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Fuzzy n-Lie Algebras

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Abstract

Properties of fuzzy subalgebras and ideals of n-ary Lie algebras are described. Methods of construction fuzzy ideals are presented. Connections with various fuzzy quotient *n*-Lie algebras are proved.

Keywords: Fuzzy set; n-ary Lie algebra; Subalgebra; Ideal; Fuzzy ideal

Introduction

In 1985 Filippov [1] proposed a generalization of the concept of a Lie algebra by replacing the binary operation by n-ary one. He defined an n-ary Lie algebra structure on a vector space L as an operation which associates with each n–tuple $(x_1,\ldots,x_n)$ of elements in L another element $[x_1,\ldots,x_n]$ which is n-linear, skew-symmetric:

$$[x_{\sigma(1)},\ldots,x_{\sigma(n)}]=sign(\sigma)[x_1,\ldots,x_n]$$

and satisfies the generalized Jacobi identity (called also the Filippov identity):

$$[[x_1,\ldots,x_n],y_2,\ldots,y_n]=\sum_{i=1}^{n}[x_1,\ldots,x_{i-1},[x_i,y_2,\ldots,y_n],x_{i+1},\ldots,x_n],$$

where $\sigma\in S_n$.

Now, such structures are also called n-Lie algebras or Filippov algebras. For n=2 we obtain a classical Lie algebras.

Note that such an n-ary operation, realized on the smooth function algebra of a manifold and additionally assumed to be an n-derivation, is an n-Poisson structure. This general concept, however, was not introduced neither by Filippov, nor by other mathematicians that time. It was done much later in 1994 by Takhtajan [2] in order to formalize mathematically the n-ary generalization of Hamiltonian mechanics proposed by Nambu [3]. Apparently Nambu was motivated by some problems of quark dynamics and the n-bracket operation he considered was:

$$[f_1,\ldots,f_n]:=det\begin{pmatrix}\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n}\\ \vdots & \ddots & \vdots\\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n}\end{pmatrix}$$

where $L=R[x_1,\ldots,x_n]$ is the vector space of polynomials in n-variables.

Nambu does not mentions that the n-bracket operation satisfies the generalized Jacobi identity but Filippov reports this operation in his paper [1] among other examples of n-Lie algebras. The formal proof is given in [4].

Ternary Lie algebras were studied [5,6]. For other generalizations and applications see ref. [7].

The study of fuzzy Lie algebras was initiated in refs. [8,9], and continued in various directions by many authors (for example [10-12]). The study of fuzzy n-ary algebras was initiated by Dudek [13]. Davvaz and Dudek described fuzzy n-ary groups as a generalization of Rosenleld's fuzzy groups [14].

In this paper we describe fuzzy n-ary Lie algebras.

Preliminaries

Let X be a non-empty set. A fuzzy subset μ of X is a function μ: $X\to[0,1]$. Let μ and λ be two fuzzy subsets of X, we say that μ is contained in λ, if $\mu(x)\le\lambda(x)$ for all $x\in X$. The set $\overline{\mu_t}=\{x\in X\,|\,\mu(x)\ge t\}$, $t\in[0,1]$ is called a level subset of μ.

Definition 2.1

Let V be a vector space over a field F. A fuzzy subset μ of V is called a fuzzy subspace of V if for all $x,y\in V$ and $\alpha\in F$, the following conditions are satisfied:

- $\mu(x+y)\ge\min\{\mu(x),\mu(y)\}$ for all $x,y\in V$,
- $\mu(\alpha x)\ge\mu(x)$ for all $x\in V$, $\alpha\in F$.

Note that the second condition implies, $\mu(-x)\ge\mu(x)$ for all $x\in V$,

Lemma 2.2

If μ is a fuzzy subspace of a vector space V, then $\mu(x)\le\mu(0)$ for all $x\in V$, and

- $\mu(x)=\mu(-x)$,
- $\mu(x-y)=\mu(0)\Rightarrow\mu(x)=\mu(y)$,
- $\mu(x)<\mu(y)\Rightarrow\mu(x-y)=\mu(x)=\mu(y-x)$

for all $x,y\in V$.

Proof. Directly from the definition we obtain $\mu(x)\le\mu(0)$ and $\mu(x)=\mu(-x)$. Moreover, for all $x,y\in V$ we have

$$\begin{aligned}\min\{\mu(x-y),\mu(y)\}&\ge\min\{\min\{\mu(x),\mu(-y)\},\mu(y)\}=\min\{\mu(x),\mu(y)\}\\&=\min\{\mu((x-y)+y),\mu(y)\}\ge\min\{\min\{\mu(x-y),\mu(y)\},\mu(y)\}\\&=\min\{\mu(x-y),\mu(y)\},\end{aligned}$$

which implies

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$$\min\{\mu(x-y),\mu(y)\}=\min\{\mu(x),\mu(y)\}.$$

Similarly

$$\min\{\mu(x-y),\mu(x)\}=\min\{\mu(x),\mu(y)\}.$$

Hence

$$\min\{\mu(x-y),\mu(y)\}=\min\{\mu(x-y),\mu(x)\}=\min\{\mu(x),\mu(y)\}.$$

This for $\mu(x-y)=\mu(0)$ gives $\mu(x)=\mu(y)$, and $\mu(x-y)=\mu(x)$ for $\mu(x)<\mu(y)$.

Theorem 2.3

For a fuzzy subset μ of a vector space V, the following statements are equivalent.

- μ is a fuzzy subspace of V.
- Each non-empty $\overline{\mu_t}$ is a subspace of V.

This theorem firstly proved in ref. [15] is a consequence of the Transfer Principle for fuzzy sets described in ref. [16].

Let $\{\mu_i\}_{i\in I}$ be a collection of fuzzy subsets of X. Then, we define the fuzzy subsets $_{i\in I}$ and $\bigcup_{i\in I}\mu_i$ by:

$$(\bigcap_{i\in I}\mu_i)(x)=\inf_{i\in I}\{\mu_i(x)\}\ \textit{for all}\ x\in X,$$

$$(\bigcup_{i\in I}\mu_i)(x)=\sup_{i\in I}\{\mu_i(x)\}\ \textit{for all}\ x\in X.$$

Fuzzy Subalgebras and Ideals

Recall that a non-empty subset S of an n-Lie algebra L is its subalgebra if it is a subspace of a vector space L and $[x_1,\dots,x_n]\in S$ for all $x_1,\dots,x_n\in S$.

A subspace S of an i-ideal of L if for all $x_1,\dots, x_{n\in L}$ and $y\in S$ we have $[x_1,\dots,x_{\{i-1\}},y, x_{\{i+1\}}, \dots, x_n]\in L$.

Two n-Lie algebras L_1, L_2 over the same field F are isomorphic if there exists a vector space isomorphism $\varphi:L_1\to L_2$ such that for all $\varphi([x_1,\dots,x_n])=[\varphi(x_1),\dots,\varphi(x_n)]$ for all $x_1,\dots,x_n\in L$.

Let L be an n-Lie algebra. Fixing in $[x_1, x_2,\dots,x_n]$ elements $x_2,\dots, x_{\{n-1\}}$ we obtain a new binary operation $\langle x,y\rangle=[x,x_2,\dots,x_{n-1},y]$ with the property $\langle x_k,y\rangle=\langle y,x_k\rangle=0$ for all $k=2,\dots,n-1$ and all $y\in L$. It is easily to see that L with respect to this new operation is an classical Lie algebra. It is called a binary retract. Fixing various $x_2,\dots,x_{n-1}$ we obtain various (generally non-isomorphic) retracts. Obviously, any subalgebra (ideal) of an n-Lie algebra is a subalgebra (ideal) of each binary retract of L. The converse is not true. Hence results obtained for n-Lie algebras are essential generalizations of results proved for Lie algebras.

Basing on the idea of fuzzyfications of algebras with one n-ary operation proposed in ref. [13] we present a fuzzyfication of n-Lie algebras.

Definition 3.1

Let L be an n-Lie algebra. A fuzzy subalgebra of L is a fuzzy subspace μ such that

$$\mu([x_1,\dots,x_n])\ge\min\{\mu(x_1),\dots,\mu(x_n)\}\quad \textit{for all}\ x_1,\dots,x_n\in L.$$

Definition 3.2

Let L be an n-Lie algebra. A fuzzy ideal of L is a fuzzy subspace μ such that

$$\mu([x_1,\dots,x_n])\ge\mu(x_i)\quad \text{for all}\ x_1,\dots,x_n\in L\ \text{and}\ 1\le i\le n.$$

The following facts are obvious. Their proofs are very similar to the proofs of analogous results for fuzzy n-ary systems [13] and fuzzy Lie algebras [9].

Proposition 3.3:

A fuzzy subspace μ of an n-Lie algebra L is its fuzzy ideal if and only if

$$\mu([x_1,\dots,x_n])\ge\max\{\mu(x_1),\dots,\mu(x_n)\}\tag{1}$$

for all $x_1,\dots,x_n\in L$.

Proposition 3.4: If μ is a fuzzy ideal of an n-Lie algebra L, then

$$L_\mu=\{x\in L|\mu(x)=\mu(0)\}$$

is an ideal of L contained in every non-empty level subset of μ.

Proposition 3.5: Let μ and λ be two fuzzy ideals of an n-Lie algebra L such that $\mu(0)=\lambda(0)$. Then $L_{\mu\cap\lambda}=L_\mu\cap L_\lambda$.

Theorem 3.6

Let $\varphi:L\to L'$ be an n-Lie algebra homomorphism of an n-Lie algebra L onto an n-Lie algebra L'. Then the following conditions hold:

- if μ is a fuzzy ideal of L, then $\varphi(\mu)$ is a fuzzy ideal of L',
- if ν is a fuzzy ideal of L' then $\varphi^{-1}(\nu)$ is a fuzzy ideal of L,
- $\overline{\varphi^{-1}(\nu)_t}=\varphi^{-1}(\overline{\nu_t})$ for every $t\in[0,1]$ and every fuzzy ideal ν of L'.

Proposition 3.7: Let L be an n-Lie algebra. Then the intersection of any family of fuzzy subalgebras (ideals) of L is again a fuzzy subalgebra (ideal) of L.

It is easy to see that the union of fuzzy subalgebras (ideals) of an n-Lie algebra L is not a fuzzy subalgebra (ideal) of L, in general. But we have the following proposition on the union of fuzzy subalgebras (ideals) of L.

Proposition 3.8: Let $\{\mu_n\}$ be a chain of fuzzy subalgebras (ideals) of an n-Lie algebra L. Then $\bigcup_n\mu_n$ is a fuzzy subalgebra (ideal) of L.

Theorem 3.9

For a fuzzy subset μ of an n-Lie algebra L, the following statements are equivalent.

- μ is a fuzzy subalgebra (ideal) of L.
- Each non-empty $\overline{\mu_t}$, is a subalgebra (ideal) of L.

Proof. Let μ be a fuzzy ideal of L. Since μ is a fuzzy subspace of L, by Theorem 2.3, each non-empty $\overline{\mu_t}$ is a subspace of L. Therefore, it is enough to prove that $[\underbrace{L,\dots,L}_{i-1},\overline{\mu_t},\underbrace{L,\dots,L}_{n-i}]\subseteq\overline{\mu_t}$. For every $y\in\overline{\mu_t}$ and $x_1,\dots,x_n\in L$ we show that $[x_1,\dots,x_{i-1},y,x_{i+1},\dots,x_n]\in\overline{\mu_t}$. Since μ is a fuzzy ideal, we have

$$t\le\mu(y)\le\mu([x_1,\dots,x_{i-1},y,x_{i+1},\dots,x_n])$$

and so $[x_1,\dots,x_{i-1},y,x_{i+1},\dots,x_n]\in\overline{\mu_t}$.

Conversely, assume that every non-empty $\overline{\mu_t}$ is an ideal of L. Therefore, $\overline{\mu_t}$ is a subspace of L and so by Theorem 2.3, μ is a fuzzy subspace of L. Now, for every $y\in L$, we put $t_0=\mu(y)$. Then, $y\in\overline{\mu_{t_0}}$. Therefore, for every $x_1,\dots x_n \in L$ we have $[x_1,\dots,x_{i-1},y,x_{i+1},\dots,x_n]\in\overline{\mu_{t_0}}$ which implies that $\mu([x_1,\dots,x_{i-1},y,x_{i+1},\dots,x_n]) \ge t_0 = \mu(y)$. So, μ is a fuzzy ideal.

For subalgebras the proof is analogous.

Proposition 3.10: Let L be an n-Lie algebra and μ be a fuzzy subalgebra of L. Let $\overline{\mu_{t_1}}$ and $\overline{\mu_{t_2}}$ (with $t_1<t_2$) be any two level subalgebras of μ. Then $\overline{\mu_{t_1}}=\overline{\mu_{t_2}}$ if and only if there is no x in L such that $t_1 \le \mu(x)<t_2$.

Theorem 3.11

Let $\{S|\lambda\in\lambda\}$, where $\varnothing\neq\lambda\subseteq[0,1]$, be a collection of ideals of an n-Lie algebra L such that

(*i*) $L=\bigcup_{\lambda\in\Lambda} S_\lambda$

(*ii*) $\alpha>\beta \Leftrightarrow S_\alpha\subset S_\beta$ for all $\alpha,\beta\in\Lambda$

Then μ defined by

$\mu(x)=\sup\{\lambda\in\Lambda \mid x\in S_\lambda\}$

is a fuzzy ideal of L.

Proof. By Theorem 3.9, it is sufficient to show that every non-empty level $\overline{\mu_\alpha}$ is an ideal of L.

Let $\overline{\mu_\alpha}\neq\varnothing$ for some fixed $\alpha\in[0,1]$. Then

$\alpha=\sup\{\lambda\in\Lambda \mid \lambda<\alpha\}=\sup\{\lambda\in\Lambda \mid S_\alpha\subset S_\lambda\}$

or

$\alpha\neq\sup\{\lambda\in\Lambda \mid \lambda<\alpha\}=\sup\{\lambda\in\Lambda \mid S_\alpha\subset S_\lambda\}$.

In the first case we have $\overline{\mu_\alpha}=\bigcap_{\lambda<\alpha} S_\lambda$, because

$x\in\overline{\mu_\alpha}\Leftrightarrow\left(x\in S_\lambda \text{ for all } \lambda<\alpha\right)\Leftrightarrow x\in\bigcap_{\lambda<\alpha}S_\lambda$.

In the second, there exists $\varepsilon>0$ such that $(\alpha-\varepsilon,\lambda)\cap\lambda=\varnothing$. In this case $\overline{\mu_\alpha}=\bigcup_{\lambda\ge\alpha}S_\lambda$. Indeed, if $x\in\bigcup_{\lambda\ge\alpha}S_\lambda$, then $x\in S_\lambda$ for some $\lambda\ge\alpha$, which gives $\mu(x)\ge\lambda\ge\alpha$. Thus $x\in\overline{\mu_\alpha}$, i.e., $\bigcup_{\lambda\ge\alpha}S_\lambda\subseteq\overline{\mu_\alpha}$.

Conversely, if $x\notin S_\lambda$, then $x\notin S_\lambda$ for all $\lambda\ge\alpha$, which implies $x\notin S_\lambda$ for all $\lambda>\alpha-\varepsilon$, i.e., if $x\in S_\lambda$ then $\lambda\le\alpha-\varepsilon$. Thus $\mu(x)\le\alpha-\varepsilon$. Therefore $x\notin\overline{\mu_\alpha}$. Hence $\overline{\mu_\alpha}\subseteq\bigcup_{\lambda\ge\alpha}S_\lambda$, and consequently $\overline{\mu_\alpha}=\bigcup_{\lambda\ge\alpha}S_\lambda$. This completes our proof.

Theorem 3.12

Let μ be a fuzzy subset defined on an n-Lie algebra L and let $Im(\mu)=\{t_0,t_1,t_2,\dots\}$, where $1\ge t_0>t_1>t_2\dots\ge 0$. If $S_0\subset S_1\subset S_2\dots$ are subalgebras (ideals) of L such that $\mu(S_k\backslash S_{k-1})=t_k$ for $k=0,1,2,\dots$, where $S_{-1}=\varnothing$, then μ is a fuzzy subalgebra (ideal) of L.

Proof. First consider the case when all S_i are subalgebras. If $[x_1,\dots,x_n]\in L\setminus\bigcup_k S_k$ then also at least one of $x_1,\dots x_n$ is in $L\setminus\bigcup_k S_k$ because in the opposite case $x_1,\dots,x_n$ and $[x_1,\dots,x_n]$ will be in some S_k. So, in this case

$\mu([x_1,\dots,x_n])=0=\min\{\mu(x_1),\dots,\mu(x_n)\}$.

It is clear that for arbitrary elements $x_1,\dots,x_n\in L$ there exists only one k such that $[x_1,\dots,x_n]\in S_k\backslash S_{k-1}$ and only one k_i such that $x_i\in S_{k_i}\setminus S_{k_i-1}$. Thus $\mu([x_1,\dots,x_n])=t_k$, $\mu(x_i)=t_{k_i}$.

Suppose $t_{k_i}>t_k$ for all $i=1,2,\dots,n$. Then, by the assumption, $k_i<k$ and $S_{k_i}\subseteq S_s\subseteq S_{k-1}\subset S_k$, where $s=\max\{k_1,\dots,k_n\}$. Hence $x_1,\dots x_n\in S_{k-1}$ and, in the consequence, $[x_1,\dots x_n]\in S_{k-1}$ because S_{k-1} is a subalgebra. This is a contradiction. Therefore there is at least one $t_{k_i}\le t_k$. In this case $\mu([x_1,\dots,x_n])=t_k\ge t_{k_i}\ge\min\{\mu(x_1),\dots,\mu(x_n)\}$. Since μ also is a fuzzy subspace of a vector space L, it is a fuzzy subalgebra of L.

Now, let all S_i be ideals and let $[x_1,\dots x_n]\in S_k\backslash S_{k-1}$ for some $x_1,\dots x_n\in L$. Then these $x_1,\dots x_n$ are in $L\backslash S_{k-1}$. If not, then there exists $x_i\in S_{k-1}$. But in this case $[x_1,\dots x_n]\in S_{k-1}$ because S_{k-1} is an ideal. This is a contradiction. So, all $x_i\in L\backslash S_{k-1}$. Hence $\max\{\mu(x_1),\dots,\mu(x_n)\}\le t_k=\mu([x_1,\dots,x_n])$. Now, if $[x_1,\dots,x_n]\in L\setminus\bigcup_k S_k$, then also all $x_1,\dots x_n$ are in $L\setminus\bigcup_k S_k$. Thus $\max\{\mu(x_1),\dots,\mu(x_n)\}=\mu([x_1,\dots,x_n])$. This completes the proof that μ is a fuzzy ideal.

Corollary 3.13

For any chain $S_0\subset S_1\subset S_2\dots$ of subalgebras (ideals) of an n-Lie algebra L and any chain of reals $1\ge t_0>t_1>\dots\ge 0$ there exists a fuzzy subalgebra (ideal) μ of L such that $\overline{\mu_{t_k}}=S_k$.

Theorem 3.14

Let $Im(\mu)=\{t_1|i\in I\}$ be the image of a fuzzy subalgebra (ideal) μ of an n-Lie algebra L. Then

(*a*) There exists a unique $t_0\in Im(\mu)$ such that $t_0\ge t_i$ for all $t_i\in Im(\mu)$,

(*b*) L is the set-theoretic union of all $\overline{\mu_{t_i}}$, $t_i\in Im(\mu)$,

(*c*) $\Omega=\{\overline{\mu_{t_i}} \mid t_i\in Im(\mu)\}$ is linearly ordered by inclusion,

(*d*) Ω contains all level subalgebras (ideals) of μ if and only if μ attains its infimum on all subalgebras (ideals) of L.

Proof. (*a*) Follows from the fact that $t_0=\mu(0)\ge\mu(x)$ for all $x\in L$.

(*b*) If $x\in L$, then $\mu(x)=t_x\in Im(\mu)$. Thus $x\in\bigcup\overline{\mu_{t_i}}\subseteq L$, where $t_i\in Im(\mu)$, which proves (*b*).

(*c*) Since $\overline{\mu_{t_i}}\subseteq\overline{\mu_{t_j}}\Leftrightarrow t_i\ge t_j$ for $i,j\in I$, then Ω linearly ordered by inclusion.

(*d*) Suppose that Ω contains all levels of μ. Let S be a subalgebra (ideal) of L. If μ is constant on S, then we are done. Assume that μ is not constant on S. We have two cases: (1) $S=L$ and (2) $S\neq L$. For $S=L$ let $\beta=\inf Im(\mu)$. Then $\beta\le t\in Im(\mu)$, i.e., $\overline{\mu_\beta}\supseteq\overline{\mu_t}$ for all $t\in Im(\mu)$. But $\overline{\mu_0}=L\in\Omega$ because Ω contains all levels of μ. Hence there exists $t'\in Im(\mu)$ such that $\overline{\mu_{t'}}=L$. It follows that $\overline{\mu_\beta}\supset\overline{\mu_{t'}}=L$ so that $\overline{\mu_\beta}=\overline{\mu_{t'}}=L$ because every level of μ is a subalgebra (resp. ideal) of L.

Now it sufficient to show that $\beta=t'$. If $\beta<t'$, then there exists $t''\in Im(\mu)$ such that $\beta\le t''<t'$. This implies $\overline{\mu_{t''}}\supset\overline{\mu_{t'}}=L$, which is a contradiction. Therefore $\beta=t'\in Im(\mu)$.

In the case $S\neq L$ we consider the fuzzy set μ_S defined by

$$\mu_S(x)=\begin{cases}\alpha & \text{for} \quad x\in S,\\ 0 & \text{for} \quad x\in L\setminus S.\end{cases}$$

Clearly μ_S is a fuzzy subalgebra (ideal) of L if S is a subalgebra (ideal).

Let

$J = \{i \in I \mid \mu(x) = t_i \text{ for some } x \in S\}$.

Then $\Omega_S = \{\overline{\mu_{t_i}} \mid i \in J\}$ contains (by the assumption) all levels of μ_S. This means that there exists $x_0 \in S$ such that $\mu(x_0) = \inf\{\mu_S(x) \mid x \in S\}$, i.e., $\mu(x_0)=\mu_S(x)$ for some $x \in S$. Hence μ attains its infimum on all subalgebras (ideals) of L.

To prove the converse let $\overline{\mu_\alpha}$ be a level subalgebra of μ. If $\alpha=t$ for some $t \in Im(\mu)$, then $\overline{\mu_\alpha} \in \Omega$. If $\alpha \neq t$ for all $t \in Im(\mu)$, then there does not exist $x \in L$ such that $\mu(x)=\alpha$.

Let $S=\{x \in L \mid \mu(x)>\alpha\}$. Obviously $0 \in S$ and $\mu(x_i)>\alpha$ for all $x_i \in S$. From the fact that μ is a fuzzy subalgebra we obtain

$$\mu([x_1,\ldots,x_n]) \geq \min\{\mu(x_1),\mu(x_2),\ldots,\mu(x_n)\} > \alpha,$$

which proves $[x_1,\ldots,x_n] \in S$. Hence S is a subalgebra. By hypothesis, there exists $y \in S$ such that $\mu(y)=\inf\{\mu(x) \mid x \in S\}$. But $\mu(y) \in Im(\mu)$ implies $\mu(y)=t'$ for some $t' \in Im(\mu)$. Hence $\inf\{\mu(x) \mid x \in S\}=t'>\alpha$.

Note that there does not exist $z \in L$ such that $\alpha \leq \mu(z) < t'$. This gives $\overline{\mu_\alpha} = \overline{\mu_{t'}}$. Hence $\overline{\mu_\alpha} \in \Omega$. Thus Ω contains all level subalgebras of μ.

Theorem 3.15

If every fuzzy subalgebra (ideal) μ defined on an n-Lie algebra L has a finite number of values, then every descending chain of subalgebras (ideals) of L terminates at finite step.

Proof. Suppose there exists a strictly descending chain

$S_0 \supset S_1 \supset S_2 \supset \ldots$

of ideals of L which does not terminate at finite step. We prove that μ defined by

$$\mu(x) = \begin{cases} \dfrac{k}{k+1} & \text{for } \quad x \in S_k \setminus S_{k+1}, \\ 1 & \text{for } \quad x \in \bigcap S_k, \end{cases}$$

where $k=0,1,2,\ldots$ and $S_0=L$, is a fuzzy ideal with an infinite number of values.

If $[x_1,\ldots,x_n] \in \bigcap S_k$, then obviously

$\mu([x_1,\ldots,x_n]) = 1 \geq \max\{\mu(x_1),\mu(x_2),\ldots,\mu(x_n)\}$.

If $[x_1,\ldots,x_n] \notin \bigcap S_k$, then $[x_1,\ldots,x_n] \in S_p \setminus S_{p+1}$ for some $p \geq 0$ and there exists at least one $i=1,2,\ldots,n$ such that $x_i \notin \bigcap S_k$, because $x_1,\ldots,x_n \in \bigcap S_k$ implies $[x_1,\ldots,x_n] \in \bigcap S_k$.

Let S_m be a maximal ideal of L such that at least one of $x_1,\ldots,x_n$ belongs to $S_m \setminus S_{m+1}$. Then $m \leq p$. Indeed, for $m>p$ we have $x_1,x_2,\ldots,x_n \in S_m \subseteq S_{p+1} \subset S_p$ and, consequently $[x_1,\ldots,x_n] \in S_{p+1}$, which is impossible. Thus $m \leq p$ and

$$\mu([x_1,\ldots,x_n]) = \frac{p}{p+1} \geq \max\{\mu(x_1),\ldots,\mu(x_n)\} = \frac{m}{m+1}.$$

This proves that μ is a fuzzy ideal and has an infinite number of different values. This is a contradiction. Hence every descending chain of ideals terminates at finite step.

For subalgebras the proof is analogous.

Theorem 3.16

Every ascending chain of subalgebras (ideals) of an n-Lie algebra L terminates at finite step if and only if the set of values of any fuzzy subalgebra (ideal) of L is a well-ordered subset of $[0,1]$.

Proof. If the set of values of a fuzzy subalgebra (ideal) μ is not well-ordered, then there exists a strictly decreasing sequence $\{t_i\}$ such that $t_i=\mu(x_i)$ for some $x_i \in L$. But in this case $\overline{\mu_{t_i}}$ form a strictly ascending chain of subalgebras (ideals) of L, which is a contradiction.

In order to prove the converse suppose that there exists a strictly ascending chain $S_1 \subset S_2 \subset S_3 \subset \ldots$ of subalgebras (ideals) of L. Then $M = \bigcup_{i \in N} S_i$ is a subalgebra (ideal) of L and μ defined by

$$\mu(x) = \begin{cases} 0 & \text{for } x \notin M, \\ \dfrac{1}{k} & \text{where } k = \min\{i \mid x \in S_i\} \end{cases}$$

is a fuzzy subalgebra (ideal) on L.

Indeed, for every $x_1,\ldots,x_n \in M$ there exist a minimal number k_i such that $x_i \in S_{k_i}$, and a minimal number p such that $[x_1,\ldots,x_n] \in S_p$. If all S_i are subalgebras, then for $k=\max\{k_1,k_2,\ldots,k_n\}$ all $x_1,\ldots,x_n$ and $[x_1,\ldots,x_n]$ are in S_k. Thus $k \geq p$. Consequently,

$$\mu([x_1,\ldots,x_n]) = \frac{1}{p} \geq \frac{1}{k} = \min\{\mu(x_1),\mu(x_2),\ldots,\mu(x_n)\}.$$

The case when at least one of $x_1,x_2,\ldots,x_n$ is not in M is obvious. Hence μ is a fuzzy subalgebra.

Now, if all S_i are ideals, then $[x_1,\ldots,x_n] \in S_m$ for $m=\min\{k_1,\ldots,k_n\}$. Thus $p \leq m$. Hence

$$\mu([x_1,\ldots,x_n]) = \frac{1}{p} \geq \frac{1}{m} = \max\{\mu(x_1),\mu(x_2),\ldots,\mu(x_n)\},$$

which means that in this case μ is a fuzzy ideal.

Since the chain $S_1 \subset S_2 \subset S_3 \subset \ldots$ is not terminating, μ has a strictly descending sequence of values. This contradicts that the set of values of any fuzzy subalgebra (ideal) is well-ordered. The proof is complete.

Definition 3.17

A fuzzy subset μ of an n-Lie algebra L is said to be normal if $\mu(0)=1$.

The following lemma is obvious.

Lemma 3.18

If μ is a fuzzy subalgebra (ideal) of an n-Lie algebra L, then μ^+ defined by

$\mu^+(x)=\mu(x)+1-\mu(0)$

is a normal fuzzy subalgebra (ideal) of L.

Corollary 3.19

Any fuzzy subalgebra (ideal) of an n-Lie algebra L is contained in some normal fuzzy subalgebra (ideal) of it.

Proof. Indeed, $\mu(x) \leq \mu(x)+1-\mu(0) = \mu^+(x)$ for every $x \in L$.

Proposition 3.20: A maximal normal fuzzy subalgebra of an n-Lie algebra L takes only two values: 0 and 1.

Proof. If $\mu(x)=1$ for all $x \in L$, then obviously μ is a maximal normal fuzzy subalgebra of L. If μ is a maximal normal fuzzy subalgebra of L and $0<\mu(a)<1$ for some $a \in L$, then a fuzzy subset ν defined by $\nu(x) = \frac{1}{2}(\mu(x)+\mu(a))$ is a fuzzy subalgebra of L. Moreover, ν^+ is a non-constant normal fuzzy subalgebra of L such that $\mu(x) \leq \nu^+(x)$ for every $x\in L$. Thus, μ is not maximal. Obtained contradiction shows that $\mu(a)=0$ for all $\mu(a)<1$.

Proposition 3.21: Let μ be a fuzzy subalgebra (ideal) of an n-Lie algebra L. If $h:[0,\mu(0)]\to[0,1]$ is an increasing function, then a fuzzy subset μ_h defined on L by $\mu_h(x)=h(\mu(x))$ is a fuzzy subalgebra (ideal). Moreover, μ_h is normal if and only if $h(\mu(0))=1$.

Proof. Straightforward.

If μ is a fuzzy subset of an n-Lie algebra L, and f is a function defined on L, then the fuzzy subset ν of $f(L)$ defined by $\nu(y)=\sup_{x\in f^{-1}(y)}\{\mu(x)\}$, for all $y\in f(L)$ is called the image of μ under f. Similarly, if ν is a fuzzy subset in $f(L)$, then the fuzzy set $\mu=\nu\circ f$ in L is called preimage of ν under f.

Theorem 3.22

An n-Lie algebra homomorphic preimage of a fuzzy ideal is a fuzzy ideal.

Proof. Let $\varphi:L_1\to L_2$ be an n-Lie algebra homomorphism, and ν be a fuzzy ideal of L_2 and μ be the preimage of ν under φ. Then, as it is not difficult to see, μ is a fuzzy subspace of L and

$$\begin{aligned}\mu([x_1,\dots,x_{i-1},y,x_{i+1},\dots,x_n]) &= \nu(\varphi([x_1,\dots,x_{i-1},y,x_{i+1},\dots,x_n]))\\ &= \nu([\varphi(x_1),\dots,\varphi(x_{i-1}),\varphi(y),\varphi(x_{i+1}),\dots,\varphi(x_n)])\\ &\geq \nu(\varphi(y))=\mu(y),\end{aligned}$$

for all $x_1,\dots,x_n,y\in L$ and $\alpha\in F$.

A fuzzy set μ of a set X is said to possess sup property if for every non-empty subset S of X, there exists $x_0\in S$ such that $\mu(x_0)=\sup_{x\in S}\{\mu(x)\}$.

Theorem 3.23

An n-Lie algebra homomorphism image of a fuzzy ideal having the sup property is a fuzzy ideal.

Proof. Suppose that $\varphi:L_1\to L_2$ is an n-Lie algebra homomorphism, μ is a fuzzy ideal of L_1 with the sup property and ν is the image of μ under φ. Suppose that $\varphi(x),\varphi(y)\in\varphi(L)$. Let $x_0\in\varphi^{-1}(\varphi((x))$ and $y_0\in\varphi^{-1}(\varphi((x))$ be such that $\mu(x_0)=\sup_{t\in\varphi^{-1}(\varphi(x))}\{\mu(t)\}$ and $\mu(y_0)=\sup_{t\in\varphi^{-1}(\varphi(y))}\{\mu(t)\}$, respectively. Then,

$$\begin{aligned}\nu(\varphi(x)+\varphi(y)) &= \sup_{t\in\varphi^{-1}(\varphi(x)+\varphi(y))}\{\mu(t)\}\geq\mu(x_0+y_0)\geq\min\{\mu(x_0),\mu(y_0)\}\\ &=\min\left\{\sup_{t\in\varphi^{-1}(\varphi(x))}\{\mu(t)\},\ \sup_{t\in\varphi^{-1}(\varphi(y))}\{\mu(t)\}\right\}\\ &=\min\{\nu(\varphi(x),\nu(\varphi(y))\},\end{aligned}$$

and

$$\begin{aligned}\nu(\varphi(-x)) &= \sup_{t\in\varphi^{-1}(\varphi(-x))}\{\mu(t)\}\geq\mu(-x_0)=\mu(x_0)=\nu(\varphi(x)),\\ \nu(\alpha\varphi(x)) &= \sup_{t\in\alpha\varphi^{-1}(\varphi(x))}\{\mu(t)\}\geq\mu(\alpha x_0)\geq\mu(x_0)=\nu(\varphi(x)).\end{aligned}$$

Finally, let $\varphi(x_1),\dots,\varphi(x_n),\varphi(y)\in\varphi(L)$ and let $a_1\in\varphi^{-1}(\varphi(x_1)),\dots,a_n\in\varphi^{-1}(\varphi(x_n))$, $b\in\varphi^{-1}(\varphi(y))$ be such that

$$\mu(a_1)=\sup_{t\in\varphi^{-1}(\varphi(x_1))}\{\mu(t)\},\dots,\mu(a_n)=\sup_{t\in\varphi^{-1}(\varphi(x_n))}\{\mu(t)\},$$
$$\mu(b)=\sup_{t\in\varphi^{-1}(\varphi(y))}\{\mu(t)\}.$$

Then,

$$\begin{aligned}&\nu([\varphi(x_1),\dots,\varphi(x_{i-1}),\varphi(y),\varphi(x_{i+1}),\dots,\varphi(x_n)])\\ &=\nu(\varphi([\varphi(x_1),\dots,\varphi(x_{i-1}),\varphi(y),\varphi(x_{i+1}),\dots,\varphi(x_n)]))\\ &=\sup_{t\in\varphi^{-1}(\varphi([\varphi(x_1),\dots,\varphi(x_{i-1}),\varphi(y),\varphi(x_{i+1}),\dots,\varphi(x_n)]))}\{\mu(t)\}\\ &\geq\mu([a_1,\dots,a_{i-1},b,a_{i+1},\dots,a_n])\geq\mu(b)=\nu(\varphi(y)).\end{aligned}$$

This proves that ν is a fuzzy ideal of $\varphi(L)$.

Fuzzy Quotient *n*-Lie Algebras

If I is an ideal of an n-Lie algebra L, then we can define a new n-Lie algebra on the quotient space L/I with the n-linear map

$$[x_1+I,\dots,x_n+I]:=[x_1,\dots,x_n]+I,$$

for all $x_1,\dots,x_n\in L$.

If I is an ideal of an n-Lie algebra L, then the quotient space L/I is also an n-Lie algebra and is the quotient n-Lie algebra.

Theorem 4.1

Let L be an n-Lie algebra.

- Let μ be a fuzzy ideal of L and let $t=\mu(0)$. Then the fuzzy subset μ^* of $L/\overline{\mu_t}$ defined by $\mu^*(x+\overline{\mu_t})=\mu(x)$ for all $x\in L$, is a fuzzy ideal of $L/\overline{\mu_t}$.
- If I is an ideal of L and ν is a fuzzy ideal of L/I such that $\nu(x+I)=\nu(I)$ only when $x\in I$, then there exists a fuzzy ideal μ of L such that $\overline{\mu_t}=I$, where $t=\mu(0)$; and $\nu=\mu^*$.

Proof. (1). Since μ is a fuzzy ideal of L, $\overline{\mu_t}$ is an ideal of L. Now, μ^* is well-defined, because if $x+\overline{\mu_t}=y+\overline{\mu_t}$ for $x,y\in L$, then $x-y\in\overline{\mu_t}$ and so $\mu(x-y)=\mu(0)$. Hence, $\mu(x)=\mu(y)$ which implies that $\mu^*(x+\overline{\mu_t})=\mu^*(y+\overline{\mu_t})$.

Now, we show μ^* is a fuzzy ideal of L. Let $x,y\in L$ and $\alpha\in F$. Then, we have

$$\begin{aligned}\mu^*((x+\overline{\mu_t})+(y+\overline{\mu_t}))=\mu^*((x+y)+\overline{\mu_t})=\mu(x+y)\geq\min\{\mu(x),\mu(y)\}\\ =\min\{\mu^*(x+\overline{\mu_t}),\mu^*(y+\overline{\mu_t})\},\end{aligned}$$

and

$$\begin{aligned}\mu^*(-x+\overline{\mu_t}) &=\mu(-x)=\mu(x)=\mu^*(x+\overline{\mu_t}),\\ \mu^*(\alpha(x+\overline{\mu_t})) &=\mu^*(\alpha x+\overline{\mu_t})=\mu(\alpha x)\geq\mu(x)=\mu^*(x+\overline{\mu_t}).\end{aligned}$$

Finally, for $x_1,\dots,x_n\in L$, we have

$$\begin{aligned}&\mu^*([x_1+\overline{\mu_t},\dots,x_{i-1}+\overline{\mu_t},y+\overline{\mu_t},x_{i+1}+\overline{\mu_t},\dots,x_n+\overline{\mu_t}])\\ &=\mu^*([x_1,\dots x_{i-1},y,x_{i+1},\dots,x_n]+\overline{\mu_t})\\ &=\mu([x_1,\dots x_{i-1},y,x_{i+1},\dots,x_n])\geq\mu(y)=\mu^*(y+\mu_t).\end{aligned}$$

(2). We define a fuzzy subset μ of L by $\mu(x)=\nu(x+I)$ for all $x\in L$. A routine computation shows that μ is a fuzzy ideal of L. Now, $\overline{\mu_t}=I$, because

$$x\in\overline{\mu_t}\Leftrightarrow\mu(x)=t=\mu(0)\Leftrightarrow\nu(x+I)=\nu(I)\Leftrightarrow x\in I.$$

Finally, $\mu^*=\nu$, since $\mu^*(x+I)=\mu^*(x+\overline{\mu_t})=\mu(x)=\nu(x+I)$.

Let μ be any fuzzy ideal of an n-Lie algebra L and let $x\in L$. The fuzzy subset μ_x^* of L defined by $\mu_x^*(a)=\mu(a-x)$ *for all* $a\in L$ is called the fuzzy coset determined by x and μ.

Let I be an ideal of L. If χ_I is the characteristic function of I, then it is easy to see that $(\chi_I)_x^*$ is the characteristic function of $x+I$.

Theorem 4.2

Let μ be any fuzzy ideal of an n-Lie algebra L. Then the set of all fuzzy cosets of μ in L, i.e., the set $L[\mu]=\{\mu_x^* | x\in L\}$, is an n-Lie algebra under the following operations:

$$\begin{array}{ll} \mu_x^*+\mu_y^*=\mu_{x+y}^* & \text{for all } x,y\in L,\\ \alpha\mu_x^*=\mu_{\alpha x}^* & \text{for all } x\in L, \alpha\in F,\\ [\mu_{x_1}^*,\ldots,\mu_{x_n}^*]=\mu_{[x_1,\ldots,x_n]}^* & \text{for all } x_1,\ldots,x_n\in L. \end{array}$$

Theorem 4.3

If μ is any fuzzy ideal of an n-Lie algebra L, then the map $\phi:L\to L[\mu]$ defined by $\varphi(x)=\mu_x^*$ for all $x\in L$, is a homomorphism with kernel $\overline{\mu_t}$, where $t=\mu(0)$.

Proof. It is easy to see that f is a homomorphism. We show $\mu(x)=\mu(0)$ implies $\mu_x^*=\mu_0^*$. For this, let $a\in L$. Then, $\mu(a)\le\mu(0)=\mu(x)$. If $\mu(a)<\mu(x)$, then $\mu(a-x)=\mu(a)$, by Lemma 2.2. On the other hand, if $\mu(a)=\mu(x)$, then $a,x\in\{y\in L|\mu(y)=\mu(0)\}$. Hence, $\mu(a-x)=\mu(0)=\mu(x)=\mu(a)$. Therefore, in either case, we have shown that $\mu(a-x)=\mu(a)$ for all $a\in L$. Consequently $\mu_x^*=\mu_0^*$. Also, $\mu_x^*=\mu_0^*$ implies that $\mu(x)=\mu(0)$. Hence, $\mu_x^*=\mu_0^*$ if and only if $\mu(x)=\mu(0)$. Now, we have

$$ker\varphi=\{x\in L|\varphi(x)=\mu_0^*\}=\{x\in L|\mu_x^*=\mu_0^*\}=\{x\in L|\mu(x)=\mu(0)\}=\overline{\mu_t},$$

where $t=\mu(0)$ $t=\mu(0)$.

Theorem 4.4

Given a homomorphism of n-Lie algebras $:L\to L'$ and fuzzy ideal μ of L and μ' of L' such that $\varphi(\mu)\subseteq\mu'$. Then, there is a homomorphism of n-Lie algebras $\varphi^*:L[\mu]\to L'[\mu']$, where $\varphi^*(\mu_x^*)=\mu'^*_{\varphi(x)}$, such that the following diagram is commutative.

$$\begin{array}{ccc} L & \overset{\varphi}{\to} & L' \\ \downarrow & & \downarrow \\ L[\mu] & \overset{\varphi^*}{\to} & L'[\mu'] \end{array}$$

Proof. If $\mu_x^*=\mu_y^*$ then $\mu(x-y)=\mu(0)$. So

$$\mu'(\varphi(x)-\varphi(y))=\mu'(\varphi(x-y))=\varphi^{-1}(\mu')(x-y)\ge\mu(x-y)=\mu(0),$$

and so $\mu'(\varphi(x)-\varphi(y))=\mu(0)$. Hence, $\mu'(\varphi(x))=\mu'(\varphi(y))$ holds. Thus, φ^* is well-defined. It is easily seen that φ^* is a homomorphism.

Let μ be a fuzzy ideal of an n-Lie algebra L. For any $x,y\in L$, we define a binary relation $\sim$ on L by $x\sim y$ if and only if $\mu(x-y)=\mu(0)$. Then $\sim$ is a congruence relation on L. We denote $[x]_\mu$ the equivalence class containing x, and $L/\mu=\{[x]\mu|x\in L\}$ the set of all equivalence classes of L. Then, L/μ is an n-Lie algebra under the following operations:

$$\begin{array}{ll} [x]_\mu+[y]_\mu=[x+y]_\mu & \text{for all } x,y\in L,\\ \alpha[x]_\mu=[cx]_\mu & \text{for all } x\in L, \alpha\in F,\\ [[x_1]_\mu,\ldots[x_n]_\mu]=[[x_1,\ldots,x_n]]_\mu & \text{for all } x,y\in L. \end{array}$$

Theorem 4.5 (Fuzzy first isomorphism theorem)

Let $\varphi:L\to L'$ be an epimorphism of n-Lie algebras and λ be a fuzzy ideal of L'. Then $L/\varphi_{-1}(\lambda)\cong L'|\lambda$.

Let I be an ideal and μ a fuzzy ideal of an n-Lie algebra L. If μ is restricted to I, then μ is a fuzzy ideal of I and $I|\mu$ is an ideal of L/μ.

Theorem 4.6 (Fuzzy second isomorphism theorem)

Let μ and λ be two fuzzy ideals of an n-Lie algebra L with $\mu(0)=\lambda(0)$. Then $\dfrac{L_\mu+L_\lambda}{\lambda}\cong\dfrac{L_\mu}{\mu\cap\lambda}$.

Theorem 4.7 (Fuzzy third isomorphism theorem)

Let μ and λ be two fuzzy ideals of an n-Lie algebra L with $\lambda\subseteq\mu$ and $\mu(0)=\lambda(0)$. Then $\dfrac{L/\lambda}{L_\mu/\lambda}\cong L/\mu$.

Conclusion

Methods of construction fuzzy ideals are presented. Connections with various fuzzy quotient n-Lie algebras are proved. Properties of fuzzy subalgebras and ideals of n-ary Lie algebras are described.

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