Fractal Functions in Interpolation and Approximation: A Bird’s-Eye View

Navascues MA1*, Chand AKB2, Viswanathan P3 and Sebastan MV4

1Departamento de Matemática Aplicada, Escuela de Ingeniería y Arquitectura, Universidad de Zaragoza, Spain
2Department of Mathematics, Indian Institute of Technology Madras, Chennai-600036, India
3Mathematical Sciences Institute, Australian National University, Australia
4Centro Universitario de la Defensa de Zaragoza Academia General Militar, Spain

Abstract

This article is targeted to provide a brief and coarse discussion on theory of fractal functions and their recent developments including some of the researches of the authors. Materials on fractal functions provided by this paper is not claimed to be exhaustive, but is intended to be read before or in parallel with technical papers available in the literature on this subject. We have made an earnest attempt to supply an overall flavor of the subject of univariate fractal approximation and useful source of links to assist a novice and perhaps to an expert in fractal functions too.

Prologue

Construction of a continuous function from given discrete data, which is referred to as interpolation, has been applied in various fields since time immemorial. In classical Numerical Analysis and Approximation Theory, there are several interpolation methods (for instance, polynomial, trigonometric, spline, rational) that can be applied to a specific data set, according to the assumptions that underlie the model to be investigated. Undoubtedly, these traditional (non-fractal) nonrecursive techniques constitute a great tool to model various physical phenomena. However, these methods produce interpolants with Hausdorff dimension one and hence if a given data set is more complex and irregular (for instance, data sampled from real-world signals such as financial series, seismic data, and bioelectric recordings), then they may not provide satisfactory results. Consequently, to model these signals, need for interpolants that are nondifferentiable in dense set of points in the domain arose quite naturally. About a quarter century has passed since Barnsley [1] introduced Fractal Interpolation Function (FIF), a concept which soon turned out to be the right frame for a constructive approximation theory of non-differentiable functions, by using the notion of Iterated Function System (IFS). In view of their diverse applications, there has been steadily increasing interest in the theory of fractal functions, and it still continues to be a hot topic of research. Following the publication of Fractals Everywhere [2], a beautiful exposition of IFS theory, fractal functions and their applications, various related issues such as calculus, Holder continuity, convergence, stability, smoothness, determination of scaling parameters, and perturbation error have been investigated in the literature [3-13]. The concept of smooth FIFs has been used to generalize the traditional splines [14-18] and to demonstrate that the interaction of classical numerical methods with fractal theory provides new interpolation schemes that supplement the existing ones. Various other extensions of FIFS include multivariable FIFS [7,9,16,19-25] generated by using higher-dimensional or recurrent IFSs, the hidden variable FIFS produced by projecting the attractors of vector valued IFSs to a lower-dimensional space.

The notion of FIFS can be used to associate a family of fractal functions $f^\alpha$ parameterized by a suitable vector $\alpha$ with a prescribed continuous function $f$ on a compact interval $I$. Using this “fractal perturbation”, Navascues introduced the operator $F^\alpha: c(I) \rightarrow c(I)$ via $F^\alpha(f) = f^\alpha$ and its extension to more general spaces $I^\alpha(f)$ using standard density arguments. These maps tend to bridge the gap between smoothness of classical mathematical objects and pseudo-randomness of experimental variables, breaking in this way their apparent diversity. Navascues and coworkers [17,18,22] contributed to the theory by defining “rough” approximants as perturbation of the functions generally used in classical approximation (polynomial, trigonometric, rational, etc.) via this operator. Recently [23], the authors investigated conditions on the elements of the IFS so that the perturbation preserves the basic shape properties inherent in the original function, thereby paving a way to shape preserving fractal approximation, an interesting area that needs further exploration at least in our opinion.

A Brief Outline of Fractal Interpolation Function

By fractal function, we mean basically a function whose graph is the attractor of an IFS, which was first constructed as an interpolant as follows. Consider a set of data points $(x_n, y_n) \in \mathbb{R}^2 : n=1,2,...,N$ with increasing abscissae. Set $I = [x, y] = [0,1]$ and let $L_i : I \rightarrow I$ be affine maps such that $L_i(x) = x_i, L_i(y) = y_i$, i.e. $K = I \times [c, d]$. Consider $N-1$ continuous maps $F_i : K \rightarrow [c, d]$ satisfying

$$|F_i(x, y) - F_i(x, y*)| \leq r_i |y - y*|; |F_i(x, y) - y| = r_i |F_i(x, y) - y| + 1 \quad (1)$$

where $x \in I, y, y* \in [c, d]$, and $0 \leq r_i < 1$. Now define $W_i ; K \rightarrow I \times [c, d]$ and let $W_i(x, y) = (L_i(x), F_i(x, y)) \forall y \in \mathbb{N}_{n+1}$

For the collection $I = \{K; W_i : i \in \mathbb{N}_{n+1}\}$, which is termed IFS, Barnsley presented the following seminal result.

Theorem 1 (Barnsley [1]) Corresponding to the IFS

$$I = \{K; W_i : i \in \mathbb{N}_{n+1}\} \text{ there exists a unique set } G \text{ satisfying }$$

$$G = \bigcup_{i=0}^{n+1} W_i(G) \text{ and } G \text{ is the graph of a continuous function } g : I \rightarrow \mathbb{R} \text{ satisfying } g(x) = y_n \text{ for } n=1, 2,...,N.$$
Fractal Functions in Approximation and Fractal Operator

Revising the spirit of the previous section, let \( f \in c(I) \) Consider (2) with \( g(x) = f \circ L(x) - \alpha f(x) \),

\[ f^\alpha(x) = f(x) + \alpha (f^\alpha - b) \circ L^\alpha(x) \forall x \in I, i \in \mathbb{N}_{-1} \]

Notice that if \( \alpha := 0 \in \mathbb{R}^{1} \) then \( f^\alpha = f \) and also that \( \alpha \in (-1,1) \) is arbitrary. Consequently, (4) associates a family of continuous functions with each fixed \( f \in c(I) \). The degrees of freedom offered by this procedure may be useful when some problems combined with approximation and optimizations have to be approached. Assume that the continuous function \( b \) depends linearly on \( f \), say for instance, \( b = b_f \) where \( L : c(I) \rightarrow c(I) \) is a linear and bounded operator with respect to the uniform norm or \( L^\alpha \) norm on \( C(I) \). Then the map \( F^\alpha : c(I) \rightarrow c(I), F^\alpha(f) = f^\alpha \) defines a linear operator referred to as an \( \alpha \)-fractal operator. For various properties of this map, its extension (3) associates a family of fractal functions with schemes available in the traditional Numerical Analysis. It is plain to see that the presence of parameters can be explored to choose best in a problem of approximation combined with optimization, (ii) the perturbation methodology described may serve to modify or preserve the properties of \( f \), for instance, to reduce the order of regularity of the classical approximant, (iii) the procedure provides new geometric possibilities, that is, in the non-smooth case, the graph owns a fractal dimension and this parameter provides an index for experimental signals.

Let us conclude this short article by hinting at the applications of fractal functions. Having provided the above brief introduction to fractal interpolation and its interconnection with classical approximants, it is not difficult to find applications of FIFs in almost every field wherein information available in finite number of grid points has to be modeled with a continuous function. The realm of applications includes geometric design, data visualization, reverse engineering, physics and chemistry, image compression, signal processing, and wavelet theory. The reason for this variety of applications lies in the underlying complicated mathematical structure of fractal functions, produced with simple recursive construction. It has been noted that for certain problems, they provide better approximants than their classical nonrecursive counterparts. For a specific application, let us mention that in reference [21], fractal interpolation of electroencephalographic recordings is used to describe the increase in the bioelectric complexity during some tests of attention in children and to compute other electroencephalographic parameters. Many areas of fractal functions and their applications are yet to be explored, for instance, calculating Hausdorff dimension of a general FIF is a challenging open problem.

We strongly believe that the field of fractal functions has bright future, however to be able to contribute to it, the reader should leave the idea of staying in the comfort of well-known classical approximation theory and start enjoying the benefits of the more versatile fractal approximants, to supplement the former if not to replace.

References