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Generalizations in Improper Integral and Hard Problems Solved using it

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Abstract

This paper's results came from a combination of complex and real analysis methods in solving improper integrals. This Result is easier method instead of the function convergent (Residue theorem) and applying Cauchy integration formula, in addition some of these results and examples are shown in the paper are very hard to be evaluated using the contour integration (Residue theorem). I'll discuss 4 results (formulas) and the modified version of the result, all of the results came from the master Result for Mohammad Hussein Abughuwaleh. For some of these results I used conditions for a, b, β, γ to let the integral convergent on it's boundaries.

Results

Result 1

Formula 1: If f (a+z) can be expanded in a series on the form:

$$f(a+z)=\sum_{k=0}^{\infty} M_K e^{-kz}$$

whether z be real or imaginary then for b ≥ 0, γ, β>0 is odd,

$$I=\int_0^{\infty}\frac{(f(a+ibx)+f(a-ibx))}{(\beta^2+x^2)(\gamma^2+x^2)}dx$$

$$=\frac{\pi}{\beta\gamma(\beta^2-\gamma^2)}\left(\beta f(a+b\gamma)-\gamma f(a-b\beta)\right)$$

Proof: If we assume that $\sum_{k=0}^{\infty} M_k$ absolutely converge then:

$$I=\int_0^{\infty}\frac{1}{(\beta^2+x^2)(\gamma^2+x^2)}\sum_{k=0}^{\infty}M_k(e^{-ikbx}++e^{ikbx})dx$$

$$I=2\int_0^{\infty}\frac{1}{(\beta^2+x^2)(\gamma^2+x^2)}\sum_{k=0}^{\infty}M_k\cos(kbx)dx$$

$$=2\sum_{k=0}^{\infty}M_k\int_0^{\infty}\frac{\cos(kbx)}{(\beta^2+x^2)(\gamma^2+x^2)}dx$$

$$=\frac{\pi}{\beta\gamma(\beta^2-\gamma^2)}\left\{\sum_{k=0}^{\infty}\beta M_k e^{(-kb\gamma)}-\sum_{k=0}^{\infty}\gamma M_k e^{(-kb\beta)}\right\}$$

$$=\frac{\pi}{\beta\gamma(\beta^2-\gamma^2)}\left(\beta f(a+b\gamma)-\gamma f(a+b\beta)\right)$$

Result 2

Formula 2: If f (a+z) can be expanded in a series on the form:

$$f(a+z)=\sum_{k=0}^{\infty} M_K e^{-kz}$$

whether z be real or imaginary then for b ≥ 0, v is odd,

$$|\arg\beta|<\frac{\pi}{2n}, 0<r<2n+1$$

$$I=\int_0^{\infty}\frac{x^{v-1}(f(a+ibx)+f(a-ibx))}{x^{2r}+\beta^{2r}}dx$$

$$f(a+b)\left[\frac{\pi\beta^{v-2r}}{r}\sum_{j=1}^{n}e^{-kb\sin\left(\frac{\pi(2j-1)}{2n}\right)}\sin\left\{\frac{v\pi(2j-1)}{2n}+kb\beta\cos\left(\frac{\pi(2j-1)}{2n}\right)\right\}\right]$$

Proof: If we assume that $\sum_{k=0}^{\infty} M_K$ absolutely converge then:

$$I=\int_0^{\infty}\frac{x^{v-1}}{x^{2r}+\beta^{2r}}\sum_{k=0}^{\infty}M_K\left(e^{-ikbx}++e^{+ikbx}\right)dx$$

$$I=2\int_0^{\infty}\frac{x^{v-1}}{x^{2r}+\beta^{2r}}\sum_{k=0}^{\infty}M_K\cos(kbx)dx$$

$$=2\sum_{k=0}^{\infty}M_K\int_0^{\infty}\frac{x^{v-1}\cos(kbx)}{x^{2r}+\beta^{2r}}dx$$

$$=\frac{\pi\beta^{v-2r}}{r}\sum_{j=1}^{r}e^{-kb\sin\left(\frac{\pi(2j-1)}{2n}\right)}\sin\left\{\frac{v\pi(2j-1)}{2r}+kb\beta\cos\left(\frac{\pi(2j-1)}{2r}\right)\right\}\sum_{k=0}^{\infty}M_K e^{-kbx}$$

$$=f(a+b)\left[\frac{\pi\beta^{v-2r}}{r}\sum_{j=1}^{r}e^{-kb\sin\left(\frac{\pi(2j-1)}{2n}\right)}\sin\left\{\frac{v\pi(2j-1)}{2r}+kb\beta\cos\left(\frac{\pi(2j-1)}{2r}\right)\right\}\right]$$

A modified version of the result 2 for Mohammad Abughuwaleh:

If f (a+z) can be expanded in a series on the form:

$$f(a+z)=\sum_{n=0}^{\infty}C_n e^{-nz}$$

whether z be real or imaginary then for b>0:

$$I=\int_0^{\infty}\frac{(f(\mathrm{a+ibx})+f(\mathrm{a-ibx}))}{1+x^2}dx=\pi f(a+b)$$

Proof: If we assume that $\sum_{n=0}^{\infty}C_n$ absolutely converge then:

$$I=\int_0^{\infty}\frac{1}{1+x^2}\sum_{n=0}^{\infty}C_n\left(e^{-inbx}++e^{inbx}\right)dx$$

$$I=2\int_0^{\infty}\frac{1}{1+x^2}\sum_{n=0}^{\infty}C_n\cos(nbx)dx$$

$$=2\sum_{n=0}^{\infty}C_n\int_0^{\infty}\frac{\cos(\mathrm{nbx})}{1+x^2}dx$$

$$=\pi\sum_{n=0}^{\infty}C_n e^{-nbx}=\pi f(a+b)$$

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Result 3

Formula 3: If f(a+z) can be expanded in a series on the form:

$$f(a+z)=\sum_{k=0}^{\infty} M_K e^{-nz}$$

whether z be real or imaginary then for b>0 , n ≥ 0

$$I=\int_0^{\infty}\frac{(f(a+ibx)+f(a-ibx))}{(x^2+1)(x^2+9)(x^2+25)...(x^2(2n+1)^2)}dx$$

$$I=f(a+b)\left\{\frac{(-1)^n\pi}{2^{2n}(2n+1)!}\sum_{k=0}^{n}(-1)^k\binom{2n+1}{k}e^{(2k-2n-1)a}\right\}$$

And for b=0, n ≥ 0

$$I=f(a+b)\left\{\frac{\pi}{(n!)^2(2n+1)2^{2n}}\right\}$$

Proof: If we assume that $\sum_{k=0}^{\infty} M_K$ absolutely converge then:

$$I=\int_0^{\infty}\frac{1}{(x^2+1)(x^2+9)(x^2+25)...(x^2(2n+1)^2)}\sum_{k=0}^{\infty}M_K(e^{-ikbx}++e^{-ikbx})\,dx$$

$$=2\sum_{n=0}^{\infty}M_K\int_0^{\infty}\frac{\cos(kbx)}{(x^2+1)(x^2+9)(x^2+25)...(x^2(2n+1)^2)}dx$$

$$I=f(a+b)\left\{\frac{(-1)^n\pi}{2^{2n}(2n+1)!}\sum_{k=0}^{n}(-1)^k\binom{2n+1}{k}e^{(2k-2n-1)a}\right\}$$

And for b=0, n≥0

$$I=f(a+b)\left\{\frac{\pi}{(n!)^2(2n+1)2^{2n}}\right\}$$

Result 4

Formula 4: Let f (z) has a maclurin series expansion with real coefficients that converges absolutely on the unit circle on the complex plane.

$$I=\Re\int_0^{\infty}\frac{e^{iax}f\left(e^{i\theta x}\right)}{x^2+\beta^2}dx \quad \text{where } \theta,\ \alpha\ge 0,\ \beta>0$$

$$I=\Re\int_0^{\infty}\frac{e^{iax}}{x^2+\beta^2}\sum_{n=0}^{\infty}\frac{f^n(0)e^{i\theta nx}}{n!}dx$$

$$I=\Re\left(\sum_{n=0}^{\infty}\frac{f^n(0)}{n!}\int_0^{\infty}\frac{e^{ix(\alpha+\theta n)}}{x^2+\beta^2}dx\right)$$

$$I=\left(\sum_{n=0}^{\infty}\frac{f^n(0)}{n!}\int_0^{\infty}\frac{\cos(x(\alpha+\theta n))}{x^2+\beta^2}dx\right)$$

$$I=\frac{\pi}{2\beta}e^{-\beta(\theta n+\alpha)}\sum_{n=0}^{\infty}\frac{f^n(0)}{n!}$$

$$I=\frac{\pi}{2\beta}e^{-\alpha\beta}\sum_{n=0}^{\infty}\frac{f^n(0)e^{-\beta\theta n}}{n!}$$

$$I=\frac{\pi}{2\beta}e^{-\alpha\beta}f(e^{-\beta\theta})$$

I'll show an example that this can be solved using contour integration but it needs a hard and along with the efforts in addition that the contour to choose is a very difficult here, but if we used the generalization we can find it in a very easy and simple way:

$$\int_0^{\infty}\frac{\ln^2\left|\tan\left(\frac{\theta x}{2}-\frac{\pi}{4}\right)\right|}{1+x^2}dx \quad \text{where } \theta>0$$

Solution:

$$f(z)=(\tan^{-1}z)^2=\frac{-1}{4}\ln^2\left(\frac{1-iz}{1+iz}\right)$$

$$f(e^{i\theta x})=\frac{-1}{4}\ln^2\left(\frac{1-e^{-i\theta x}}{1+e^{+i\theta x}}\right)$$

$$f(e^{i\theta x})=\frac{-1}{4}\ln^2\left(\frac{1+\sin(\theta x)-i\cos(\theta x)}{1-\sin(\theta x)+i\cos(\theta x)}\right)$$

$$=\frac{-1}{4}\left(\ln\left(1+\sin(\theta x)-i\cos(\theta x)\right)-\ln\left(1-\sin(\theta x)+i\cos(\theta x)\right)\right)^2$$

$$=\frac{-1}{4}\left(\frac{\pm i\pi}{2}-\ln\left|\tan\left(\frac{\theta x}{2}-\frac{\pi}{4}\right)\right|\right)^2$$

By taking the real part of f($e^{i\theta x}$) we will get:

$$\Re\left(f\left(e^{i\theta x}\right)\right)=\frac{\pi^2}{16}-\frac{1}{4}\ln^2\left|\tan\left(\frac{\theta x}{2}-\frac{\pi}{4}\right)\right|$$

And note that: f(z) has a maclurin series expansion with real coefficients that converges absolutely on the unit circle on the complex plane so f(z) can be written as:

$$f(z)=\sum_{n=1}^{\infty}a_n z^n$$

So by applying on the result 4 and let α=0, β=1 we will get

$$I=\Re\int_0^{\infty}\frac{f\left(e^{i\theta x}\right)}{1+x^2}dx=\int_0^{\infty}\frac{\frac{\pi^2}{16}-\frac{1}{4}\ln^2\left|\tan\left(\frac{\theta x}{2}-\frac{\pi}{4}\right)\right|}{1+x^2}dx$$

$$I=\frac{\pi}{2}f(e^{-\theta})=\frac{\pi}{2}\left(\tan^{-1}e^{-\theta}\right)^2$$

but we must reach to our desired integral

$$I=\int_0^{\infty}\frac{\frac{\pi^2}{16}}{1+x^2}-\frac{1}{4}\int_0^{\infty}\frac{\ln^2\left|\tan\left(\frac{\theta x}{2}-\frac{\pi}{4}\right)\right|}{1+x^2}dx=\frac{\pi}{n}\left(\tan^{-1}e^{-\theta}\right)^2$$

$$\frac{\pi}{2}\left(\tan^{-1}e^{-\theta}\right)^2=\frac{\pi^3}{32}-\frac{1}{4}\int_0^{\infty}\frac{\ln^2\left|\tan\left(\frac{\theta x}{2}-\frac{\pi}{4}\right)\right|}{1+x^2}dx$$

We will get:

$$\int_0^{\infty}\frac{\ln^2\left|\tan\left(\frac{\theta x}{2}-\frac{\pi}{4}\right)\right|}{1+x^2}dx=\frac{\pi^3}{8}-2\pi\left(\tan^{-1}e^{-\theta}\right)^2$$

We can use the formula to generalize and find unlimited of hard integrals that can be solved using this very easy way another example and application

$$\int_0^{\infty} \frac{e^{\cos(\theta x)}\left(\cos\left(\alpha x+\sin\left(\theta x\right)\right)\right)}{x^2+\beta^2}dx$$

And this is a straight forward example using formula 4

$$\Re\int_0^{\infty} \frac{e^{iax} f\left(e^{i\theta x}\right)}{x^2+\beta^2}dx = \frac{\pi}{2\beta}e^{-\alpha\beta} f\left(e^{-\beta\theta}\right)$$

Let $f(z)=e^z$

$$\Re\left(e^{iax} f\left(e^{i\theta x}\right)\right) = e^{\cos(\theta x)}\left(\cos\left(\alpha x+\sin\left(\theta x\right)\right)\right)$$

$$I = \frac{\pi}{2\beta}e^{-\alpha\beta} f\left(e^{-\beta\theta}\right) = \frac{\pi}{2\beta}e^{-\alpha\beta}e^{e^{-\beta\theta}}$$

Another example of using the modified version of the result 2:

$$\int_0^{\infty} \frac{e^{a\tan^{-1}(\theta x)}+e^{-a\tan^{-1}(\theta x)}}{x^2+1}\cos\left(\frac{a}{2}\ln\left(1+\theta^2x^2\right)\right)dx$$

Solution: Let $f(z)=\cos\left(a\ln(1+z)\right)$

$$\int_0^{\infty} \frac{\cos\left(a\ln\left(1+i\theta x\right)\right)+\cos\left(a\ln\left(1-i\theta x\right)\right)}{1+x^2}dx$$

$$=\int_0^{\infty} \frac{\cos\left(\frac{a}{2}\ln\left(1+\theta^2x^2\right)+i\,a\tan^{-1}\theta x\right)+\cos\left(\frac{a}{2}\ln\left(1+\theta^2x^2\right)-i\,a\tan^{-1}\theta x\right)}{1+x^2}dx$$

$$=\int_0^{\infty} \frac{\cosh\left(a\tan^{-1}(\theta x)\right)}{1+x^2}\cos\left(\frac{a}{2}\ln\left(1+\theta^2x^2\right)\right)dx = \pi\cos\left(a\ln(1+\theta)\right)$$

And a lot of hard questions can be generated using these formulas and for example of one of the hard questions that can be generated:

$$\int_0^{\infty} \frac{\ln^2\left(\frac{1-\sin^2(\theta x)}{1+\sin^2(\theta x)}\right)}{\left(x^2+1\right)\left(x^2+9\right)\left(x^2+25\right)\ldots\left(x^2+(2n+1)^2\right)}dx$$

Which can be solved using Result 3.

Conclusion

These Generalizations are very easy methods to solve hard Improper integrals instead of using “contour integration” or other methods that sometimes you cannot find solution using the other methods and we can use these generalizations to generate thousands of hard questions that all CAS “computer algebra systems” will not found solutions for these difficult questions, so these results have a lot of benefits if we used it in CAS which will improve the results in computations.

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