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Fixed Point Theory for Three φ – Weak Contraction Functions

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Abstract

In 2009 Qingnian Zhang and Yisheng Song proved fixed point theory for two φ–weak contraction functions. We generalize this result by finding fixed point and coincidence point for three single-valued φ–weak contraction T_1, T_2, T_3 defined on a complete metric space.

Keywords: Complete metric space; Coincidence point; Fixed point; ϕ–Weak contraction function

Introduction and Preliminaries

Let metric space be E. A map $T: E \to E$ is a contraction if for each $x, y \in$

E, with a constant $k \in (0,1)$ so that

$d(Tx, Ty) \le kd(x,y)$.

A map $T: E \to E$ is a φ–weak contraction, there exists a function

$\varphi: [0,\infty)\to[0,\infty)$

$\varphi(0)=0$ and $d(T_x,T_y) \le d(x,y)-\varphi(d(x,y))$

for each $\varphi(d(x,y))$ for each $x, y \in E$

Alber and Guerre-Delabriere [1] demonstrated the "concept of weak contraction" in 1997. Actually in concept of weak contraction, the authors defined such mappings for single-valued maps on Hilbert spaces and proved the existence of fixed points. Rhoades [2] showed that most results of concept of weak contraction are still true for any Banach space along with that he has proved the following very interesting fixed point theorem which is one of generalizations of the Banach contraction principle because it contains contractions as special cases ($\varphi(t)=(1-k)t$).

Theorem 1.1

Let A be a φ–weak contraction on a complete metric space (E, d). *If* $\varphi: [0,\infty)\to[0,\infty)$ is a continuous and no decreasing function with $\varphi(t)> 0$ for all $t(0,\infty)$ and $t\varphi(0)=0$ then A has a unique fixed point [2].

In fact, the weak contractions are also closely related to maps of Boyd and Wong type ones [3] and Reich type ones [4].

The aim of this work is to prove that there is a unique common fixed point for three single-valued φ–weak T_1, T_2, T_3 defined on a complete metric space E.

Main Results

Theorem 2.1

Let (E, d) be a complete metric space, and T_1, T_2, T_3: E → E three mappings such that for all x, y ∈E

$$\left.\begin{aligned} d(T_1x,T_2y)\le M_1(x,y)-\varphi\big(M_1(x,y)\big),\\ d(T_2x,T_3y)\le M_2(x,y)-\varphi\big(M_2(x,y)\big)\end{aligned}\right\} \tag{2.1}$$

where $\varphi: [0,+\infty) \to [0,+\infty)$ is a lower semi-continuous function with

$\varphi(t)>0$ for $t \in (0,+\infty)$, $\varphi(0)=0$, and

$$\left.\begin{aligned} M_1(x,y)=\max\left\{d(x,y),d(T_1x,x)d(T_2y,y)\tfrac{1}{2}\big[d(y,T_1x)+d(x,T_2y)\big]\right\}\\ M_2(x,y)=\max\left\{d(x,y),d(T_2x,x)d(T_1y,y)\tfrac{1}{2}\big[d(y,T_2x)+d(x,T_3y)\big]\right\}\end{aligned}\right\} \tag{2.2}$$

then there exists a unique coincidence point u∈E such that u=T_1u=T_2u=T_3u.

Proof

First, we show that $M_1(x,y)=0$ f and only if $x=y$ is a common fixed point of T_1 and T_2. Clearly, if $x=y=T_1x=T_1y=T_2x=T_2y$. then

On the other hand, Let $M_1(x,y)=0$. Then from

$d(x,y)=d(T_1x,x)=d(T_2y,y)=d(y,T_1x)=d(y,T_2y)=0$

So, we get

$x=y=T_1x=T_1y=T_2x=T_2y$.

Similarly, $M_2(x,y)=0$ if and only if $x=y$ is a common fixed point of T_1 and T_2.

For x_0E, Putting $x_1=T_2x_0$ and $x_2=T_1x_1$, then let $x_3=T_2x_2$ and $x_4=T_1x_3$. Inductively, choose a sequence $\{x_n\}$in E so that $x_{2n+2}=T_1x_{2n+1}$ and $x_{2n+1}=T_2x_{2n}$ for all $n \ge 0$.

It follows from the property of the φ that if n is odd and for $x_n \ne x_{n-1}$, then

$d(x_{n+1},x_n)=d(T_1x_n,T_2x_{n-1}) \le M_1(x_n,x_{n-1})-\varphi(M_1(x_n,x_{n-1}))\le$

$$\max\left\{d(x_n,x_{n-1}),d(T_1x_n,x_n)d(T_2x_{n-1},x_{n-1}),\tfrac{1}{2}\big[d(x_{n-1},T_1x_n)+d(x_{n-1},T_2x_{n-1})\big]\right\}$$

$$\max\left\{d(x_n,x_{n-1}),d(x_{n+1},x_n)d(x_n,x_{n-1}),\tfrac{1}{2}\big[d(x_{n-1},x_{n+1})+d(x_n,x_n)\big]\right\}$$

$$\max\left\{d(x_n,x_{n-1}),d(x_{n+1},x_n),\tfrac{1}{2}\big[d(x_{n-1},x_n)+d(x_n,x_{n+1})\big]\right\}$$

$$\max\left\{d(x_n,x_{n-1}),d(x_{n+1},x_n)\right\}$$

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We show $d(x_n,x_{n-1}) \geq d(x_{n+1},x_n)$.

Otherwise if $d(x_n,x_{n-1}) < d(x_{n+1},x_n)$ then from (2.1), we get, and furthermore

$M_1(x_n,x_{n-1})=d(x_{n+1},x_n)$

$d(x_{n+1},x_n)=d(T_1x_n,T_2x_{n-1}) \leq M_1(x_n,x_{n-1})-\varphi(M_1(x_n,x_{n-1}))$

then

$d(x_{n+1},x_n) \leq d(x_{n+1},x_n) - \varphi(d(x_{n+1},x_n))$,

Which gives a contradiction.

Similarly, if n is even, we also obtain that

$d(x_{n+1},x_n) \leq M_1(x_n,x_{n-1}) \leq \mathrm{d}(x_n,x_{n-1})$.

Therefore, for all $n \geq 0$, $d(x_{n+1},x_n) \leq M_1(x_n,x_{n-1}) \leq \mathrm{d}(x_n,x_{n-1})$ and

$\{d(x_{n+1},x_n)\}$is monotone nonincreasing and bounded from below. So,

$\exists r\geq 0$ such that

$$\varphi(r) \leq \liminf_{n\to\infty} \varphi\left(M1\left(x_n,x_{n-1}\right)\right)$$

Then (by the lower semi-continuity φ)

$$\varphi(r) \leq \liminf_{n\to\infty} \varphi\left(M1\left(x_n,x_{n-1}\right)\right)$$

We claim that r=0. In fact, taking upper limits as $n \to \infty$ on either side of the following inequality:

we get

$d(x_{n+1},x_n) \leq M_1(x_n,x_{n-1})-\varphi(M_1(x_n,x_{n-1}))$

$$r \leq r - \liminf_{n\to\infty} \varphi\left(M_1\left(x_n,x_{n-1}\right)\right) \leq r - \varphi(r)$$

i.e. $\varphi(r) \leq 0$. Thus $\varphi(r)$=0 by the property of the function φ, hence

$$r = \lim_{n\to\infty} d\left(x_{n+1},x_n\right) = 0 \tag{2.3}$$

Next we show that $\{x_n\}$ is a Cauchy sequence. Let

Then $\{c_n\}$ is decreasing. Suppose contrarily that $\lim_{n\to\infty} c_n = c > 0$. Choose $\varepsilon > \frac{c}{8}$ small enough and

$c_n=sup\{d(x_i,x_j):i,j\geq n\}$.

select N such that for all $n\geq N$,

and

$d(x_{n+1},x_n)<\varepsilon$; $cn<c+\varepsilon$

By the definition of C_{N+1}, there exists $n,m\geq N+1$ such that

$d(x_m,x_n)>c_n-\varepsilon \geq c-\varepsilon$

Replace x_m by x_{m+1} if necessary. We may assume that m is even, n is odd, $d(x_m,x_n) > c-2\varepsilon$. Then, and

$d(x_m,x_n)=d(T_1x_{m-1},T_2x_{n-1}) \leq M_1(x_{m-1},x_{n-1})\ \varphi(M_1(x_{m-1},x_{n-1})) \leq$

$$\max\left\{d\left(x_{m-1},x_{n-1}\right),d\left(T_1x_{m-1},x_{m-1}\right)d\left(T_2x_{n-1},x_{n-1}\right),\frac{1}{2}\left[d\left(x_{n-1},T_1x_{m-1}\right)+d\left(x_{m-1},T_2x_{n-1}\right)\right]\right\}-\varphi\left(M_1\left(x_{m-1},x_{n-1}\right)\right)$$

$$\max\left\{d\left(x_{m-1},x_{n-1}\right),\varepsilon,\varepsilon,\frac{1}{2}\left[d\left(x_{n-1},x_{m-1}\right)+d\left(x_{m-1},T_2x_{n-1}\right)\right]\right\}-\varphi\left(\frac{c}{2}\right)$$

We have proved that $C_N - \varphi\left(\frac{c}{2}\right) > C_{N+1}$ (if ε is small enough). This is impossible. Thus, we must have c=0. That is, the sequence $\{x_n\}$ is a Cauchy sequence. $M_1(x_n,x_{n-1})$

It follows from the completeness of E that there exists $u\in E$ such that

$X_n \to u$ as $n \to \infty$

Now we proved that $u=T_1u=T_2u$. Indeed, suppose $u\neq T_1u$; then for $d(u) > 0$, $\exists N_1\in\mathbb{N}$ such that for any $n > N1$,

$$d(x_{2n+1},u) < \frac{1}{2}d\left(u,T_1u\right), d(x_{2n},u) < \frac{1}{2}d\left(u,T_1u\right),$$

$$d(x_{2n},x_{2n+1}),\frac{1}{2}d\left(u,T_1u\right)$$

Accordingly,

$$d\left(u,T_1u\right)=d\left(T_2x_{2n},T_1u\right)\leq M1\left(u,x_{2n}\right)$$

$$=\max\left\{d\left(u,x_{2n}\right),d\left(u,T_1u\right),d\left(x_{2n},x_{2n+1}\right)\frac{1}{2}\left[d\left(u,x_{2n+1}\right)+d\left(x_{2n},T_1u\right)\right]\right\}$$

$$\leq\max\left\{\frac{1}{2}d\left(u,T_1u\right),d\left(u,T_1u\right),\frac{1}{2}d\left(u,T_1u\right),\frac{1}{2}\left[\frac{1}{2}d\left(u,T_1u\right)+\frac{1}{2}d\left(u,T_1u\right)+d\left(u,T_1u\right)\right]\right\}$$

$$=d\left(u,T_1u\right),$$

that is, $M_1(u_nx_{2n})=d(u,T_1u)$.

Since,

$d(x_{2n+1},T_1u)=d(T_2x_{2n},T_1u)\leq M_1(u,x_{2n})-\varphi(M_1(u,x_{2n}))$

then letting $n \to \infty$, we have

$d(u,T_1u)\leq d(u,T_1u)-\varphi(d(u,T_1u))$

We obtain a contradiction. Hence $u=T_1u$. As

$d(u,T_2u)=d(T_1u, T_2u)\leq M_1(u,u)-\varphi(M_1(u,u))$

$\leq d(u,T_2u)-\varphi(d(u,T_2u))$

$< d(u,T_2u)$

Therefore, $d(u,T_2u)$=0, i.e., $u=T_1u=T_2u$. If there exists another point $v\in E$ such that $v=T_1v=T_2v$, then using an argument similar to the above we get

$d(u,v)=d(T_1u,T_2v) \leq M_1(u,v) - \varphi(M_1(u,v))$

$\leq d(u,v)-\varphi(d(u,v))$

Hence $u=v$.

By the same way over $M_2(x, y)$ there exists a unique point $w\in E$ such that w=T_2w=T_2w

We will show that u=w.

d(u, T_1u, T_2w) $\leq$ M1(u, w) $-$ φ(M1(u, w))

$$< \max\left\{d\left(u,w\right),0,0,\frac{1}{2}\left[d\left(w,T_1u\right)+d\left(u,T_2w\right)\right]\right\}$$

$=d(u,w)$

Then $u=w$ so $u=T_1$u=T_2u=T_3u.

The proof is completed.

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