,\ldots,g$. This can be written as:

$$S^{(g)}\begin{bmatrix}\theta^{(g)} \\ \phi^{(g)}\end{bmatrix}(x,y|\Omega^{(g)}) = \Theta^{(g)}\begin{bmatrix}\alpha^{(g)} \\ \beta^{(g)}\end{bmatrix}\left(x,y;\nu^{(g)},0\,|\,\Omega^{(g)}\right)\left(\mathcal{E}^{(g)}(x,y)\right)^{-1},$$

with the functional:

$$\Theta^{(g)}\begin{bmatrix}\alpha^{(g)} \\ \beta^{(g)}\end{bmatrix}\left(x,y;f_1,f_2\,|\,\Omega^{(g)}\right) = \frac{\vartheta^{(g)}\begin{bmatrix}\alpha^{(g)} \\ \beta^{(g)}\end{bmatrix}\left(\int_y^x f_1\,|\,\Omega^{(g)}\right)}{\vartheta^{(g)}\begin{bmatrix}\alpha^{(g)} \\ \beta^{(g)}\end{bmatrix}\left(f_2\,|\,\Omega^{(g)}\right)}.$$

The Szegö Kernel in the ε-formalism

In Appendix we recall the ε-formalism due to Yamada [7] of sewing of two Riemann surfaces of genuses g_1 and g_2 to form a genus g_1+g_2 Riemann surface. In a study [8] we determined the Szegö kernel on the Riemann surface $\Sigma^{(g_1+g_2)}$ in terms of data coming from Szegö kernel:

$$S^{(g_a)}(x,y) = S^{(g_a)}[\theta^{(g_a)} \tag{6}$$

on the surface $\Sigma^{(g_a)}$ for a=1,2. Let us recall that construction here. We adopt the abbreviated notation of the left hand side of eqn. (6)

***Corresponding author:** Alexander Zuevsky, Institute of Mathematics, Czech Academy of Sciences, Prague, Czech Republic, Tel: +420222090731; E-mail: zuevsky@yahoo.com

Received July 14, 2018; **Accepted** July 31, 2018; **Published** August 07, 2018

Citation: Zuevsky A (2018) Geometric Formula for Prime form on a Sewn Riemann Surface. J Generalized Lie Theory Appl 12: 292. doi: 10.4172/1736-4337.1000292

Citation: Zuevsky A (2018) Geometric Formula for Prime form on a Sewn Riemann Surface. J Generalized Lie Theory Appl 12: 292. doi: 10.4172/1736-4337.1000292

when there is no ambiguity. Similarly, the Szegö kernel on $\Sigma^{(g_1+g_2)}$ is denoted by:

$$S^{(g_1+g_2)}(x,y)=S^{(g_1+g_2)}[\theta^{(g_1+g_2)} \tag{7}$$

with periodicities $(\theta_{s_a}^{(g_1+g_2)},\phi_{s_a}^{(g_1+g_2)})=(\theta_{s_a}^{(g_a)},\phi_{s_a}^{(g_a)})$ on the inherited homology basis. Note that we exclude those Riemann theta characteristics for which eqn. (7) exists but where either of the lower genus theta functions vanishes, i.e., we assume that eqn. (6) exists for a=1,2.

Following a study [8] we define weighted moments for $S^{(g_1+g_2)}$ by:

$$X_{ab}(k,l,\varepsilon)=X_{ab}[\theta^{(g_1+g_2)}=\frac{\varepsilon^{\frac{1}{2}(k+l-1)}}{(2\pi i)^2}\oint_{C_a(x)}\oint_{C_b(y)}x^{-k}y^{-l}S^{(g_1+g_2)}(x,y)dx^{\frac{1}{2}}dy^{\frac{1}{2}}, \tag{8}$$

for $k, l \geq 1$. It follows that:

$$X_{ab}[\theta^{(g_1+g_2)} \tag{9}$$

We denote by $X_{ab}=(X_{ab}(k,l,\varepsilon))$ the infinite matrix indexed by $k, l \geq 1$.

We also define various moments for $S^{(g_a)}(x,y)$. These provide the data used to construct $S^{(g_1+g_2)}(x,y)$. Define holomorphic $\frac{1}{2}$-forms on $\hat{\Sigma}^{(g_a)}$ by:

$$h_a(k,x,\varepsilon)=h_a[\theta^{(g_a)} \tag{10}$$

$$\overline{h}_a(k,y,\varepsilon)=\overline{h}_a[\theta^{(g_a)} \tag{11}$$

and introduce infinite row vectors $h_a(x)=(h_a(k,x))$, $h_a(x)=(h_a(k,x))$, indexed by $k \geq 1$. It follows that:

$$\overline{h}_a[\theta^{(g_a)} \tag{12}$$

Finally, we define the moment matrix:

$$\begin{aligned} F_a(k,l,\varepsilon)=F_a[\theta^{(g_a)} &= \frac{\varepsilon^{\frac{1}{2}(k+l-1)}}{(2\pi i)^2}\oint_{C_a(x)}\oint_{C_a(y)}x^{-k}y^{-l}S^{(g_a)}(x,y)dx^{\frac{1}{2}}dy^{\frac{1}{2}} \\ &= \frac{\varepsilon^{\frac{k}{2}-\frac{1}{4}}}{2\pi i}\oint_{C_a(x)}x^{-k}h_a(l,x)dx^{\frac{1}{2}}=\frac{\varepsilon^{\frac{l}{2}-\frac{1}{4}}}{2\pi i}\oint_{C_a(y)}y^{-l}\overline{h}_a(k,y)dy^{\frac{1}{2}}. \end{aligned} \tag{13}$$

F_a (k, l, ε) obeys a skew-symmetry property similar to eqn. (9).

We are now in a position to express $S^{(g_1+g_2)}(x,y)$ in terms of the lower genus data. From the sewing relation eqn. (23) we have $dz_a=-\varepsilon\frac{dz_{\bar{a}}}{z_{\bar{a}}^2}$ so that:

$$dz_a^{\frac{1}{2}}=(-1)^{\bar{a}}\xi\varepsilon^{\frac{1}{2}}\frac{dz_{\bar{a}}^{\frac{1}{2}}}{z_{\bar{a}}}, \tag{14}$$

where $\xi\in\{\pm\sqrt{-1}\}$ determines the square root branch chosen. We then find:

Proposition 1: $S^{(g_1+g_2)}(x,y)$ *is given by*:

$$S^{(g_1+g_2)}(x,y)=\begin{cases} S^{(g_a)}(x,y)+h_a(x)X_{\bar{a}\bar{a}}\overline{h}_a^T(y), & for\ x,y\in\hat{\Sigma}^{(g_a)}, \\ h_a(x)\left(\xi(-1)^{\bar{a}}I-X_{\bar{a}a}\right)\overline{h}_{\bar{a}}^T(y), & for\ x\in\hat{\Sigma}^{(g_a)},\ y\in\hat{\Sigma}^{(g_{\bar{a}})}, \end{cases} \tag{15}$$

Where I denotes the infinite identity matrix and T denotes the transpose.

In a study [8] we computed the explicit form of the moment matrix X_{ab} in terms of the moments F_a of $S^{(g_a)}(x,y)$. It is useful to introduce infinite block matrices:

$$X=\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix},\quad F=\begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix},\Xi=\begin{pmatrix} 0 & \xi I \\ -\xi I & 0 \end{pmatrix},\quad Q=F\Xi=\begin{pmatrix} 0 & \xi F_1 \\ -\xi F_2 & 0 \end{pmatrix}. \tag{16}$$

Then one finds:

Proposition 2: X is given by:

$$X=(I-Q)^{-1}F \tag{17}$$

where $(I-Q)^{-1}=\sum_{n\geq 0}Q^n$ is convergent for $|\varepsilon|<|r_1r_2|$.

Propositions 1 and 2 imply:

Theorem 1: $S^{(g_1+g_2)}(x,y)$ *is given by*:

$$S^{(g_1+g_2)}(x,y)=\delta_{ab}S^{(g_a)}(x,y)+h_a(x)\left(\Xi(I-Q)^{-1}\right)_{ab}\overline{h}_b^T(y),$$

For $x\in\hat{\Sigma}^{(g_a)}, y\in\hat{\Sigma}^{(g_b)}$. Equivalently,

$$S^{(g_1+g_2)}(x,y)=\begin{cases} S^{(g_a)}(x,y)+h_a(x)\left(I-F_{\bar{a}}F_a\right)^{-1}F_{\bar{a}}\overline{h}_a^T(y), & for\ x,y\in\hat{\Sigma}^{(g_a)}, \\ \xi(-1)^{\bar{a}}h_a(x)\left(I-F_{\bar{a}}F_a\right)^{-1}\overline{h}_{\bar{a}}^T(y), & for\ x\in\hat{\Sigma}^{(g_a)},\ y\in\hat{\Sigma}^{(g_{\bar{a}})}. \end{cases} \tag{18}$$

Similarly to a study [9] we define the determinant of I-Q as a formal power series in $\varepsilon^{\frac{1}{2}}$ by:

$$\log\det\left(I-Q\right)=\mathrm{Tr}\log\left(I-Q\right)=-\sum_{n\geq 1}\frac{1}{n}\mathrm{Tr}(Q^n).$$

Clearly $\mathrm{Tr}(Q^{2k})=2\mathrm{Tr}((F_1F_2)^k)$ for $k \geq 0$ whereas $\mathrm{Tr}(Q^n)=0$ or n odd. Furthermore, from eqn. (13) the diagonal terms $(F_1F_2)^k$ have integral power series in ε. Thus it follows that [8]:

Lemma 1: $\det(I-Q)=\det(I-F_1F_2)$ and is a formal power series in ε.

The determinant has the following holomorphic properties:

Theorem 2: $\det(I$-$Q)$ is non-vanishing and holomorphic in ε or $|\varepsilon|<r_1r_2$.

Geometric Formulas for Prime Forms

In this section we derive formulas for the prime form on a Reimann surface of genus g_1+g_2 sewn of two Reimann surfaces of genuses g_1 and g_2. Let us denote:

$$J_{a\bar{a}}=\begin{cases} h_a(x)\left(\xi(-1)^{\bar{a}}I-X_{\bar{a}a}\right)\overline{h}_{\bar{a}}^T(y), & for\ x\in\hat{\Sigma}^{(g_a)},\ y\in\hat{\Sigma}^{(g_{\bar{a}})}, \\ h_a(x)X_{\bar{a}\bar{a}}\overline{h}_a^T(y), & for\ x,y\in\hat{\Sigma}^{(g_a)}. \end{cases}$$

Then we obtain:

Proposition 3

$$\begin{aligned} \mathcal{E}^{(g_1+g_2)}(x,y)=J_{a\bar{a}}^{-1}\Bigg[&\Theta^{(g_1+g_2)}\begin{bmatrix}\alpha^{(g_1+g_2)} \\ \beta^{(g_1+g_2)}\end{bmatrix}\left(x,y;\nu^{(g_1+g_2)},0\,|\,\Omega^{(g_1+g_2)}\right) \\ &-\delta_{a\bar{a}}\Theta^{(g_a)}\begin{bmatrix}\alpha^{(g_a)} \\ \beta^{(g_a)}\end{bmatrix}\left(x,y;\nu^{(g_a)},0\,|\,\Omega^{(g_a)}\right)e^{-\frac{1}{2}b_aX_{\bar{a}\bar{a}}b_a^T}\Bigg], \end{aligned} \tag{19}$$

or, equivalently,

$$\mathcal{E}^{(g_1+g_2)}(x,y)=\Theta^{(g_1+g_2)}\begin{bmatrix}\alpha^{(g_1+g_2)} \\ \beta^{(g_1+g_2)}\end{bmatrix}\left(x,y;\nu^{(g_1+g_2)},0\,|\,\Omega^{(g_1+g_2)}\right)$$

$$\cdot\left[\delta_{a\bar{a}}\left(\vartheta^{(g_a)}\begin{bmatrix}\alpha^{(g_a)} \\ \beta^{(g_a)}\end{bmatrix}\left(0\,|\,\Omega^{(g)}\right)\right)^{-1}\left(\zeta^{(g_a)}(x)\right)^{\frac{1}{2}}\left(\zeta^{(g_a)}(y)\right)^{\frac{1}{2}}+J_{a\bar{a}}\right]^{-1}, \quad (20)$$

Proof. In a study [9] it was proved that:

$$\mathcal{E}^{(g_1+g_2)}(x,y)=\mathcal{E}^{(g_a)}(x,y)e^{-\frac{1}{2}b_a X_{\bar{a}\bar{a}} b_a^T}.$$

Using eqn. (22) we obtain:

$$\Theta^{(g_1+g_2)}\begin{bmatrix}\alpha^{(g_1+g_2)} \\ \beta^{(g_1+g_2)}\end{bmatrix}\left(x,y;\nu^{(g_1+g_2)},0\,|\,\Omega^{(g_1+g_2)}\right)\left(\mathcal{E}^{(g_1+g_2)}(x,y)\right)^{-1} \quad (21)$$

$$=\delta_{a\bar{a}}\Theta^{(g_a)}\begin{bmatrix}\alpha^{(g_a)} \\ \beta^{(g_a)}\end{bmatrix}\left(x,y;\nu^{(g_a)},0\,|\,\Omega^{(g_a)}\right)(\mathcal{E}^{(g_1+g_2)}(x,y))^{-1}e^{-\frac{1}{2}b_a X_{\bar{a}\bar{a}} b_a^T}+J_{a\bar{a}},$$

and therefore:

$$\mathcal{E}^{(g_1+g_2)}(x,y)=\Theta^{(g_1+g_2)}\begin{bmatrix}\alpha^{(g_1+g_2)} \\ \beta^{(g_1+g_2)}\end{bmatrix}\left(x,y;\nu^{(g_1+g_2)},0\,|\,\Omega^{(g_1+g_2)}\right)$$

$$\left[\delta_{a\bar{a}}\Theta^{(g_a)}\begin{bmatrix}\alpha^{(g_a)} \\ \beta^{(g_a)}\end{bmatrix}\left(x,y;\nu^{(g_a)},0\,|\,\Omega^{(g_a)}\right)\left(\mathcal{E}^{(g_1+g_2)}(x,y)\right)^{-1}e^{-\frac{1}{2}b_a X_{\bar{a}\bar{a}} b_a^T}+J_{a\bar{a}}\right]^{-1}.$$

Thus we get eqns. (19) and (20) using and the definition eqn. (4).

Remark 1: Note that in eqns. (19) and (20), $J_{a\bar{a}}$ can be expressed as in eqn. (18), i.e.,

$$J_{a\bar{a}}=\begin{cases}h_a(x)\left(I-F_{\bar{a}}F_a\right)^{-1}F_{\bar{a}}\bar{h}_a^T(y), & for\ x,y\in\hat{\Sigma}^{(g_a)}, \\ \xi(-1)^{\bar{a}}h_a(x)\left(I-F_{\bar{a}}F_a\right)^{-1}\bar{h}_{\bar{a}}^T(y), & for\ x\in\hat{\Sigma}^{(g_a)},\ y\in\hat{\Sigma}^{(g_{\bar{a}})},\end{cases} \quad (22)$$

where I denotes the infinite identity matrix and T the transpose.

Acknowledgement

Research of the author was supported by the GACR project 18-00496S and RVO: 67985840.

References

1. Farkas HM, Kra I (1980) Riemann Surfaces. Springer-Verlag (New York).
2. Springer G (1959) Introduction to Riemann Surfaces. Addison-Wesley.
3. Mumford D (1983) Tata lectures on Theta I. Boston: Birkhäuser.
4. Fay JD (1973) Theta Functions on Riemann Surfaces. Lecture Notes in Mathematics, Vol. 352. Berlin-New York: Springer-Verlag.
5. Schier M (1966) Half-order differentials on Riemann surfaces. SIAM J.Appl. Math. 14: 922-934.
6. Hawley NS, Schier M (1966) Half-order differentials on Riemann surfaces. Acta Math. 115: 199-236.
7. Yamada A (1980) Precise variational formulas for abelian differentials. Kodai Math.J.3:114-143.
8. Tuite MP, Zuevsky A (2011) The Szegö kernel on a sewn Riemann surface. Commun. Math Phys 306: 617-645.
9. Mason G, Tuite MP (2007) On genus two Riemann surfaces formed from sewn tori. Commun. Math. Phys. 270: 587-634.