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Introduction

1) A digraph $D$ consists of a set of elements, called vertices and a list of ordered pairs of these elements, called arcs. The set of vertices is called the vertex set of $D$, denoted by $V(D)$, and the list of arcs is called the arc list of $D$, denoted by $A(D)$. If $v$ and $w$ are vertices of $D$, then an arc of the form $vw$ is said to be directed from $v$ to $w$ [1].

2) A dibipartite graph is a digraph whose vertex set can be split into sets $A$ and $B$ in such a way that each arc (directed edge) of the digraph runs from a vertex in $A$ to a vertex in $B$ (or a vertex of $B$ to a vertex of $A$). We can distinguish the vertices in $A$ from those in $B$ by drawing the former in black and the latter in white, so that each arc is incident from a black (or white) vertex to a white (or black) vertex (Figure 1).

3) A complete dibipartite graph is a dibipartite graph in which each black (or white) vertex is joined to each white (or black) vertex by exactly one arc. The complete dibipartite graph with $r$ black vertices and $s$ white vertices is denoted by $K_{rs}$. We call a complete dibipartite graph of the form $K_{1,s}$ star sink or star source digraphs (Figure 2).

4) Let $D_1$ and $D_2$ be digraphs and $f: D_1 \rightarrow D_2$ is a continuous function. Then $f$ is called a digraph map if,

- a) For each vertex $v \in V(D_1)$, $f(v)$ is a vertex in $V(D_2)$.
- b) For each arc $e \in A(D_1)$, $\dim(f(e)) \leq \dim(e)$.

Folding of Dibipartite Graphs

Definition

Let $D_1$ and $D_2$ be simple digraphs, we call a digraph map $f: D_1 \rightarrow D_2$ a digraph folding if $f$ maps vertices to vertices and arcs to arcs, i.e., for each $v \in V(D_1)$, $f(v) \in V(D_2)$ and for each $e \in A(D_1)$, $f(e) \in A(D_2)$.

If the digraph contains loops, then the digraph folding must send loops to loops but of the same direction. We denote the set of digraph foldings between digraphs $D_1$ and $D_2$ by $\mathcal{F}(D_1, D_2)$ and the set of digraph foldings of $D$ into itself by $\mathcal{F}(D)$. In the case of a digraph folding $f$ the set of singularities, $\Sigma_f$, consists of vertices only [2]. The digraph folding is non-trivial if $\Sigma_f \neq \emptyset$. In this case the no. $V(f(D_1)) \leq \text{no. } V(D_1)$, also no. $A(f(D_1)) \leq \text{no. } A(D_1)$.

Example

Let $D$ be the digraph shown in Figure 3. Then the graph map $f: D \rightarrow D$ defined by $f(v_1, \ldots, v_4) = (v_3, v_2, v_3, v_4)$ and $f(e_1, e_2, e_3, e_4, e_5) = (e_4, e_3, e_3, e_4, e_5)$ is a digraph folding. The image $f(D)$ is shown in Figure 3. From now on the omitted vertices or arcs will be mapped into themselves.

Theorem

Any dibipartite graph $D$ can be folded.

Proof:

Let $D$ be a dibipartite graph, then the vertex set $V(D)$ can be split into two sets $A$ and $B$. Let $f: D \rightarrow D$ be a digraph map such that $f$ maps vertices of $A$ to vertex of $A$, say $u$, and vertices of $B$ to a vertex of $B$, say $v$. Thus each arc $e$ will be mapped to the arc $f(e) = (u, v)$, where $u \in V(A)$ and $v \in V(B)$ and hence $f$ is a digraph folding.
Example
Let $D_1$ be the dibipartite graph shown in Figure 4. A digraph folding $f : D_1 \rightarrow f(D_1)$ can be defined as follows $f(v_1,v_2,v_3,v_4) = (v_5,v_2,v_2)$ and $f(e_1,e_2,e_5,e_6) = (e_4,e_3,e_4,e_4)$. The image $f(D_1) = D_2$ is shown in Figure 4.

Theorem
Any complete dibipartite graph $D$ can be folded to an arc.

Proof:
Let $D$ be a complete dibipartite graph with vertex set $V(D) = \{v_1,v_2,\ldots,v_r,v_{r+1},\ldots,v_{r+s}\}$. This set again can be split into two sets, $A = \{v_1,v_2,\ldots,v_r\}$ and $B = \{v_{r+1},\ldots,v_{r+s}\}$, such that each vertex of $A$ is joined to each vertex of $B$ by exactly one arc.

Thus
$$A(D) = \{v_1,v_2,\ldots,v_r\} \times \{v_{r+1},\ldots,v_{r+s}\} = \{v_1,v_2,\ldots,v_r\} \times \{v_{r+1},\ldots,v_{r+s}\} \times \{v_{r+1},\ldots,v_{r+s}\}$$

Now let $f : D \rightarrow D$ be a digraph map defined by

$$f(v_i) = \begin{cases} v_1, & \text{if } k = 1,\ldots,r \\ v_{i+1}, & \text{if } k = r + 1,\ldots,r \end{cases}$$

Thus the image of any arc of $A(D)$ will be the arc $(v_i,v_{i+1})$. Of course, this map is a digraph folding.

Example
Consider the complete dibipartite graph $K_{4,4}$ shown in Figure 5. A digraph folding $f$ of $K_{4,4}$ into itself may be defined as follows $f(v_1,v_2,v_3,v_4) = (v_1,v_2,v_3,v_4)$ and $f(e_1,e_2) = (e_1,e_2)$. This may be done by the composition of the two digraph folding $f_1$ and $f_2$ shown in Figure 5.

The Diagraph Folding and Adjacency Matrix

Definition
Let $D$ be a digraph without loops, with $n$ vertices labeled 1,2,3,...,$n$. The adjacency matrix $M(D)$ is the $n \times n$ matrix in which the entry in row $i$ and column $j$ is the number of arcs from vertex $v_i$ to vertex $v_j$ [1]. For example if $D$ is the digraph shown in Figure 6, then the matrix $M(D)$ will be given by

$$M(D) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}$$

Note that every entry on the main diagonal (top-left to bottom-right) is 0, since the digraph has no loops.

Proposition
Let $D$ be a connected digraph without loops with $n$ vertices. Then a digraph folding of $D$ into itself may be defined, if there is any, as a digraph map $f$ of $D$ to an image $f(D)$ by mapping:

1) The multiple arcs into one of its arc.

2) a. The vertex $v_i$ to the vertex $v_j$ if the numbers appearing in the adjacency matrix in the $i$th and $j$th rows (or columns) are the same.

b. The vertex $v_i$ to the vertex $v_j$ if the entries of the $i$th and $j$th rows are zeros and if the $i$th and $j$th columns are the same, or there exists a row $k$ which has numbers 1 in the $i$th and $j$th columns.

3) a. The arc $(v_i,v_k)$ to the arc $(v_j,v_k)$ if the $i$th and $j$th rows (or columns) are the same.

b. The arc $(v_i,v_j)$ to the arc $(v_i,v_k)$ if the $j$th and $k$th columns (or rows) are the same.

In general the arc $(v_i,v_j)$ will be mapped to the arc $(v_i,v_k)$ if $v_i$ mapped to $v_j$ and $v_k$ mapped to $v_j$.

Examples
(a) Let $D_1$ be the digraph shown in Figure 7.

The adjacency matrix $M(D_1)$ is given by Figure 7.

The adjacency matrix $M(D_1)$ is given by

$$M(D_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}$$

Figure 3: Digraph.

Figure 4: Dibipartite graph.

Figure 5: Digraph folding.

Figure 6: Digraph without loops.
Then we can fold first $D_1$ by folding the multiple arc into itself to get the digraph $D_2$. In this case $M(D_2)$ is nothing but $M(D_1)$ after replacing the number 2 by the number 1. Then a digraph folding $g : D(D_2) \to D(D_1)$ can be defined by using $M(D_1)$ by mapping the vertex $v_i$ to the vertex $v_j$ since the first and the third row of $M(D_1)$ have the same entries. Thus the arcs $(v_i,v_j)$ and $(v_j,v_i)$ will be mapped to the arcs $(v_i,v_j)$ and $(v_j,v_i)$ respectively, since the first and the third row are the same.

(b) Let $D$ be the digraph shown in Figure 8.

The adjacency matrix $M(D)$ is given by

$$M(D) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Then a digraph folding $g : D \to D$ can be defined by using $M(D)$ by mapping the vertices $v_i$, $v_j$, and $v_k$ to $v_{i'}$, $v_{j'}$, and $v_{k'}$ respectively. Also the arcs $(v_i,v_j)$ and $(v_j,v_i)$ will be mapped to the arcs $(v_{i'},v_{j'})$ and $(v_{j'},v_{i'})$ respectively since $g(v_i)=v_{i'}$, $g(v_j)=v_{j'}$, $g(v_k)=v_{k'}$. Also the image of the arc $(v_i,v_j)$ is $(v_{i'},v_{j'})$ since the first and third columns are the same. Finally the image of the arc $(v_j,v_i)$ is $(v_{j'},v_{i'})$ since the second and fourth rows are the same, and so on. See the adjacency matrix $M(D)$.

References