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Research Article

Generalized Matric Massey Products for Graded Modules⋆

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Abstract The theory of generalized matric Massey products has been applied for some time to A-modules M, A being a k-algebra. The main application is to compute the local formal moduli $\hat{H}_M$, isomorphic to the local ring of the moduli of A-modules. This theory is also generalized to $\mathcal{O}_X$-modules $\mathcal{M}$, X being a k-scheme. In these notes, we consider the definition of generalized Massey products and the relation algebra in any obstruction situation (a differential graded k-algebra with certain properties), and prove that this theory applies to the case of graded R-modules, R being a graded k-algebra and k algebraically closed. When the relation algebra is algebraizable, that is, the relations are polynomials rather than power series, this gives a combinatorial way to compute open (étale) subsets of the moduli of graded R-modules. This also gives a sufficient condition for the corresponding point in the moduli of $\mathcal{O}_{\mathrm{Proj}(R)}$-modules to be singular. The computations are straightforwardly algorithmic, and an example on the postulation Hilbert scheme is given.

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1 Introduction

The theory of generalized matric Massey products (GMMP) for A-modules, A being a k-algebra, is given by Laudal in [4], and applied to the theory of moduli of global and local modules in [6,7]. This theory can obviously be applied also to the study of various Hilbert schemes, leading to GMMP for graded R-modules M, R being a graded k-algebra. Let $(A^\bullet, d_\bullet)$ be a differential graded k-algebra, and let $\underline{\alpha} = \{\alpha_{e_1}, \ldots, \alpha_{e_d}\}$ be a set of elements in $H^1(A^\bullet)$. For $\underline{n} \in (\mathbb{N} - \{0\})^d$ and $|\underline{n}| = n_1 + \cdots + n_d = 2$ we have the ordinary cup-products $\underline{\alpha} \otimes_k \underline{\alpha} \to H^2(A^\bullet)$ given by

$$\langle \underline{\alpha}; \underline{n} \rangle = \sum_{\substack{\underline{m}_1 + \underline{m}_2 = \underline{n} \\ \underline{m}_i \in (\mathbb{N} \cup \{0\})^d}} \alpha_{\underline{m}_1} \cdot \alpha_{\underline{m}_2}.$$

For example,

$$\big\langle \underline{\alpha}; (1, 0, \ldots, 0, 1) \big\rangle = \alpha_{e_1} \cdot \alpha_{e_d} + \alpha_{e_d} \cdot \alpha_{e_1}, \quad \big\langle \underline{\alpha}; (1, 1, 0, \ldots, 0) \big\rangle = \alpha_{e_1} \cdot \alpha_{e_2}.$$

By inductively adding elements $\alpha_{\underline{m}} \in A^1$, $\underline{m} \in \overline{B} \subseteq (\mathbb{N} \cup \{0\})^d$ due to some relations, we define the higher-order generalized matric Massey products $\langle \underline{\alpha}; \underline{n} \rangle$, $\underline{n} \in B' \subseteq (\mathbb{N} \cup \{0\})^d$, for some $\underline{n}$ of higher-order $|\underline{n}|$, provided $A^\bullet$ satisfies certain properties. The inductive definition of GMMP is controlled at each step by the relations between the monomials in an algebra $\hat{H}_{\underline{\alpha}}$ constructed in parallel. We call this algebra the relation algebra of $\underline{\alpha}$. It is interesting in its own right to study the GMMP structure and the relation algebra of various sets of $\underline{\alpha} \in H^1(A^\bullet)$.

Deformation theory is introduced as a tool for studying local properties of various moduli spaces. It is well known that the prorepresenting hull of the deformation functor of a point M in moduli is the completion of the local ring in that point [5]. Consider a graded R-module M, R being a graded k-algebra. Choose a minimal resolution $0 \leftarrow M \leftarrow L_\bullet$ of M and consider the degree zero part $\mathrm{Hom}^\bullet_{R,0}(L_\bullet, L_\bullet)$ of the Yoneda complex. Then, $(\mathrm{Hom}^\bullet_{R,0}(L_\bullet, L_\bullet), d_\bullet)$ is a differential graded k-algebra. Let

$$\underline{x}^* = \{x_1^*, \ldots, x_d^*\} \subseteq H^1\big(\mathrm{Hom}^\bullet_{R,0}\left(L_\bullet, L_\bullet\right)\big) \cong \mathrm{Ext}^1_{R,0}(M, M)$$

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be a k-basis. Then, the relation algebra $\hat{H}_{\underline{x}^*}$ is isomorphic to the prorepresenting hull $\hat{H}_M$ of the (graded) deformation functor Def_M, that is $\hat{H}_{\underline{x}^*} \cong \hat{H}_M$. In addition to the definition of the graded GMMP, this is the main result of the paper, implying that general results about the GMMP give local information about moduli. In addition, we get the following result, telling us how GMMP on R can be used to study the singular locus of sheaves on $\mathrm{Proj}(R)$.

Proposition 1. *Let $M = \Gamma_*(\mathcal{M})$ for $\mathcal{M}$ being a coherent $\mathcal{O}_{\mathrm{Proj}(R)}$-module. Assume that*

$$\mathrm{ext}^1_{R,0}(M, M) = \mathrm{ext}^1_{\mathrm{Proj}(R)}(\mathcal{M}, \mathcal{M}).$$

Then, the hulls of the two deformation functors $\mathrm{Def}_{R,\mathcal{M}}$ and $\mathrm{Def}_{R,M}$ are isomorphic, that is $H_{\mathcal{M}} \cong H_M$.

We conclude the paper with an explicit application to the postulation Hilbert scheme, suggested by Professor Jan Kleppe.

2 Classical graded theory

2.1 Notation

We let $R = \oplus_{d\in\mathbb{Z}} R_d$ be a graded k-algebra, k being algebraically closed of characteristic 0 and R finitely generated in degree 1. We let $M = \oplus_{d\in\mathbb{Z}} M_d$ be a graded R-module, and let $M(p)$ denote the twisted module of M with grading $M_d(p) = M_{p+d}$.

We will reserve the name S for the free polynomial k-algebra $S = k[x_1, \dots, x_n]$, so that R is a quotient of some S by some homogenous ideal I, that is $R = S/I$.

By $\underline{\ell}$ we mean the category of local Artinian k-algebras with residue field k. If $U \in \mathrm{ob}(\underline{\ell})$, we will use the notation $\underline{m}_U$ for the maximal ideal in U. A surjective morphism $\pi : U \twoheadrightarrow V$ in $\underline{\ell}$ is called small if $\ker\pi \cdot \underline{m}_U = 0$. The ring of dual numbers is denoted by $k[\varepsilon]$, that is $k[\varepsilon] = k[\varepsilon]/(\varepsilon^2)$.

If V is a vector space, V^* denotes its dual.

2.2 Homomorphisms

Classically, homomorphisms of graded k-algebras R are *homogenous of degree* 0. This is also the case with morphisms of graded R-modules. We might extend this definition by giving a grading to the homomorphisms:

$$\mathrm{Hom}_R\Big(\underset{d\in\mathbb{Z}}{\oplus} M_d, \underset{d\in\mathbb{Z}}{\oplus} N_d\Big) = \underset{d\in\mathbb{Z}}{\oplus} \mathrm{Hom}_{R,d}(M, N),$$

where $\phi_d \in \mathrm{Hom}_{R,d}(M, N) \subseteq \mathrm{Hom}_R(M, N)$ has the additional property $\phi_d(M_p) \subseteq N_{p+d} = N_p(d)$.

2.3 Construction of graded S-modules

For the graded R-modules M and N, $M \oplus N$ does certainly not inherit a total grading by $(M \oplus N)_d = \oplus_{d'+d''=d}(M_{d'} \oplus N_{d''})$. Thus, the sentence "a free graded R-module" does simply not make any sense. In this section, we clarify how the grading is given.

Recall that if M, N are graded R-modules and $f : N \to M$ is a homogenous homomorphism (of degree 0), then $\ker(f)$ and $\mathrm{im}(f)$ are both graded submodules.

Lemma 2. *Let $N = \oplus_{d\in\mathbb{Z}} N_d$ be a graded R-module, M any R-module and $f : M \twoheadrightarrow N$ a surjective R-module homomorphism. Then, $\mathrm{gr}(f^{-1}) = \oplus_{d\in\mathbb{Z}} f^{-1}(N_d)$ has a natural structure of* ***graded*** *R-module.*

Remark 3. The proof of the above lemma is immediate, but it is not always the case that $M \cong \mathrm{gr}(f^{-1})$ for some f. In fact, this is equivalent with M being graded.

For the sake of simplicity, assume that $M = \oplus_{d\in\mathbb{Z}} M_d$ is a finitely generated graded module, generated by a finite number of homogenous elements $m_1, \dots, m_n$ of degrees $p_1, \dots, p_n$, respectively. Then, we have a surjective homomorphism

$$\varepsilon : R\big(-p_1\big) \oplus R\big(-p_2\big) \oplus \cdots \oplus R\big(-p_n\big) \longrightarrow M \longrightarrow 0$$

sending e_i of degree p_i to m_i (also of degree p_i).

We easily see that $R(-p_1) \oplus R(-p_2) \oplus \cdots \oplus R(-p_n) \cong \mathrm{gr}(\varepsilon^{-1})$, making $R(-p_1) \oplus R(-p_2) \oplus \cdots \oplus R(-p_n)$ into a graded module. As the kernel is also generated by a finite number of homogenous elements, say by $(g_1, \dots, g_l)$, where $g_i = \sum_j g_{ij}$ with g_{ij} being homogenous in R, we have the following proposition.

Proposition 4. *Every finitely generated, graded R-module M has a minimal resolution of the form*

$$\cdots \longrightarrow \bigoplus_{i=1}^{m_n} R\big(-d_i^n\big)^{\beta_{n,i}} \longrightarrow \cdots \longrightarrow \bigoplus_{i=1}^{m_1} R\big(d_i^1\big)^{\beta_{1,i}} \longrightarrow M \longrightarrow 0.$$

Conversely, given homogenous elements $g_1, \ldots, g_n \in R$ of degrees $d_1, \ldots, d_n$, respectively, then $R(-d_1) \oplus \cdots \oplus R(-d_n)$ maps surjectively onto the graded module $(g_1, \ldots, g_n) \subseteq R$, and so it is a graded R-module.

2.4 Families of graded modules

A finitely generated graded R-module M defines a coherent sheaf $\tilde{M}$ of $\mathcal{O}_{\mathrm{Proj}(R)}$-modules. In the same way, an ideal $I \subseteq R$ defines a sheaf of ideals on $\mathrm{Proj}(R)$, and this gives a subscheme of $\mathrm{Proj}(R)$. Thus, the study of various moduli spaces is influenced by the study of graded R-modules.

Let us denote $\underline{R} = \mathrm{Spec}(R)$ for short, and let X be a scheme/k. Then, a sheaf of graded $\mathcal{O}_{\underline{R}\times_k X}$-modules is an $\mathcal{O}_{\underline{R}\times_k X}$-module $\mathcal{G}$ such that $\mathcal{G}(\underline{R} \times_k U)$ is a graded $R \otimes_k \mathcal{O}_X(U)$-module for every open $U \subseteq X$. We define the contravariant functor $\mathrm{Gr}_R : \underline{\mathrm{Sch}}_k \to \underline{\mathrm{Sets}}$ by

$$\mathrm{Gr}_R(X) = \big\{\text{coherent graded } \mathcal{O}_{\underline{R}\times_k X}\text{-modules } \mathcal{G}_X \mid \mathcal{G}_X \text{ is } X\text{-flat}\big\}/\cong .$$

The moduli spaces that we want to give results about are the schemes representing various restrictions of the functor $\mathrm{C}_R : \underline{\mathrm{Sch}}_k \to \underline{\mathrm{Sets}}$ given by

$$\mathrm{C}_R(X) = \big\{\text{coherent } \mathcal{O}_{\mathrm{Proj}(R)\times_k X}\text{-modules } \mathcal{F} \mid \mathcal{F} \text{ is } X\text{-flat}\big\}/\cong .$$

The restrictions can be that $\mathcal{F}$ is an ideal sheaf with fixed Hilbert polynomial $p(t)$. Then, the above functor in the case where $R = k[t_1, \ldots, t_n]$ is the Hilbert functor $\mathrm{Hilb}^{p(t)}_{\mathbb{P}^n_k}$. If $\mathcal{F}$ is locally free of rank r with fixed chern classes, we get $\mathcal{M}(\mathbb{P}^n_k; c_1, \ldots, c_r)$ and so forth.

The two functors Gr_R and C_R are usually not equivalent, that is, there exist graded modules M such that $\Gamma_*(\tilde{M}) \ncong M$. However, $\Gamma_*(\mathcal{F}) \cong \Gamma_*(\mathcal{G}) \Rightarrow \mathcal{F} \cong \mathcal{G}$, and this, as we will see, is sufficient for given applications.

In general, for a contravariant functor $\mathcal{F} : \underline{\mathrm{Sch}} \to \underline{\mathrm{Sets}}$, and an element $x \in \mathcal{F}(\mathrm{Spec}(k))$, we define the fiber functor $\mathcal{F}_x : \underline{\mathrm{Sch}}/k \times \{\underline{pt}\} \to \underline{\mathrm{Sets}}$ from the category of pointed schemes over k to the category of sets by

$$\mathcal{F}_x(X) = \big\{F \in \mathcal{F}(X) \mid \mathcal{F}\big(\mathrm{Spec}(k) \xrightarrow{\underline{pt}} X\big)(F) = F_{\underline{pt}} = x\big\}.$$

If $\mathcal{F}$ is represented by a scheme $\mathbb{M}$, and if $x \in \mathbb{M}$ is a geometric point, then the tangent space in this point is

$$\big(\underline{m}_x/\underline{m}_x^2\big)^* \cong \mathrm{Hom}_x \,\mathrm{Spec}\,\big(k[\varepsilon], X\big) \cong \mathcal{F}_x\Big(\mathrm{Spec}\,\big(k[\varepsilon]\big)\Big).$$

The fiber functors define covariant functors $D_x : \underline{\ell} \to \underline{\mathrm{Sets}}$ and $D_x(V) = \mathcal{F}_x(\mathrm{Spec}(V))$. In our situation, we obtain the two deformation functors

$$D_M^{\mathrm{Gr}_R} = \mathrm{Def}_{R,M}, \quad D_{\mathcal{M}}^{\mathrm{C}_R} = \mathrm{Def}_{R,\mathcal{M}} : \underline{\ell} \longrightarrow \underline{\mathrm{Sets}}$$

given by

$$\mathrm{Def}_{R,M}(V) = \big\{\text{f.g. graded } R \otimes_k V\text{-modules } M_V \mid M_V \text{ is } V\text{-flat},\ M_{V,0} \cong M\big\}/\cong,$$
$$\mathrm{Def}_{R,\mathcal{M}}(V) = \big\{\text{coherent } \mathcal{O}_{\mathrm{Proj}(R\otimes_k V)}\text{-modules } \mathcal{M}_V \mid \mathcal{M}_V \text{ is } V\text{-flat},\ \mathcal{M}_{V,0} \cong \mathcal{M}\big\}/\cong .$$

For the rest of this section, we assume that $\mathcal{M} = \tilde{M}$. We will use the notations Def_M and $\mathrm{Def}_{\mathcal{M}}$ when no confusion is possible.

By definition, the tangent spaces of the moduli spaces are

$$\mathrm{Def}_M\,\big(k[\varepsilon]\big) \cong \mathrm{Ext}^1_{R,0}(M, M), \quad \mathrm{Def}_{\mathcal{M}}\,\big(k[\varepsilon]\big) \cong \mathrm{Ext}^1_{\mathrm{Proj}(R)}(\mathcal{M}, \mathcal{M}),$$

respectively ($\mathrm{Ext}^1_{R,0}(M, M)$ will be defined below). For every $V \in \underline{\ell}$, the morphism $\tilde{\ }$ is surjective with section Γ_*. That is, we have the diagram

$$\mathrm{Def}_{R,M}(V) \underset{\sim}{\overset{\Gamma_*}{\rightleftarrows}} \mathrm{Def}_{R,\mathcal{M}}(V)\,, \quad \text{where } \Gamma_*(\mathcal{F})\tilde{\ } = \mathcal{F}.$$

For a surjective small morphism $\pi : U \twoheadrightarrow V$ in $\underline{\ell}$, given a diagram

$$\begin{array}{ccc} M_U \in \mathrm{Def}_M(U) & \xrightarrow[\sim]{} & \mathrm{Def}_{\mathcal{M}}(U) \ni \tilde{M}_U \\ \downarrow & \overset{\Gamma_*}{\longleftarrow} & \downarrow \\ \Gamma_*\big(\mathcal{F}_V\big) \in \mathrm{Def}_M(V) & \xrightarrow{\sim} & \mathrm{Def}_{\mathcal{M}}(V) \ni \mathcal{F}_V \\ & \underset{\Gamma_*}{\longleftarrow} & \end{array}$$

with M_U mapping to $\Gamma_*(\mathcal{F}_V)$, $\Gamma_*(\mathcal{F}_V)$ mapping to $\mathcal{F}_V$, then it follows by functoriality of $\tilde{}$ that $\tilde{M}_U$ is a lifting of $\mathcal{F}_V$.

This has obvious consequences, and we will eventually prove the following.

Proposition 5. *Let $M = \Gamma_*(\mathcal{M})$ for $\mathcal{M} \in \mathrm{C}_R(\mathrm{Spec}(k))$. Assume that $\mathrm{ext}^1_{R,0}(M, M) = \mathrm{ext}^1_{\mathrm{Proj}(R)}(\mathcal{M}, \mathcal{M})$. Then the hulls of the two deformation functors $\mathrm{Def}_{R,\mathcal{M}}$ and $\mathrm{Def}_{R,M}$ are isomorphic, that is $H_{\mathcal{M}} \cong H_M$.*

Remark 6. This proof says that if $\Gamma_*(\mathcal{F}_2) = M_2$, where $\mathcal{F}_2$, M_2, are the liftings to $k[\varepsilon]$ corresponding to the canonical morphism $\hat{H} \to \hat{H}/\underline{m}^2$, then $H_{\mathcal{M}} \cong H_M$. This can be used correspondingly in the higher-order liftings, but it is very hard to check.

3 Deformation theory

3.1 Generalized Massey products

In this subsection, we consider a differential graded k-algebra $(A^\bullet, d_\bullet)$ with certain properties. We will assume that $0 \in \mathbb{N}$, and for $\underline{n} \in \mathbb{N}^d$, we will use the notation $|\underline{n}| = \sum_{i=1}^d n_i$. For $\underline{\alpha} = (\alpha_{e_1}, \ldots, \alpha_{e_d}) \in (H^1(A^\bullet))^d$, $d \in \mathbb{N}$, we will define some generalized Massey products $\langle \underline{\alpha}; \underline{m} \rangle \in H^2(A^\bullet)$, $\underline{m} \in B'$, where $B' \subseteq \{\underline{n} \in \mathbb{N}^d : |\underline{n}| \geq 2\}$. Notice that the Massey products may not be defined for all (if any) $n \in \mathbb{N}^d$. The overall idea is the following.

Let $\alpha_{e_1}, \ldots, \alpha_{e_d}$ be a set of d elements in $H^1(A^\bullet)$, let $B'_2 = \{\underline{n} \in \mathbb{N}^d : |\underline{n}| = 2\}$, and put $\overline{B}_1 = \{\underline{n} \in \mathbb{N}^d : |\underline{n}| \leq 1\}$. The first-order Massey products are then the ordinary cup-products in $A^\bullet$. That is

$$\langle \underline{\alpha}; \underline{n} \rangle = \overline{y(\underline{n})}, \quad y(\underline{n}) = \sum_{\substack{\underline{m}_1 + \underline{m}_2 = \underline{n} \\ |\underline{m}_i| = 1}} \alpha_{\underline{m}_1} \cdot \alpha_{\underline{m}_2}, \quad \underline{n} \in B'_2.$$

Definition 7. One will say that the Massey product is *identically zero* if $y(\underline{n}) = 0$.

The higher-order Massey products are defined inductively. Assume that the Massey products are defined for $\underline{n} \in B'_N$, $B'_N \subseteq \{\underline{n} \in \mathbb{N}^d : |\underline{n}| \leq N\}$, $N \in \mathbb{N}$. For each $\underline{m} \in B_N \subseteq B'_N$, assume that there exists a fixed linear relation

$$b_{\underline{m}} = \sum_{l=0}^{N-2} \sum_{\underline{n} \in B'_{2+l}} \beta_{\underline{n},\underline{m}} \langle \underline{\alpha}; \underline{n} \rangle = 0,$$

and choose an $\alpha_{\underline{m}} \in A^1$ such that $d(\alpha_{\underline{m}}) = l(\underline{m})$. The set $\{\alpha_{\underline{m}}\}_{\underline{m} \in \bar{B}_N}$, $\bar{B}_N = \bar{B}_{N-1} \cup B_N$, is called ***a defining system*** for the ***Massey products***

$$\langle \underline{\alpha}; \underline{n} \rangle = \overline{y(\underline{n})}, \quad y(\underline{n}) = \sum_{|\underline{m}| \leq N+1} \sum_{\substack{\underline{m}_1 \underline{m}_2 = \underline{m} \\ \underline{m}_i \in \bar{B}_N}} \beta'_{\underline{n},\underline{m}} \alpha_{\underline{m}_1} \cdot \alpha_{\underline{m}_2}, \quad \underline{m} \in B'_{N+1},$$

where $\beta'_{\underline{n},\underline{m}}$ are chosen linear coefficients for each pair $\underline{m}$, $\underline{n}$ such that $\langle \underline{\alpha}; \underline{n} \rangle \in H^2(A^\bullet)$.

One way to construct Massey products, that is to construct the relations b given above, is as follows. Let $\alpha_{e_1}, \ldots, \alpha_{e_d}$ be a set of representatives of d elements in $H^1(A^\bullet)$. Let

$$S_2 = k\big[[u_1, \ldots, u_d]\big]/\underline{m}^2 = k\big[[\underline{u}]\big]/\underline{m}^2, \quad R_3 = k\big[[\underline{u}]\big]/\underline{m}^3,$$
$$\bar{B}_1 = \big\{\underline{n} \in \mathbb{N}^d : |n| \leq 1\big\}, \quad B'_2 = \big\{\underline{n} \in \mathbb{N}^d : |n| = 2\big\}, \quad \bar{B}'_2 = \bar{B}_1 \cup B'_2.$$

Definition 8. The first-order Massey products are the ordinary cup-products in $A^\bullet$. That is, for $\underline{n} \in B_2'$,

$$\langle \underline{\alpha}; \underline{n} \rangle = \overline{y(\underline{n})}, \quad y(\underline{n}) = \sum_{\substack{\underline{m}_1 + \underline{m}_2 = \underline{n} \\ |\underline{m}_i| = 1}} \alpha_{\underline{m}_1} \cdot \alpha_{\underline{m}_2}.$$

Choose a k-basis $\{y_1^*, \ldots, y_r^*\}$ for $H^2(A^\bullet)$, and put

$$f_j^2 = \sum_{\underline{n} \in B_2'} y_j\big(\langle \underline{\alpha}; \underline{n} \rangle\big) \underline{u}^{\underline{n}}, \quad j = 1, \ldots, r.$$

Let $S_3 = R_3/(f_1^2, \ldots, f_r^2)$, and choose $B_2 \subseteq B_2'$ such that $\{\underline{u}^{\underline{n}}\}_{\underline{n} \in B_2}$ is a monomial basis for

$$\underline{m}^2/\underline{m}^3 + \big(f_1^2, \ldots, f_r^2\big).$$

Put $\overline{B}_2 = \overline{B}_1 \cup B_2$. For each $\underline{n} \in \mathbb{N}^d$ with $|\underline{n}| \leq 3$, we have a unique relation in S_3:

$$\underline{u}^{\underline{n}} = \sum_{\underline{m} \in \overline{B}_2} \beta_{\underline{n}, \underline{m}} \underline{u}^{\underline{m}}$$

and for each $\underline{m} \in B_2$ we have

$$b_{\underline{m}} = \sum_{\underline{n} \in B_2'} \beta_{\underline{n}, \underline{m}} \langle \underline{\alpha}; \underline{n} \rangle = 0.$$

Choose for each $\underline{m} \in B_2$ an $\alpha_{\underline{m}} \in A^1$ such that $d(\alpha_{\underline{m}}) = -b_{\underline{m}}$. Put

$$R_4 = k\big[[\underline{u}]\big]/\underline{m}^4 + \underline{m}\big(f_2, \ldots, f_r^2\big).$$

Choose a monomial basis $\{\underline{u}^{\underline{n}}\}_{\underline{n} \in B_3'}$ for $\underline{m}^3/\underline{m}^4 + \underline{m}^3 \cap (f_1^2, \ldots, f_r^2)$ such that for each $\underline{n} \in B_3'$, $\underline{u}^{\underline{n}} = u_k \cdot \underline{u}^{\underline{m}}$ for some $0 \leq k \leq d$ and some $\underline{m} \in B_2$.

Definition 9. The set $\{\alpha_{\underline{m}}\}_{\underline{m} \in \overline{B}_2}$ is called a defining system for the Massey products $\langle \underline{\alpha}; \underline{n} \rangle$, $\underline{n} \in B_3'$.

Assume that the k-algebras

$$\begin{array}{c} R_{n+1} = k\big[[\underline{u}]\big]/\big(\underline{m}^{n+1} + \underline{m}\big(f_1^{n-1}, \ldots, f_r^{n-1}\big)\big) \\ \Big\downarrow {\scriptstyle \pi'_{n+1}} \\ S_n = k\big[[\underline{u}]\big]/\big(\underline{m}^n + \big(f_1^{n-1}, \ldots, f_r^{n-1}\big)\big) \end{array}$$

and the sets B_{n-1}, $\overline{B}_{n-1}$, B_n', $\{\alpha_{\underline{m}}\}_{\underline{m} \in \overline{B}_{n-1}}$ have been constructed for $1 \leq n \leq N$ according to the above, in particular

$$\begin{aligned} \ker \pi'_{n+1} &= \underline{m}^n + \big(f_1^{n-1}, \ldots, f_r^{n-1}\big)/\big(\underline{m}^{n+1} + \underline{m}\big(f_1^{n-1}, \ldots, f_r^{n-1}\big)\big) \\ &\cong \underline{m}^n/\big(\underline{m}^{n+1} + \underline{m}^n \cap \underline{m}\big(f_1^{n-1}, \ldots, f_r^{n-1}\big) \oplus \big(f_1^{n-1}, \ldots, f_r^{n-1}\big)/\underline{m}\big(f_1^{n-1}, \ldots, f_r^{n-1}\big)\big), \end{aligned}$$

and we assume (by induction) that $\{\underline{u}^{\underline{n}}\}_{\underline{n} \in B_n'}$ is a basis for

$$I_{n+1} = \underline{m}^n/\big(\underline{m}^{n+1} + \underline{m}^n \cap \underline{m}\big(f_1^{n-1}, \ldots, f_r^{n-1}\big)\big).$$

Put $\overline{B}'_{N+1} = \overline{B}_N \cup B'_{N+1}$. For each $\underline{n} \in B'_{N+1}$ we have a unique relation in R_{N+1}:

$$\underline{u}^{\underline{n}} = \sum_{\underline{m} \in \overline{B}'_{N+1}} \beta'_{\underline{m}, \underline{n}} \underline{u}^{\underline{m}} + \sum_j \beta_{\underline{n}, j} f_j^{n-1}.$$

Definition 10. The Nth-order Massey products are

$$\langle \underline{\alpha}; \underline{n} \rangle = \overline{y(\underline{n})}, \quad y(\underline{n}) = \sum_{|\underline{m}| \leq N+1} \sum_{\substack{\underline{m}_1 + \underline{m}_2 = \underline{m} \\ \underline{m}_i \in \overline{B}_N}} \beta'_{\underline{m}, \underline{n}} \alpha_{\underline{m}_1} \cdot \alpha_{\underline{m}_2}, \quad \underline{n} \in B'_{N+1}.$$

For this to be well defined, we need both that $y(\underline{n})$ is a coboundary, and that its class is independent of the choices of α. We will only consider algebras $A^\bullet$ that obey this, and we call $A^\bullet$ ***an obstruction situation algebra***, or concisely an ***OS-algebra***.

Put

$$f_j^N = f_j^{N-1} + \sum_{\underline{n} \in B'_N} y_j\big(\langle \underline{\alpha}; \underline{n} \rangle\big)\underline{u}^{\underline{n}},$$

$$\begin{array}{c} R_{N+2} = k\big[[\underline{u}]\big]/\big(\underline{m}^{N+2} + \underline{m}(f_1^N, \ldots, f_r^N)\big), \\ \big\downarrow \pi'_{N+1} \\ S_{N+1} = R_{N+1}/\big(f_1^N, \ldots, f_r^N\big) = k\big[[\underline{u}]\big]/\big(\underline{m}^{N+1} + \big(f_1^N, \ldots, f_r^N\big)\big) \xrightarrow{\pi_{N+1}} S_N, \end{array}$$

pick a monomial basis $\{\underline{u}^{\underline{n}}\}_{\underline{n} \in B_N}$ for $\ker \pi_{N+1}$ such that $B_N \subseteq B'_N$, and put $\overline{B}_N = \overline{B}_{N-1} \cup B_N$. For each $\underline{n} \in \mathbb{N}^d$, $|\underline{n}| \le N$, we have a unique relation in S_{N+1}, $\underline{u}^{\underline{n}} = \sum_{\underline{m} \in \overline{B}_N} \beta_{\underline{n},\underline{m}} \underline{u}^{\underline{m}}$, and for each $\underline{n} \in B_N$,

$$b_{\underline{n}} = \sum_{l=0}^{N-1} \sum_{\underline{n} \in B'_{2+l}} \beta_{\underline{n},\underline{m}} \langle \underline{\alpha}; \underline{n} \rangle = 0.$$

For each $\underline{m} \in B_N$, choose $\alpha_{\underline{m}} \in A^1$ such that $d(\alpha_{\underline{m}}) = -b_{\underline{m}}$.

Definition 11. The set $\{\alpha_{\underline{m}}\}_{\underline{m} \in \overline{B}_N}$ is called a defining system for the Massey products $\langle \underline{\alpha}; \underline{n} \rangle$, $\underline{n} \in B'_{N+1}$.

Choose a monomial basis $\{\underline{u}^{\underline{n}}\}_{\underline{n} \in B'_{N+1}}$ for

$$\underline{m}^{N+1}/\underline{m}^{N+2} + \underline{m}^{N+2} + \underline{m}^{N+1} \cap \underline{m}\big(f_1^N, \ldots, f_r^N\big)$$

such that for each $\underline{n} \in B'_{N+1}$, we have $\underline{u}^{\underline{n}} = u_k \cdot \underline{u}^{\underline{m}}$ for some $1 \le k \le d$ and $\underline{m} \in B_{N+1}$. The construction then continues by induction.

Definition 12. Let $(A^\bullet, d_\bullet)$ be a differential graded OS k-algebra, and let $\alpha_1, \ldots, \alpha_d = \underline{\alpha}$ be a set of elements in $H^1(A^\bullet)$. Let $\{y_1^*, \ldots, y_r^*\}$ be a k-basis for $H^2(A^\bullet)$, and for $1 \le j \le r$, let

$$f_j = \sum_{l=2}^{\infty} \sum_{\underline{n} \in B'_l} y_j\big(\langle \underline{\alpha}; \underline{n} \rangle\big)\underline{u}^{\underline{n}}.$$

Then, one defines

$$\hat{H}_{\underline{\alpha}} = k\big[[u_1, \ldots, u_d]\big]/\big(f_1, \ldots, f_r\big)$$

and calls it the **relation algebra** of $\underline{\alpha}$.

3.2 Obstruction theory

In this section, fix once and for all a minimal (graded) resolution of the graded R-module M:

$$0 \longleftarrow M \xleftarrow{\varepsilon} L_0 \xleftarrow{\delta_1} L_1 \xleftarrow{\delta_2} L_2 \xleftarrow{\delta_3} \cdots$$

with $L_n \cong \oplus_{i=1}^{m_n} R(-d_{i,n})^{\beta_{i,n}}$. Consider a small surjective morphism $\pi : U \twoheadrightarrow V$ in $\underline{\ell}$, and let $M_V \in \mathrm{Def}_M(V)$. Then, an element $M_U \in \mathrm{Def}_M(U)$ such that $\mathrm{Def}_M(\pi)(M_U) = M_V$ is called a lifting of M_V to U.

Lemma 13. *Giving a lifting $M_U \in \mathrm{Def}_M(U)$ of the graded R-module M is equivalent to giving a lifting of complexes:*

$$\begin{array}{ccccccccccc} 0 & \longleftarrow & M_U & \xleftarrow{\varepsilon^U} & L_0 \otimes_k U & \xleftarrow{\delta_1^U} & L_1 \otimes_k U & \xleftarrow{\delta_2^U} & L_2 \otimes_k U & \xleftarrow{\delta_3^U} & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longleftarrow & M & \xleftarrow[\varepsilon]{} & L_0 & \xleftarrow[\delta_1]{} & L_1 & \xleftarrow[\delta_2]{} & L_2 & \xleftarrow[\delta_3]{} & \cdots \end{array}$$

One also has that $\varepsilon^U(l \otimes_k 1) \in M \otimes_k \underline{m}_U$, $\delta_i^U(l \otimes_k 1) \in L_{i-1} \otimes_k \underline{m}_U$ for all $i \ge 1$, and that the top row is exact.

Proof. Because U is Artinian, its maximal ideal $\underline{m}_U$ is nilpotent. We will prove the lemma by induction on n such that $\underline{m}_U^n = 0$, the case $n = 1$ being obvious. Assuming the result true for n, then assume $\underline{m}_U^{n+1} = 0$ and put $V = U/\underline{m}_U^n$. Thus, the sequence of U-modules $0 \to I \to U \overset{\pi}{\to} V \to 0$ with $I = \underline{m}_U^n = \ker \pi$ is exact with π being a small morphism, and such that $\underline{m}_V^n = 0$. Notice that $M_U \otimes_U V$ is V-flat and that $(M_U \otimes_U V) \otimes_V k \cong M_U \otimes_U k \cong M$ such that $M_V := M_U \otimes_U V \in \operatorname{Def}_M(V)$. Also notice that $M_U \otimes_U I \cong (M_U \otimes_U k) \otimes_k I \cong M \otimes_k I$. Thus, tensorizing $0 \to I \to U \overset{\pi}{\to} V \to 0$ over U with M_U we get the exactness of the first vertical sequence in the diagram

$$
\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \leftarrow & M \otimes_k I & \xleftarrow{\varepsilon \otimes \mathrm{id}} & L_0 \otimes_k I & \xleftarrow{\delta_1 \otimes \mathrm{id}} & L_1 \otimes_k I & \xleftarrow{\delta_2 \otimes \mathrm{id}} & \text{Ł}_2 \otimes_k I & \xleftarrow{\delta_3 \otimes \mathrm{id}} & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \leftarrow & M_U & \xleftarrow{\varepsilon^U} & L_0 \otimes_k U & \xleftarrow{\delta_1^U} & L_1 \otimes_k U & \xleftarrow{\delta_2^U} & L_2 \otimes_k U & \xleftarrow{\delta_3^U} & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \leftarrow & M_V & \xleftarrow[\varepsilon^V]{} & L_0 \otimes_k V & \xleftarrow[\delta_1^V]{} & L_1 \otimes_k V & \xleftarrow[\delta_2^V]{} & L_2 \otimes_k V & \xleftarrow[\delta_3^V]{} & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0. & &
\end{array}
$$

The exactness of the horizontal top row follows from exactness of I over k, the bottom row is exact by assumption, and the middle row is constructed as follows: let ε^U be a lifting of ε^V, which obviously exists. By assumption, $\varepsilon^V(l \otimes 1) = \varepsilon(l) \otimes 1 + h$, $h \in M \otimes_k \underline{m}_V$. Thus, $\varepsilon^V(l \otimes 1) = \varepsilon(l) \otimes 1 + h + v$, $h \in \underline{m}_U$, $v \in I$, that is $\varepsilon^V(l \otimes 1) = \varepsilon(l) \otimes 1 + u$, $u \in \underline{m}_U$. The commutativity of the first rectangle then follows from the fact that $\underline{m}_U \cdot I = 0$, that is, π is a small morphism.

Now, choose a lifting $\tilde{\delta}_1^U$ of δ_1^V. As above, $\tilde{\delta}_1^U(l \otimes 1) \in L_0 \otimes_k \underline{m}_U$ by the induction hypothesis, and therefore it commutes with $\delta_1 \otimes \mathrm{id}$.

For each generator l of $L_1 \otimes_k U$, choose an $x \in L_0 \otimes_k I$ such that $(\varepsilon \otimes \mathrm{id})(x) = \varepsilon^U(\tilde{\delta}_1(l))$, and put $\delta_1(l) = \tilde{\delta}_1(l) - x$. Then

$$\varepsilon^U\big(\delta_1(l)\big) = \varepsilon^U\big(\tilde{\delta}_1(l)\big) - \varepsilon^U(x) = \varepsilon^U\big(\tilde{\delta}_1(l)\big) - (\varepsilon \otimes \mathrm{id})(x) = 0.$$

This gives the desired lifting, and we may continue this way with δ_i^U, $i > 1$. We have proved that the middle sequence is a complex.

Conversely, given a lifting of complexes as in the lemma, then taking the tensor product over U with $V = U/m_U^n$ in the top row, we get a lifting as in the above diagram. By the induction hypothesis, the bottom row is exact with $M_V = H_0(L_\bullet \otimes_k V)$ being a lifting of M.

Writing up the long exact sequence of the short exact sequence of complexes, we have that the sequence in the middle is also exact, $M_U = H_0(L_\bullet \otimes_k U)$ is flat over U because $M_U \cong M \otimes_k U$ as k-vector space implies that M_U is U-free and thus flat. □

As the category of graded R-modules is the (abelian) category of representations of the graded k-algebra R, a homomorphism $\phi : M \to N$ between the two graded R-modules M and N is by definition homogenous of degree 0. This implies that the derived functors of Hom_R in the category of graded R-modules are the derived functors of $\operatorname{Hom}_{R,0}$, where $\operatorname{Hom}_{R,0}$ denotes R-linear homomorphisms of degree 0. Thus, we use the notation $\operatorname{Ext}_{R,0}^p(M, N)$.

We have fixed the minimal graded resolution $0 \leftarrow M \leftarrow L_\bullet$ of M, and we define the graded Yoneda complex by $(\operatorname{Hom}_{R,0}^\bullet(L_\bullet, L_\bullet), \delta_\bullet)$, where

$$\operatorname{Hom}_{R,0}^p\big(L_\bullet, L_\bullet\big) = \prod_{n \geq p} \operatorname{Hom}_{R,0}\big(L_n, L_{n-p}\big)$$

and where the differential

$$\delta_p : \operatorname{Hom}_{R,0}^p\big(L_\bullet, L_\bullet\big) \longrightarrow \operatorname{Hom}_{R,0}^{p+1}\big(L_\bullet, L_\bullet\big)$$

is given by

$$\delta_p\big(\{\xi_n\}_{n \geq p}\big) = \big\{\delta_n \circ \xi_{n-1} - (-1)^p \xi_n \circ \delta_{n-p}\big\}_{n \geq p+1},$$

where the composition is given by $\xi \circ \delta(x) = \delta(\xi(x))$. It is straightforward to prove the following.

Lemma 14. $H^n(\mathrm{Hom}^{\bullet}_{R,0}(L_\bullet, L_\bullet)) \cong \mathrm{Ext}^n_{R,0}(M,N)$, $n \geq 0$.

Proposition 15. *Let $\pi : U \twoheadrightarrow V$ be a small morphism in $\underline{\ell}$ with kernel I. Let $M_V \in \mathrm{Def}_M(V)$ correspond to the lifting $(L_\bullet \otimes_k V, \delta^V_\bullet)$ of the complex $(L_\bullet, \delta)$. Then, there is a uniquely defined obstruction*

$$o\big(M_V, \pi\big) \in \mathrm{Ext}^2_{R,0}(M,M) \otimes_k I,$$

given in terms of the 2-cocycle $o \in \mathrm{Hom}^{\bullet}_{R,0}(L_\bullet, L_\bullet) \otimes_k I$, such that $o(M_V, \pi) = 0$ if and only if M_V may be lifted to U. Moreover, if $o(M_V, \pi) = 0$, then the set of liftings of M_V to U is a principal homogenous space over $\mathrm{Ext}^1_{R,0}(M,M)$.

Proof. Since L_i is free for each i, we can choose a lifting $\tilde{\delta}^R_i$ making the following diagram commutative for each i:

$$\begin{array}{ccc} L_{i-1} \otimes_k U & \xleftarrow{\ \tilde{\delta}^U_i\ } & L_i \otimes_k U \\ \big\downarrow & & \big\downarrow \\ L_{i-1} \otimes_k V & \xleftarrow[\ \delta^V_i\ ]{} & L_i \otimes_k V. \end{array}$$

As π is small, the composition $\tilde{\delta}^U_i \circ \tilde{\delta}^U_{i-1} : L_i \otimes_k U \to L_{i-2} \otimes_k U$ is induced by a unique morphism $o_i : L_i \to L_{i-2} \otimes_k I$, and so

$$o = \big\{o_i\big\} \in \mathrm{Hom}^2_{R,0}\,\big(L_\bullet, L_\bullet\big) \otimes_k I.$$

Also, o is a cocycle, and

$$o\big(M_V, \pi\big) = \bar{o} \in \mathrm{Ext}^2_{R,0}(M,M).$$

Another choice $\tilde{\delta}^R$ leads to an $o \in \mathrm{Hom}^2_{R,0}(L_\bullet, L_\bullet) \otimes_k I$ differing by the image of an element in $\mathrm{Hom}^1_{R,0}(L_\bullet, L_\bullet) \otimes_k I$ such that $o(M_V, \pi)$ is independent of the choice of liftings. This also proves the "only if" part.

If $o = o(M_V, \pi) = 0$, then there is an element $\xi \in \mathrm{Hom}^1_{R,0}(L_\bullet, L_\bullet) \otimes_k I$ such that $o = -d_1(\xi)$. Put $\delta^U_i = \tilde{\delta}^U_i + \xi_i$, and one finds that $\delta^U_i \circ \delta^U_{i-1} = 0$. Thus, it follows from Lemma 13 that M_V can be lifted to U.

For the last statement, given two liftings M^1_U and M^2_U corresponding to $(l_\bullet \otimes_k U, \delta^{U,1}_\bullet)$ and $(l_\bullet \otimes_k U, \delta^{U,2}_\bullet)$. Then their difference induces morphisms $\eta_i = \delta^{U,1}_i - \delta^{U,2}_i : L_i \to L_{i-1} \otimes_k I$, and (for each choice of basis element in I) $\eta = \{\eta_i\} \in \mathrm{Hom}^{\bullet}_{R,0}(L_\bullet, L_\bullet)$ is a cocycle and thus defining $\bar{\eta} \in \mathrm{Ext}^1_{R,0}(M,M)$. This gives the claimed surjection

$$\big\{\text{Liftings of } M_V \text{ to } U\big\} \times \mathrm{Ext}^1_{R,0}(M,M) \twoheadrightarrow \big\{\text{Liftings of } M_V \text{ to } U\big\}.$$

□

Notice that this proves that $\mathrm{Hom}^{\bullet}_{R,0}(L_\bullet, L_\bullet)$ is an OS-algebra as well.

We now combine the theory of Massey products and the theory of obstructions. We let $\hat{H}_M = \hat{H}$ denote the prorepresenting hull (the local formal moduli) of Def_M. All the way we will use the notations and constructions in Section 3.1.

Pick a basis

$$\big\{x_1, \ldots, x_d\big\} \in \mathrm{Ext}^1_{R,0}(M,M)^*$$

and a basis

$$\big\{y_1, \ldots, y_r\big\} \in \mathrm{Ext}^2_{R,0}(M,M)^*.$$

Denote by $\{x^*_i\}$ and $\{y^*_i\}$ the corresponding dual bases. Put

$$S_2 = k\big[u_1, \ldots, u_d\big]/\underline{m}^2 = k\big[[\underline{u}]\big]/\underline{m}^2, \quad \overline{B}_1 = \big\{\underline{n} \in \mathbb{N}^d : |n| \leq 1\big\}.$$

We set $\alpha_{\underline{0}} = \{d_i\}$ and $\alpha_{e_j} = \{x^*_{j,i}\}$. Let $t_{\hat{H}}$ and t_{Def_E} denote the tangent spaces of $\hat{H}$ and Def_M, respectively. A deformation $E_2 \in \mathrm{Def}_E(S_2)$ corresponding to an isomorphism $t_{\hat{H}} \to t_{\mathrm{Def}_E}$ is represented by the lifting $\{L_\bullet \otimes_k S_2, d^{S_2}_\bullet\}$ of $\{L_\bullet, d_\bullet\}$ where

$$d^{S_2}_\bullet\big|_{L_\bullet \otimes 1} = \sum_{\underline{m} \in \overline{B}_1} \alpha_{\underline{m}\bullet} \otimes \underline{u}^{\underline{m}}.$$

Now, put $\pi_3' : R_3 = k[[\underline{u}]]/\underline{m}^3 \to S_2$, choose B_2' as in Section 3.1 and put $\overline{B}_2' = \overline{B}_1 \cup B_2'$. Then

$$o\big(E_2, \pi_3'\big) = \operatorname{cl}\left\{d_i^{S_2} \circ d_{i-1}^{S_2}\right\} = \sum_{\underline{n}\in B_2'} \overline{y(\underline{n})} \otimes \underline{u}^{\underline{n}} \in \operatorname{Ext}^2_{R,0} \otimes_k \ker\left(\pi_3'\right)$$

with

$$y(\underline{n}) = \sum_{\substack{\underline{m}_1+\underline{m}_2=\underline{n}\\ \underline{m}_i\in\overline{B}_1}} \alpha_{\underline{m}_1,i} \circ \alpha_{\underline{m}_2,i-1}.$$

This is to say $\langle \underline{x}^*; \underline{n}\rangle = \overline{y(\underline{n})}$ for each $\underline{n} \in B_2'$. Translating, we get

$$o\big(E_2; \pi_3'\big) = \sum_{\underline{n}\in B_2'} \langle \underline{x}^*; \underline{n}\rangle \otimes_k \underline{u}^{\underline{n}} = \sum_{i=1}^{r} y_i \otimes \left(\sum_{\underline{n}\in B_2'} y_i\big(\langle \underline{x}; \underline{n}\rangle\big)\underline{u}^{\underline{n}}\right) = \sum_{i=1}^{r} y_i \otimes f_i^2.$$

Following the construction in Section 3.1, for each $\underline{m} \in B_2$, we pick a 1-cochain $\alpha_{\underline{m}} \in \operatorname{Hom}^1_{R,0}(L_\bullet, L_\bullet)$ such that

$$d\big(\alpha_{\underline{m}}\big) = -b_{\underline{m}} = -\sum_{\underline{n}\in B_2'} \beta_{\underline{n},\underline{m}} y(\underline{n}).$$

Then the family $\{\alpha_{\underline{m}}\}_{\underline{m}\in\overline{B}_2}$ is a defining system for the Massey products $\langle \underline{x}^*; \underline{n}\rangle$, $\underline{n} \in B_3'$. Define d^{S_3} by

$$d^{S_3}\big|_{L_i\otimes 1} = \sum_{\underline{m}\in\overline{B}_2} \alpha_{\underline{m},i} \otimes \underline{u}^{\underline{m}}.$$

Then $(d^{S_3})^2 = 0$, and so, by Lemma 13, $\{L_\bullet \otimes_k S_3, d_i^{S_3}\}$ corresponds to a lifting $E_3 \in \operatorname{Def}_M(S_3)$. We continue by induction: given a defining system $\{\alpha_{\underline{m}}\}_{\underline{m}\in\overline{B}_N}$ for the Massey products $\langle \underline{x}^*; \underline{n}\rangle$, $\underline{n} \in B'_{N+1}$, assume that d^{S_N} is defined by $d^{S_N}|_{L_i\otimes 1} = \sum_{\underline{m}\in\overline{B}_N} \alpha_{\underline{m},i} \otimes \underline{u}^{\underline{m}}$. Then it follows that

$$o\big(E_N, \pi'_{N+1}\big) = \sum_{\underline{n}\in B'_{N+1}} \left(\sum_{\substack{\underline{m}_1+\underline{m}_2=\underline{m}\\ \underline{m}_i\in\overline{B}_N}} \beta'_{\underline{n},\underline{m}} \alpha_{\underline{m}_1,i} \circ \alpha_{\underline{m}_2,i-1}\right) \otimes \underline{u}^{\underline{n}} + \sum_{j=1}^{r} y_j^* \otimes f_j^N.$$

For $1 \le j \le r$, letting

$$f_j^{N+1} = f_j^N + \sum_{\underline{n}\in B'_{N+1}} y_j\big(\langle \underline{x}^*; \underline{n}\rangle\big)$$

as in Section 3.1 gives

$$o\big(E_N, \pi'_{N+1}\big) = \sum_j y_j^* \otimes f_j^{N+1}.$$

Dividing out by the obstructions, that is letting $S_{N+1} = R_{N+1}/(f_1^{N+1}, \ldots, f_r^{N+1})$, makes the obstruction 0, that is $\sum_{\underline{n}\in B'_{N+1}} \beta_{\underline{n},\underline{m}}\langle \underline{x}^*; \underline{n}\rangle = 0$ for each $\underline{m} \in B_{N+1}$, such that the next order defining system can be chosen.

Thus we have proved the following.

Proposition 16. *Let R be a graded k-algebra and M a graded R-module. Let $\{\underline{x}^*\} = \{x_1^*, \ldots, x_d^*\} \subseteq \operatorname{Ext}^1_{R,0}(M, M) = H^1(\operatorname{Hom}^\bullet_{R,0}(M, M))$ be a k-vector space basis. Then, the relation algebra of $\{\underline{x}^*\}$ is isomorphic to the prorepresenting hull $\hat{H}_M$ of Def_M, that is $\hat{H}_{\{\underline{x}^*\}} \cong \hat{H}_M$.*

Proof. This follows directly from Schlessinger's article [5]. □

Proof of Proposition 5. For each small morphism $\pi : U \twoheadrightarrow V$, if M_V is unobstructed, so is $\tilde{M}_V$. Thus, if there are no relations in H_M, there are none in $H_{\tilde{M}}$ either. □

Proposition 17. *Let R be a graded k-algebra and $[\mathcal{M}]$ a point in the moduli space $\mathbb{M}$ of $\mathcal{O}_{\operatorname{Proj}(R)}$-modules corresponding to $\mathcal{M}$. If all cup-products of $M = \Gamma_*(\mathcal{M})$ are identically zero and $\operatorname{ext}^1_{R,0}(M, M) = \operatorname{ext}^1_{\mathcal{O}_{\operatorname{Proj}(R)}}(\mathcal{M}, \mathcal{M})$, then $\mathbb{M}$ is nonsingular in the point $[\mathcal{M}]$.*

Proof. We can choose all defining systems for M equal to zero so that there are no relations in H_M. The result then follows from Proposition 5. □

4 An example of an obstructed determinantal variety in the postulation Hilbert scheme

The ***postulation Hilbert scheme*** is the scheme $\mathrm{GradAlg}$ parameterizing graded algebras with fixed Hilbert function. The following example is given to me by Jan Kleppe. The theory is treated in [2,3].

Let $R = k[x_0, x_1, x_2, x_3]$, $k = \overline{k}$ and consider the two R-matrices

$$G_I = \begin{pmatrix} x_0 & x_1 & x_2 & x_3^3 \\ x_2 & x_0 & x_1 & x_2^3 \end{pmatrix}, \quad G_J = \begin{pmatrix} x_0 & x_1 & x_2 & x_3^3 \\ x_3 & x_0 & x_1 & x_2^3 \end{pmatrix}.$$

We let I and J be the ideals generated by the minors of G_I and G_J, respectively. Then the graded modules $M_I = R/I$ and $M_J = R/J$ belong to the same component in $\mathrm{GradAlg}$. (This is because the irreducible variety of fixed degree polynomials maps to an irreducible subset of $\mathrm{GradAlg}$, contained in the same component.) Thus if the dimension of the tangent space of the two modules differs, the one with the highest dimension necessarily has to be obstructed (meaning that it corresponds to a singular point). Computing with Singular [1], we find that $\mathrm{ext}^1_{R,0}(M_I, M_I) = 24$, $\mathrm{ext}^1_{R,0}(M_J, M_J) = 22$. We then know that the first is an example of an obstructed module.

Notice that a computer program (a library in Singular [1]) can be made for these computations. This will be clear in this example. However, for large tangent space dimensions, it seems that the common computers of today are too small.

In this section, we will cut out the tangent space by a hyperplane where the variety in question is obstructed. This will give readable information about the relations in the point corresponding to the variety, and the example will be possible to read.

We put

$$s_1 = x_1^2 - x_0x_2, \quad s_2 = x_0x_1 - x_2^2, \quad s_3 = x_0^2 - x_1x_2,$$
$$s_4 = x_2^4 - x_1x_3^3, \quad s_5 = x_1x_2^3 - x_0x_3^3, \quad s_6 = x_0x_2^3 - x_2x_3^3.$$

Then $I = (s_1, \dots, s_6)$ and $M = M_I = R/I$ are given by the minimal resolution

$$0 \longleftarrow M \longleftarrow R \overset{d_0}{\longleftarrow} R(-2)^3 \oplus R(-4)^4 \overset{d_1}{\longleftarrow} R(-2)^2 \oplus R(-5)^6 \overset{d_2}{\longleftarrow} R(-6)^3 \longleftarrow 0$$

with

$$d_0 = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \end{pmatrix},$$

$$d_1 = \begin{pmatrix} x_0 & -x_2 & x_3^3 & 0 & 0 & x_2^3 & 0 & 0 \\ -x_1 & x_0 & 0 & x_3^3 & 0 & 0 & x_2^3 & 0 \\ x_2 & -x_1 & 0 & 0 & x_3^3 & 0 & 0 & x_2^3 \\ 0 & 0 & x_1 & x_0 & 0 & x_0 & x_2 & 0 \\ 0 & 0 & -x_2 & 0 & x_0 & -x_1 & 0 & x_2 \\ 0 & 0 & 0 & -x_2 & -x_1 & 0 & -x_1 & -x_0 \end{pmatrix},$$

$$d_2 = \begin{pmatrix} x_3^3 & x_2^3 & 0 \\ 0 & -x_3^3 & x_2^3 \\ -x_0 & -x_2 & 0 \\ x_1 & x_0 & 0 \\ -x_2 & -x_1 & 0 \\ 0 & -x_0 & x_2 \\ 0 & x_1 & -x_0 \\ 0 & -x_2 & x_1 \end{pmatrix}.$$

To compute a basis for $\operatorname{Ext}^1_{R,0}(M,M)$, we apply the functor $\operatorname{Hom}_{R,0}(-,M)$, resulting in the sequence

$$M \xrightarrow{d_0^T} M(2)^3 \oplus M(4)^4 \xrightarrow{d_1^T} M(2)^2 \oplus M(5)^6 \xrightarrow{d_2^T} M(6)^3 \longrightarrow 0.$$

We then notice that $d_0^T = 0$ and so $\operatorname{Ext}^1_{R,0}(M,M) = (\ker d_1^T)_0$.

Programming in Singular's work [1], a basis for $\operatorname{Ext}^1_{R,0}(M,M)$ is given by the columns in the following 6×24-matrix:

$$\left(\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_0x_3^3 & x_0x_2x_3^2 & x_0x_2^2x_3 & x_0x_2^3 & x_0x_1x_3^2 & x_0x_1x_2x_3 & x_0x_1x_2^2 & x_0x_1^2x_3 \\
x_2x_3^3 & x_2^2x_3^2 & x_2^3x_3 & x_2^4 & x_1x_2x_3^2 & x_1x_2^2x_3 & x_1x_2^3 & x_1^2x_2x_3 \\
x_1x_3^3 & x_1x_2x_3^2 & x_1x_2^2x_3 & x_1x_2^3 & x_1^2x_3^2 & x_1^2x_2x_3 & x_1^2x_2^2 & x_1^3x_3
\end{matrix}\right.$$

$$\begin{matrix}
0 & -x_0x_1 & -x_1^2 & -x_1x_2 & -x_1x_3 & x_0x_2 & x_1x_2 & x_2^2 \\
0 & -x_0^2 & -x_0x_1 & -x_0x_2 & -x_0x_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -x_0^2 & -x_0x_1 & -x_0x_2 \\
x_0^2x_3^2 & x_0x_3^3 & x_1x_3^3 & x_2x_3^3 & x_3^4 & 0 & 0 & 0 \\
x_0x_2x_3^2 & 0 & 0 & 0 & 0 & x_0x_3^3 & x_1x_3^3 & x_2x_3^3 \\
x_0x_1x_3^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{matrix}$$

$$\left.\begin{matrix}
x_2x_3 & 0 & -x_0x_3 & x_1x_3 & 0 & -x_2x_3 & x_0x_3 & x_1x_3 \\
0 & x_2x_3 & -x_2x_3 & 0 & x_1x_3 & -x_1x_3 & 0 & x_0x_3 \\
-x_0x_3 & x_1x_3 & 0 & -x_2x_3 & x_0x_3 & 0 & -x_1x_3 & 0 \\
0 & 0 & x_2^3x_3 & 0 & 0 & x_1x_2^2x_3 & 0 & 0 \\
x_3^4 & 0 & 0 & x_2^3x_3 & 0 & 0 & x_1x_2^2x_3 & x_2^3x_3 \\
0 & x_3^4 & 0 & 0 & x_2^3x_3 & 0 & 0 & x_1x_2^2x_3
\end{matrix}\right).$$

Following the algorithm and notation given in Section 3.1, we compute cup-products. Of the 300 computed, 79 are identically zero in the meaning that

$$0 \equiv \alpha_{i_1} \circ \alpha_{i'_2} + \alpha_{i'_1} \circ \alpha_{i_2} : R(-3)^2 \oplus R(-5)^6 \longrightarrow R.$$

Of the remaining 221, 205 are zero in cohomology, giving in total 16 nonzero cup-products. With respect to a basis $\{\tilde{y}_i\}_{i=1}^{33}$ for $\operatorname{Ext}^2_{R,0}(M,M)$, these products can be expressed by

$$\begin{aligned}
&v_{13}v_{23} = y_1, && v_{13}v_{24} = y_2, && v_{17}v_{22} = -y_1, && v_{17}v_{24} = y_3, && v_{18}v_{22} = -y_2, && v_{18}v_{23} = -y_3, \\
&v_{19}v_{22} = -y_2, && v_{19}v_{23} = -y_3, && v_{20}v_{23} = y_1, && v_{20}v_{24} = y_2, && v_{21}v_{22} = -y_1, && v_{21}v_{24} = y_3, \\
&v_{22}^2 = y_1, && v_{22}v_{23} = y_2, && v_{22}v_{24} = -y_3, && v_{23}^2 = y_3.
\end{aligned}$$

Letting

$$\begin{aligned}
f_1 &= v_{13}v_{23} - v_{17}v_{22} + v_{20}v_{23} - v_{21}v_{22} + v_{22}^2, \\
f_2 &= v_{13}v_{24} - v_{18}v_{22} - v_{19}v_{22} + v_{20}v_{24} + v_{22}v_{23}, \\
f_3 &= v_{17}v_{24} - v_{18}v_{23} - v_{19}v_{23} + v_{21}v_{24} - v_{22}v_{24} + v_{23}^2,
\end{aligned}$$

we may conclude from Proposition 16 the following.

Proposition 18. *The determinantal scheme given by the minors of the matrix G_I has first-order relations given by its second-order local formal moduli*

$$\hat{H}/\underline{m}^3 \cong k\big[\big[v_1,\ldots,v_{24}\big]\big]/\big(\big(f_1,f_2,f_3\big)+\underline{m}^3\big).$$

From [3] it follows that the obstruction space for $M = R/I$ is $H^2(M,M,R)$, where M is considered as a graded R-algebra. This k-vector space has dimension 3, and so we may conclude the following.

Corollary 19. *The determinantal scheme given by the minors of the matrix G_I is maximally obstructed.*

We are now going to put most (21) of the variables above to zero. That is, we choose the most interesting of the 24 variables above; $t_1 = v_{22}$, $t_2 = v_{23}$, $t_3 = v_{24}$, all others are put to zero. We follow the algorithm given in Section 3.1 and we work in the Yoneda complex $\mathrm{Hom}^\bullet(L_\bullet, L_\bullet)$, where $L_\bullet$ denotes the R-free resolution of M given above.

Notice that for $\alpha = \{\alpha_i\} \in \mathrm{Hom}^1(L_\bullet, L_\bullet)$, it is always sufficient to have the two leading morphisms $\alpha_1 : L_1 \to L_0$, $\alpha_2 : L_2 \to L_1$. Also, it is known that finding these by the methods below, they can always be extended to the full complex; see [7].

We find

$$\alpha_{e_1,1} = \begin{pmatrix} -x_2x_3 & -x_1x_3 & 0 & x_1x_2^2x_3 & 0 & 0 \end{pmatrix},$$

$$\alpha_{e_1,2} = \begin{pmatrix} -x_3 & 0 & -x_2^2x_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_3 & 0 & -x_2^2x_3 & 0 & -x_2^2x_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x_3 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\alpha_{e_2,1} = \begin{pmatrix} x_0x_3 & 0 & -x_1x_3 & 0 & x_1x_2^2x_3 & 0 \end{pmatrix},$$

$$\alpha_{e_2,2} = \begin{pmatrix} 0 & -x_3 & 0 & 0 & 0 & x_2^2x_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x_2^2x_3 & 0 & 0 & 0 \\ -x_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x_3 & 0 & 0 & 0 \\ 0 & 0 & x_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\alpha_{e_3,1} = \begin{pmatrix} x_1x_3 & x_0x_3 & 0 & 0 & x_2^3x_3 & x_1x_2^2x_3 \end{pmatrix},$$

$$\alpha_{e_3,2} = \begin{pmatrix} 0 & 0 & 0 & 0 & x_2^2x_3 & 0 & x_2^2x_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_2^2x_3 \\ 0 & -x_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This given, it is an easy match to compute the cup products (or the first order generalized Massey products). These are

$$\begin{aligned}
&\big\langle \underline{v}^*; (2,0,0)\big\rangle = \alpha_{e_1,1}\cdot\alpha_{e_1,2} = x_3^2\big(x_2, -x_1, x_2^3, 0, 0, x_1x_2^2, 0, 0\big),\\
&\big\langle \underline{v}^*; (1,1,0)\big\rangle = \alpha_{e_1,1}\cdot\alpha_{e_2,2} + \alpha_{e_2,1}\cdot\alpha_{e_1,2} = x_3^2\big(-x_0, x_2, -x_0x_2^2, 0, 0, -x_2^3, 0, 0\big),\\
&\big\langle \underline{v}^*; (1,0,1)\big\rangle = \alpha_{e_1,1}\cdot\alpha_{e_3,2} + \alpha_{e_3,1}\cdot\alpha_{e_1,2} = x_3^2\big(-x_1, x_0, -x_1x_2^2, -x_0x_2^2, -x_2^3, -x_0x_2^2, -x_2^3, -x_1x_2^2\big),\\
&\big\langle \underline{v}^*; (0,2,0)\big\rangle = \alpha_{e_2,1}\cdot\alpha_{e_2,2} = x_3^2\big(x_1, -x_0, x_1x_2^2, 0, 0, x_0x_2^2, 0, 0\big),\\
&\big\langle \underline{v}^*; (0,1,1)\big\rangle = \alpha_{e_2,1}\cdot\alpha_{e_3,2} + \alpha_{e_3,1}\cdot\alpha_{e_2,2} = x_3^2\big(0, 0, x_2^3, x_1x_2^2, 0, x_1x_2^2, x_0x_2^2, 0\big),\\
&\big\langle \underline{v}^*; (0,0,2)\big\rangle = \alpha_{e_3,1}\cdot\alpha_{e_3,2} = x_3^2\big(0, 0, 0, x_2^3, x_1x_2^2, 0, x_1x_2^2, x_0x_2^2\big).
\end{aligned}$$

As classes in cohomology, we find (as we already knew)

$$\begin{aligned}
&\big\langle \underline{v}^*; (2,0,0)\big\rangle = y_1, \quad \big\langle \underline{v}^*; (1,1,0)\big\rangle = y_2, \quad \big\langle \underline{v}^*; (1,0,1)\big\rangle = -y_3,\\
&\big\langle \underline{v}^*; (0,2,0)\big\rangle = y_3, \quad \big\langle \underline{v}^*; (0,1,1)\big\rangle = 0, \quad \big\langle \underline{v}^*; (0,0,2)\big\rangle = 0.
\end{aligned}$$

We put

$$f_1^2 = t_1^2, \quad f_2^2 = t_1 t_2, \quad f_3^2 = t_2^2 - t_1 t_3,$$

and the *restricted* local formal moduli to the second order is

$$\hat{H}'_M/\underline{m}^3 = k\big[[t_1, t_2, t_3]\big]/\big(f_1^2, f_2^2, f_3^3\big).$$

We follow the algorithm given in Section 3.1 further. We choose a basis B_2 for

$$\underline{m}^2/\big(\underline{m}^3 + \big(f_1^2, f_2^2, f_3^2\big)\big),$$

for example,

$$B_2 = \big\{(0,2,0), (0,1,1), (0,0,2)\big\}$$

and choose a third-order defining system. Notice that by $\langle \underline{v}^*; \underline{n}\rangle$ we mean a *representative* of the cohomology class

$$b_{(0,2,0)} = \big\langle \underline{v}^*; (0,2,0)\big\rangle + \big\langle \underline{v}^*; (1,0,1)\big\rangle = x_3^2\big(0,0,0,-x_0x_2^2, -x_2^3, 0, -x_2^3, -x_1x_2^2\big).$$

Similarly,

$$\begin{aligned} b_{(0,1,1)} &= x_3^2\big(0,0,x_2^3, x_1x_2^3, x_1x_2^2, 0, x_1x_2^2, x_0x_2^2, 0\big),\\ b_{(0,0,2)} &= x_3^2\big(0,0,0,x_2^3, x_1x_2^2, 0, x_1x_2^2, x_0x_2^2\big). \end{aligned}$$

Writing up why these are cocycles, we find what $\alpha_{\underline{n}}$ to choose for $d(\alpha_{\underline{n}}) = -b_{\underline{n}}$:

$$\begin{aligned} \alpha_{(0,2,0),1} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -x_0x_2x_3^2 \end{pmatrix},\\ \alpha_{(0,2,0),2} &= \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&-x_2x_3^2&0&-x_2x_3^2&0\\ 0&0&0&0&0&0&0&-x_2x_3^2\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix},\\ \alpha_{(0,1,1),1} &= \begin{pmatrix} 0 & 0 & 0 & -x_0x_2x_3^2 & 0 & 0 \end{pmatrix},\\ \alpha_{(0,1,1),2} &= \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&x_2x_3^2&0&0&0&0&0\\ 0&0&0&x_2x_3^2&0&x_2x_3^2&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix},\\ \alpha_{(0,0,2),1} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & x_2^2x_3^2 \end{pmatrix},\\ \alpha_{(0,0,2),2} &\equiv 0. \end{aligned}$$

Again, following the algorithm given in Section 3.1, we choose a monomial basis $B'_3 = \{(0,2,1), (0,1,2), (0,0,3)\}$ for $\underline{m}^3/\underline{m}^4 + \underline{m}^3 \cap \underline{m}(f_1^2, f_2^2, f_3^2)$, and can compute the Massey products (notice again that the next to last expression is the representative in the Yoneda complex of its cohomology class):

$$\begin{aligned} \big\langle \underline{v}^*; (0,2,1)\big\rangle &= \alpha_{(0,2,0)} \cup \alpha_{e_3} + \alpha_{(0,1,1)} \cup \alpha_{e_2} + \alpha_{(0,0,2)} \cup \alpha_{e_1}\\ &= \big(0,0,-x_2^2x_3^3, -x_1x_2x_3^3, 0, -x_1x_2x_3^3, -x_0x_2x_3^3, 0\big) = 0,\\ \big\langle \underline{v}^*; (0,1,2)\big\rangle &= \alpha_{(0,1,1)} \cup \alpha_{e_3} + \alpha_{(0,0,2)} \cup \alpha_{e_2} \equiv 0,\\ \big\langle \underline{v}^*; (0,0,3)\big\rangle &= \alpha_{(0,0,2)} \cup \alpha_{e_3} \equiv 0. \end{aligned}$$

We now put

$$f_1^3 = f_1^2, \quad f_2^3 = f_2^2, \quad f_3^3 = f_3^2$$

and so

$$\hat{H}'/\underline{m}^4 \cong k\big[[t_1, t_2, t_3]\big]/\big(f_1^3, f_2^3, f_3^3\big) + \underline{m}^4.$$

We put $B_3 = B_3' = \{(0,2,1),(0,1,2),(0,0,3)\}$, and the next order defining system is easy to find, only one of the representations of the elements $b_{\underline{n}}$ is different from zero:

$$b_{(0,2,1)} = x_0x_3^3\big(0,0,-x_1,-x_0,0,-x_0,-x_2,0\big) + \big(0,0,x_3^3s_2,x_3^3s_3,0,x_3^3s_3,0,0\big).$$

We choose

$$\alpha_{(0,2,1),1} = \begin{pmatrix}0 & 0 & 0 & x_0x_3^3 & 0 & 0\end{pmatrix}, \quad \alpha_{(0,2,1),2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -x_3^3 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -x_3^3 & 0 & -x_3^3 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The rest of the elements in the defining system are chosen identically zero, and we put $B_4' = \{(0,2,2),(0,1,3),(0,0,4)\}$ and compute the fourth-order Massey products:

$$\begin{aligned}
\big\langle \underline{v}^*;(0,2,2)\big\rangle &= \alpha_{(0,1,2)} \cup \alpha_{e_2} + \alpha_{(0,2,1)} \cup \alpha_{e_3} + \alpha_{(0,2,0)} \cup \alpha_{(0,0,2)} + \alpha_{(0,1,1)} \cup \alpha_{(0,1,1)} + \alpha_{(0,0,3)} \cup \alpha_{e_1} \equiv 0,\\
\big\langle \underline{v}^*;(0,1,3)\big\rangle &= \alpha_{(0,1,2)} \cup \alpha_{e_3} + \alpha_{(0,0,3)} \cup \alpha_{e_2} + \alpha_{(0,1,1)} \cup \alpha_{(0,0,2)} \equiv 0,\\
\big\langle \underline{v}^*;(0,0,4)\big\rangle &= \alpha_{(0,0,3)} \cup \alpha_{e_3} + \alpha_{(0,0,2)} \cup \alpha_{0,0,2} \equiv 0.
\end{aligned}$$

Now, put $f_i^4 = f_i^3$, $i = 1,2,3$. Because these then are homogenous of degree two, the next order defining systems involves only fourth-order Massey products, and these can all be chosen identically zero. Then, the fifth-order Massey products involve $\alpha_{\underline{m}_1} \cup \alpha_{\underline{m}_2}$ with at least one of $|\underline{m}_i| = 3$. We see that $\alpha_{(0,2,1)} \cup \alpha_{\underline{m}} \equiv 0$ for all $\underline{m}$ with $|\underline{m}| = 2$, and so all fifth-order Massey products are zero. Noting also that $\alpha_{(0,2,1)} \cup \alpha_{(0,2,1)} \equiv 0$, we are ready to conclude the following proposition.

Proposition 20. *Let $f_1 = t_1^2$, $f_2 = t_1t_2$, $f_3 = t_2^2 - t_1t_3$. Then, there exists an open subset of the component of* GradAlg*, the moduli scheme of graded R-algebras, containing the determinantal scheme corresponding to the matrix G_I such that its intersection with the hyperplane $t_4 = \cdots = t_{24} = 0$ is isomorphic to*

$$k\big[t_1, t_2, t_3\big]/\big(f_1, f_2, f_3\big)$$

with the versal family

$$M_{(t_1,t_2,t_3)} = k\big[x_0, x_1, x_2, x_3\big]/I\big((t_1, t_2, t_3)\big)$$

for $\underline{t} \in Z(f_1, f_2, f_3)$ with

$$\begin{aligned}
I\big(t_1, t_2, t_3\big) = \big(&s_1 - x_2x_3t_1 + x_0x_3t_2 + x_1x_3t_3, s_2 - x_1x_3t_1 + x_0x_3t_3,\\
&s_3 - x_1x_3t_2, s_4 + x_1x_2^2x_3t_1 - x_0x_2x_3^2t_2t_3 + x_0x_3^3t_2^2t_3,\\
&s_5 + x_1x_2^2x_3t_2 + x_2^3x_3t_3, s_6 + x_1x_2^2x_3t_3 - x_0x_2x_3^2t_2^2 + x_2^2x_3^2t_3^2\big).
\end{aligned}$$

Remark 21. When the local formal moduli with its formal family is algebraizable in this way, we get an open subset of the moduli (at least étale). Thus, we get a lot more than just the local formal information. The conditions for when $\hat{H}_M$ is algebraizable is an interesting question.

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