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From Monge-Ampere-Boltzman to Euler Equations

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Abstract

This paper concerns with the convergence of the Monge-Ampere-Boltzman system to the in compressible Euler Equations in the quasi-neutral regime.

Keywords: Vlasov-Monge-Ampère-Boltzmann system; Euler equations of the incompressible fluid

Introduction and Main Results

In this paper, we are interested in the hydrodynamical limit of the Boltzman-Monge-Ampere system (BMA)

$$f^{\varepsilon}+\xi.\nabla x f^{\varepsilon}+\nabla x\phi\varepsilon.\nabla_{\xi} f^{\varepsilon}=Q(f^{\varepsilon},f^{\varepsilon}) \tag{1.1}$$

$$\det(\mathbb{I}_d+\varepsilon^2D^2\phi^{\varepsilon})=\rho^{\varepsilon} \tag{1.2}$$

where $f^{\varepsilon}(t,x,\xi)\geq 0$ the electronic density at time $t\geq 0$ point $x\in[0,1]^d=\mathbb{T}d$, and with a velocity $\xi\in\mathbb{R}^d$, and Id is the identity matrix defined by

$$\begin{pmatrix} 1 & 0 & \cdots & 0\\ 0 & 1 & 0 & \vdots\\ \vdots & 0 & \ddots & 0\\ 0 & \cdots & 0 & 1\end{pmatrix}$$

The spatially periodic electric potential is coupled with ϕ^{ε} through the nonlinear Monge-Ampere equation (1.2). The quantities $\varepsilon>0$ and

$\rho^{\varepsilon}(t,x)\geq 0$ denote respectively the vacuum electric permittivity and

$$\rho^{\varepsilon}(t,x)=_{\mathbb{R}d}f^{\varepsilon}(t,x,\xi)d\xi. \tag{1.3}$$

$Q(f^{\varepsilon},f^{\varepsilon})$ is the Boltzman collision integral. This integral operates only on the ξ–argument of the distribution f^{ε} and is given by

$$Q\left(f^{\varepsilon},f^{\varepsilon}\right)(t,x,x)=\iint_{S_+^{d-1}\times\mathbb{R}^d}\left(\left(f^{\varepsilon}\right)'\left(f_1^{\varepsilon}\right)'-f^{\varepsilon}f_1^{\varepsilon}\right)b(\xi-\xi_1,\sigma)\,d\sigma\,d\xi_1,$$

where the terms f_1^{ε}, $(f^{\varepsilon})'$ and $\left(f_1^{\varepsilon}\right)'$ defines, respectively the values

$f^{\varepsilon}(t,x,\xi_1)$, $f^{\varepsilon}(t,x,\xi')$ and $f^{\varepsilon}(t,x,\xi_1')$ with $\xi'$and ξ_1' given in terms of

$\xi,\xi_1\in\mathbb{R}^d$, and $\sigma\in S_+^{d-1}=\left\{\sigma\in S^{d-1}/\sigma.\xi\geq\sigma\xi_1\right\}$ by

$$\xi'=\frac{\xi+\xi_1}{2}+\frac{\xi-\xi_1}{2}\sigma,\xi_1'=\frac{\xi+\xi_1}{2}-\frac{\xi-\xi_1}{2}\sigma.$$

The aim of this work is to investigate the hydrodynamic limit of the (BMA) system with optimal transport techniques.

Note that

$$\int_{\mathbb{R}^d}Q(f^{\varepsilon},f^{\varepsilon})d\xi=\int_{\mathbb{R}^d}\xi_iQ(f^{\varepsilon},f^{\varepsilon})d\xi=\int_{\mathbb{R}^d}|\xi|^2Q(f^{\varepsilon},f^{\varepsilon})d\xi=0, i=1,2,...,d.$$

The linearization of the determinant about the identity matrix gives

$\det(\mathbb{I}_d+\varepsilon^2D^2\phi^{\varepsilon})=1+\varepsilon^2\,\Delta\phi^{\varepsilon}+O(\varepsilon^4)$.

Where $\mathbb{I}_d$ represents the identity matrix.

So, one can see that the BMA system is considered as a fully non-linear version of the Vlasov Poisson-Boltzman (VPB) system defined by

$$\partial_t f^{\varepsilon}+\xi.\nabla_x f\varepsilon+\nabla_x\phi^{\varepsilon}.\nabla_{\xi}f^{\varepsilon}=Q(f^{\varepsilon},f^{\varepsilon}) \tag{1.4}$$

$$\varepsilon^2\Delta\phi^{\varepsilon}=\rho^{\varepsilon}-1 \tag{1.5}$$

The analysis of the VPB system has been considered by many authors and many results can be found in a vast literature [1-10].

In Hsiao et al. [11] study the convergence of the VPB system to the Incompressible Euler Equations. Bernier and Grégoire show that weak solution of Vlasov-Monge-Ampère converge to a solution of the incompressible Euler equations when the parameter goes to 0, Brenier [12] and Loeper [13] for details. So, is a ligitim question to look for the convergence of a weak solution of BMA (of course if such solution exists) to a solution of the incompressible Euler equations when the parameter goes to 0.

The study of the existence and uniqueness of solution to the BMA system seems a difficult matter. Here we assume the existence and uniqueness of smooth solution to the BMA and we just look to the asymptotic analysis of this system.

Definitions and Recalls

Definition 1

For a fixed bounded convex open set W of $\mathbb{R}^d$ and a positive measure on $\mathbb{R}^d$ of total mass |W|, we note by F[,ρ] the unique up to a constant convex function on satisfying

$$\forall g\in C^0\left(\mathbb{R}^d\right)\cap L^1(d\rho),\int_{\mathbb{R}^d}g(x)\,d\rho(x)=\int_{\Omega}g\left(\Phi[\Omega,\rho]\right)(y)\,dy.$$

Its Legendre-Fenchel transform denoted $\Psi[\Omega,\rho]$ the function satisfying (1.6) $\forall g\in C^0(\mathbb{R}^d)$ Ç $L^1(\Omega,dy)$ $\int_{\mathbb{R}^d}g(\nabla\Psi[\Omega,\rho])d\rho(x)=\int_{\Omega}g(y)dy$. we may write Φ (resp. Ψ) instead of Φ [Ω,] (resp. $\Psi[\Omega,\rho]$) if no confusion is possible.

Remark 2

- Existence and uniqueness of Φ is due to the polar factorization theorem.
- By setting the change of variables $y=\nabla\Psi(x)$, we get dy=det D2Ψ(x)dx. So (1.6) can be transformed to:

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$\forall g \in C^0(\mathbb{R}^d) \subset L^1(\Omega, dy)$,

$$\int_{\mathbb{R}^d} g(\nabla\Psi[\Omega,\rho])\rho(x)dx = \int_\Omega g(\nabla\Psi(x))\det D^2\Psi(x)dxc \qquad (1.6)$$

Which is a weak version of the Monge–Ampere equation

$\det D^2\Psi(x)=\rho(x)$

∇ mapps supp (ρ) in Ω

We assume that BMA system has a renormalized solution in the sense of DiePerna and Lions [3].

For simplicity, we set

$$\tilde{\varphi}[\rho] = \frac{|x|^2/2 - \Psi[\rho]}{\varepsilon}$$

and

$$\ddot{o}[\rho] = \frac{\Phi[\rho] - |x|^2/2}{\varepsilon}$$

So that, $\nabla\phi[\rho] = \nabla\tilde{\varphi}[\rho] o \nabla\Phi[\rho]$, and the (BMA$_p$) ($p$ stands for periodic) system takes the following form

$$\begin{cases} \partial_t f^\varepsilon + \xi.\nabla_x f^\varepsilon - \frac{1}{\varepsilon}\nabla\tilde{\varphi}[\rho]\nabla_\xi f^\varepsilon = Q(f^\varepsilon, f^\varepsilon), \\ f^\varepsilon(0,.,.) = f_0^\varepsilon \end{cases} \qquad (1.7)$$

The energy is given by

$$E(t) = \frac{1}{2}\int f(t,x,\xi)|\xi|^2 dxd\xi + \frac{1}{2}\int|\nabla\varphi|^2 dx$$

$$= \frac{1}{2}\int f(t,x,\xi)|\xi|^2 dxd\xi + \frac{1}{2}\int\rho|\nabla\tilde{\varphi}|^2 dx$$

It has been shown |2| that the energy is conserved.

The Euler equation for incompressible fluids reads

$$\begin{cases} \partial_t\bar{v} + \bar{v}.\nabla\bar{v} = -\nabla p, \\ \nabla.\bar{v} = 0. \end{cases} \qquad (1.8)$$

One can find in Loeper [13] more details for this kind of equations.

Theorem 3

Let f^ε be a weak solution of (1.7) with finite energy, let $(t,x) \mapsto \bar{v}(t,x)$ be a smooth $C^2([0,T] \times T^d)$ solution of (1.8) for $t \in [0,T]$, and $p(t,x)$ the corresponding pressure, let

$$H\varepsilon(t) = \frac{1}{2}\int f(t,x,\xi)|\xi - \bar{v}(t,x)|^2 dxd\xi + \frac{1}{2}\int\left|\nabla\varphi(t,x)^2\right|dx$$

Then

$$H\varepsilon(t) \le C\exp(Ct)\left(H\varepsilon(0) + \varepsilon^2\right), \forall t \in [0,T]$$

The constant C depends only on

$$\sup_{0\le s\le T} \| \bar{v}(s,.), p(s,.), \partial_t p(s,.), \nabla p(s,.) \|_{W^{1,\infty}(T^d)}$$

Proof of the Theorem 3

Later, in the section, we need the following Lemma

Lemma 4:

Let G: $\mathbb{T}^d \to \mathbb{R}$ be Lipschitz continuous such that $\int_{\mathbb{T}^d} G = 0$ then for all R>0 has one

$$\left|\int_\rho G = 0\right| \le \frac{1}{2}\|\nabla G\|_{L^\infty}\left(\frac{1}{R}\varepsilon^2 + RH_\varepsilon\right)$$

We have

$$\frac{d}{dt}H_\varepsilon = \frac{d}{dt}\left[\frac{1}{2}\int f(t,x,\xi)\left(|\bar{v}|^2 - 2\xi.\bar{v}\right)dxd\xi\right]$$

$$= \int f(t,x,\xi)(\partial_t\bar{v}.\bar{v} - \partial_t\bar{v}.\xi) + \frac{1}{2}\int\partial_t f\left(|\bar{v}|^2 - 2\xi.\bar{v}\right)dxd\xi$$

From the BMA we have

$$\int\partial_t f\left(|\bar{v}|^2 - 2\xi.\bar{v}\right)dxd\xi = -\int\nabla_x.(f(t,x,\xi)\xi)\left(|\bar{v}|^2 - 2\xi.\bar{v}\right)dxd\xi$$
$$+\int\nabla_\xi.\left(\frac{1}{\varepsilon}\nabla\bar{\varphi}f(t,x,\xi)\xi\right)\left(|\bar{v}|^2 - 2\xi.\bar{v}\right)dxd\xi$$
$$+\int Q(f^\varepsilon, f^\varepsilon)\left(|\bar{v}|^2 - 2\xi.\bar{v}\right)dxd\xi$$

The last term is equal to zero from the property of Boltzman Operator [1,3,5,7-9,11].

It follows by integrating by party that

$$\int\partial_t f\left(|\bar{v}|^2 - 2\xi.\bar{v}\right)dxd\xi = 2\int f(t,x,\xi)\xi\nabla\bar{v}(\bar{v} - \xi)dxd\xi + 2\int f(t,x,\xi)\frac{1}{\varepsilon}\nabla\tilde{\varphi}.\bar{v}\xi dxd\xi.$$

Thus

$$\frac{d}{dt}H\varepsilon = \int f(t,x,\xi)(\partial_t\bar{v}.\bar{v} - \partial_t\bar{v}.\xi)dxd\xi + \int f(t,x,\xi)\xi\nabla\bar{v}(\bar{v} - \xi)dxd\xi + \int f(t,x,\xi)\frac{1}{\varepsilon}\nabla\tilde{\varphi}.\bar{v}\xi dxd\xi$$
$$= -\int f(t,x,\xi)(\bar{v} - \xi)\nabla\bar{v}(\bar{v} - \xi)dxd\xi + \int f(t,x,\xi)(\partial_t\bar{v}.(\bar{v} - \xi))dxd\xi + \int f(t,x,\xi)\bar{v}\nabla\bar{v}(\bar{v} - \xi)dxd\xi + \int f(t,x,\xi)\frac{1}{\varepsilon}\nabla\tilde{\varphi}.\bar{v}\xi dxd\xi$$
$$= -\int f(t,x,\xi)(\bar{v} - \xi)\nabla\bar{v}(\bar{v} - \xi)dxd\xi + \int f(t,x,\xi)(\bar{v} - \xi)(\partial_t\bar{v} + \bar{v}\nabla\bar{v})dxd\xi + \int f(t,x,\xi)\frac{1}{\varepsilon}\nabla\tilde{\varphi}.\bar{v}\xi dxd\xi$$
$$= -\int f(t,x,\xi)(\bar{v} - \xi)\nabla\bar{v}(\bar{v} - \xi)dxd\xi - \int f(t,x,\xi)(\bar{v} - \xi)\nabla p dxd\xi + \int f(t,x,\xi)\frac{1}{\varepsilon}\nabla\tilde{\varphi}.\bar{v}\xi dxd\xi$$
$$= A + B + D$$

Let us begin with the first term A. Use Holder inequality and that $|\nabla\bar{v}| \le C$ to decompose

$$A = \int f^{\frac{1}{2}}(t,x,\xi)(\bar{v} - \xi).f^{\frac{1}{2}}(t,x,\xi)\nabla\bar{v}(\xi - \bar{v})dxd\xi$$

$$\le \left[\int f(t,x,\xi)|\xi - \bar{v}|^2 dxd\xi\right]^{\frac{1}{2}}.\left[C\left(\int f(t,x,\xi)|\xi - \bar{v}|^2 dxd\xi\right)\right]^{\frac{1}{2}}$$

$$\le C.H_\varepsilon^{\frac{1}{2}}.H_\varepsilon^{\frac{1}{2}}. = C.H_\varepsilon$$

From the second term D, one has

$$D = \iint f(t,x,\xi)\frac{1}{\varepsilon}\nabla\tilde{\varphi}.\bar{v}\xi dxd\xi = \frac{1}{\varepsilon}\int\rho\bar{v}\nabla\tilde{\varphi} = \frac{1}{\varepsilon}\int\bar{v}\nabla\tilde{\varphi}d\rho$$

From the definition of Φ, we have

$$D = \frac{1}{\varepsilon}\int\bar{v}(\nabla\Phi).\nabla\tilde{\varphi}(\nabla\Phi)dx = \frac{1}{\varepsilon}\int\bar{v}(x + \varepsilon\nabla\varphi)\nabla\varphi$$

$$= \frac{1}{\varepsilon}\int\bar{v}\nabla\Phi + \left(\bar{v}(x + \varepsilon\nabla\varphi) - \bar{v}(x)\right).\nabla\varphi.$$

Since $\bar{v}$ is divergence free, once gets

$$B \le 0 + C.\int|\nabla\varphi|^2 \le C.H_\varepsilon$$

Consider now the last term D.

$$D = -\int f(t,x,\xi)\bar{\upsilon}.\nabla p dxd\xi + \frac{1}{2}\int f(t,x,\xi)\xi.\nabla p dxd\xi$$

$$= -\int f(t,x)\bar{\upsilon}.\nabla p dxd\xi + \int f(t,x,\xi)\xi.\nabla p dxd\xi$$

But since $\bar{v}$ is divergence fre we have $\int\bar{v}.\nabla p = 0$. Thus form Lemma 4

$(G = \bar{v}.\nabla p)$, once has

$-\int\rho\bar{\upsilon}.\nabla p \le C.(\varepsilon^2 + H_\varepsilon)$.

Since it costs no generality to suppose that for all $t \in [0,T]$, $\int P(t,x)dx = 0$,

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we get from the equation of conservation of mass

$$\int f(t,x,\xi)\xi.\nabla p = -\int \nabla(f(t,x,\xi)\xi)p = \int \partial_t \rho.p = \frac{d}{dt}\int \rho p - \int \rho \frac{\partial \rho}{\partial t}$$

By Lemma4 and setting $Q(t)=-\int \rho p$ we can deduct that

$$\int f(t,x,\xi)\xi.\nabla p \le C.(\varepsilon^2 + H_\varepsilon) - \frac{dQ}{dt}$$

Thus

$$D \le C.(H_\varepsilon + \varepsilon^2) - \frac{dQ}{dt}$$

We deduce then the following inequality

$$\frac{d}{dt} = (H_\varepsilon + Q) \le CH_\varepsilon + \theta(\varepsilon^2) \quad (2.1)$$

Still using 4,

$$|Q(t)| = |\int \rho p| \le C\varepsilon^2 + \frac{1}{2}H_\varepsilon(t)$$

Thus

$$H_\varepsilon + Q \ge \frac{1}{2}H_\varepsilon(t) - C\varepsilon^2$$

So once can transform (2.1) as

$$\frac{d}{dt}(H_\varepsilon + Q) \le C(H_\varepsilon + Q) + C\varepsilon^2$$

And by Gronwall's inequality [11] yields

$$H_\varepsilon(t) + Q(t) \le (H_\varepsilon(0) + Q(0))exp(Ct)$$

Finally we conclude that

$$H_\varepsilon(t) \le C(H_\varepsilon(0) + C\varepsilon^2)exp(Ct)$$

Which achieves the proof of the theorem.

References

1. Cercignani C (1988) The Boltzmann equation. The Boltzmann Equation and Its Applications. Springer New York, pp: 40-103.
2. Cercignani C, Illner R, Pulvierenti M, Bardos C (1995) The mathematical theory of dilute gases. SIAM Review 37: 622-623.
3. DiPerna RJ, Lions PL (1989) On the Cauchy problem for Boltzmann equations: global existence and weak stability. Annals of Mathematics 130: 321-366.
4. DiPerna RJ, Lions PL (1989) Global weak solutions of Vlasov-Maxwell systems. Communications on Pure and Applied Mathematics 42: 729-757.
5. Desvillettes L, Dolbeault J (1991) On long time asymptotics of the Vlasov-Poisson-Boltzmann equation. Communications in Partial Differential Equations 16: 451-489.
6. Glassey RT (1996) The Cauchy problem in kinetic theory. Society for Industrial and Applied Mathematics.
7. Guo Y (2002) The Vlasov-Poisson-Boltzmann system near Maxwellians. Communications on Pure and Applied Mathematics 55: 1104-1135.
8. Yang T, Zhao HJ (2006) A new energy method for the Boltzmann equation. Journal of Mathematical Physics 47: 053301.
9. Yang T, Yu H, Zhao H (2006) Cauchy problem for the Vlasov Poisson Boltzmann system. Archive for Rational Mechanics and Analysis 182: 415-470.
10. Sone Y (2012) Kinetic theory and fluid dynamics. Springer Science & Business Media, Boston.
11. Hsiao L, Li FC, Wang S (2007) Convergence of the Vlasov–Poisson–Boltzmann system to the incompressible Euler equations. Acta Mathematica Sinica English Series 23: 761-768.
12. Brenier Y (1991) Polar factorization and monotone rearrangement of vector-valued functions. Communications on Pure and Applied Mathematics 44: 375-417.
13. Loeper G (2004) A geometric approximation to the Euler equations: the Vlasov–Monge–Ampere system. Geometric & Functional Analysis GAFA 14: 1182-1218.