Fixed Point Theorem of Iterated Function System

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Abstract
In 2012, Bessem, Calogero and Pasquale proved some fixed point theorem which improved the Banach contraction theorem. Iterated function systems (IFS) were introduced by Hutchinson in 1981 as a natural generalization of the well-known Banach contraction principle. The purpose of this paper is to study the iterated function system using Bessem, Calogero and Pasquale mappings.

Keywords: Fixed points; α-φ-contraction mapping; Iterated function systems; α-admissible

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Introduction
The most well-known result in the theory of fixed points is Banach’s contraction mapping principle. Bessem, Calogero and Pasquale [1] proved the theorem of existence of fixed point of α - φ - contraction mapping in complete metric space. They discussed the Banach contraction principle with some generalized contraction conditions and weakened the usual contraction condition.

Many researches [1-4] have studied the fixed point theorem on the complete metric space (X, d), however, there are few results for the existence of fixed point on the complete metric space (H(X), h) with the use of fixed point theorem on (X, d) [5-14].

The aim of this paper is to obtain the fixed point theorems of the some generalized contractions in complete metric space (H(X), h). Before we establish the fixed point theorems in metric space (H(X), h), we discuss some basic results. In [1], Bessem Samet, Calogero and Pasquale Vetro proved the following results in the complete metric space.

Denote with Φ the family of non-decreasing functions \( \phi: [0, +\infty) \rightarrow [0, +\infty) \) such that \( \sum_{n=0}^{\infty} \phi^n(t) < +\infty \) for each \( t > 0 \), where \( \phi^n \) the n-th iterate of \( \phi \).

Definition 3.1
Let \((X, d)\) be a metric space and \( f: X \rightarrow X \) be a given mapping.

We say that \( f \) is an α - φ - contraction mapping if there exist two functions \( \alpha: X \times X \rightarrow [0, +\infty) \) and \( \phi \in \Phi \) such that \( \alpha(x, y) \delta(f(x), f(y)) \leq \phi(d(x, y)) \), for all \( x, y \in X \). If \( \alpha(x, y) = 1 \) for all \( x, y \in X \) and \( \phi(t) = kt \) for all \( t \geq 0 \) and some \( k \in [0,1) \), then \( f: X \rightarrow X \) satisfies the Banach contraction principle. There is example involving a function \( f \) that is not continuous [1].

If \( \alpha(x, y) = 1 \) for all \( x, y \in X \) and \( \lim_{t \to +\infty} \phi(t) = 0 \) for all \( t > 0 \) (not necessarily \( \sum_{n=0}^{\infty} \phi^n(t) < +\infty \) for each \( t > 0 \)), then \( f: X \rightarrow X \) satisfies a condition of the Matkowski’s contraction theorem [14].

Definition 3.2
Let \( f: X \rightarrow X \) and \( \alpha: X \times X \rightarrow [0, +\infty) \). We say that \( f \) is α - admissible if \( x, y \in X \), \( \alpha(x, y) \geq 1 \implies \alpha(f(x), f(y)) \geq 1 \) [1].

Theorem 3.1
Let \((X,d)\) be a complete metric space and \( f: X \rightarrow X \) be an α - φ - contraction mapping satisfying the following conditions [1]:

1. \( f \) is α - admissible;
2. There exists \( x_0 \in X \) such that \( \alpha(x_0, f(x_0)) \geq 1 \);
3. \( f \) is continuous.

Then, \( f \) has a fixed point, that is, there exists \( x^* \in X \) such that \( f(x^*) = x^* \).

This fixed point theorem extended the results of Banach’s contraction principle and Matkowski’s fixed point theorem.

Main Results
We now give the theorems of fixed point of the some generalized contractions in the complete metric space \((H(X), h)\). Let \((X, d)\) be a metric space and \( H(X) \) the class of all nonempty compact subsets of \( X \). That is, \( H(X) \) denotes the space whose points are the compact subsets of \( X \), other than the empty set.

Define
\[ d(A,B) = \max\{ d(x, B) : x \in A \} \]
\[ d(A,B) = \text{the distance from the set } A \in H(X) \text{ to the set } B \in H(X) \]

Define \( h: H(X) \times H(X) \rightarrow [0, +\infty) \) by \( h(A, B) = \max(\max\{d(x, y) : x \in A \text{ and } y \in B \} , \max\{d(x, y) : x \in B \text{ and } y \in A \}) \)

for all \( A, B \in H(X) \). The metric space \((H(X), h)\) is complete provided that \((X, d)\) is complete [6-8]. The classical iterated function system (IFS) consists of a finite family of Banach contractions on \( X \) to itself. Then there is a unique nonempty compact invariant subset of \( X \) with respect to these contractions. In what follows, we extend these IFS by considering a family of α - φ - contractions.

We assume that \((X, d)\) is a complete metric space and \( f: X \rightarrow X \) is a continuous mapping on the metric space \((X, d)\) such that

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\[ h(A, B) = d(x', y') \Leftrightarrow h(f(A), f(B)) = d(f(x'), f(y')) \] for all \( A, B \in H(X) \).

Because \( f : X \to X \) is a continuous mapping on the metric space \((X, d)\), \( F \) maps \( H(X) \) into itself.

Define \( F_i : H(X) \to H(X) \) by \( F_i(f) = f(A) = f(A, B) = f(A) \) for all \( A, B \in H(X) \).

**Example 4.1**

Shows that there are continuous functions such that \( h(A, B) = d(x', y') \Leftrightarrow h(f(A), f(B)) = d(f(x'), f(y')) \) for all \( A, B \in H(X) \).

**Example 4.2**

Let \( X = (\infty, +\infty) \) and let \( f(x) = \frac{1}{3} x \), \( x \in (\infty, +\infty) \).

Then \( h(A, B) = d(x', y') \Leftrightarrow h(f(A), f(B)) = d(f(x'), f(y')) \) for all \( A, B \in H(X) \).

In the following example, we show that there are continuous functions on a complete metric space which are not continuous functions such that \( h(A, B) = d(x', y') \Leftrightarrow h(f(A), f(B)) = d(f(x'), f(y')) \) for all \( A, B \in H(X) \).

**Example 4.3**

Let \( X = (\infty, +\infty) \).

Let \( f(x) = 4x(1-x) \), \( A = [0, \frac{1}{8}] \) and \( B = [\frac{5}{8}, \frac{7}{9}] \). Then \( h(A, B) = \frac{7}{16} d(\frac{5}{8}, \frac{7}{9}) \) and
\[ h(f(A), f(B)) = \frac{3}{4} d\left(\frac{7}{16}, \frac{7}{9}\right) = \frac{7}{16} d\left(\frac{7}{9}, \frac{7}{8}\right) \cdot \]

That is, for some \( A, B \in H(X) \), \( h(A, B) = d(x', y') \Leftrightarrow h(f(A), f(B)) = d(f(x'), f(y')) \).

**Lemma 4.1**

If \( f \) is \( \alpha \) - admissible then \( F \) is \( \alpha \) - admissible.

**Proof:** Let \( \alpha(A, B) \geq 1 \).

Since \( \alpha(A, B) = \inf\{\alpha(x', y')| d(x', y') = h(A, B), x' \in X, y' \in Y \} \geq 1 \), for all \( x' \in A, y' \in B \) such that \( h(A, B) = d(x', y') \), \( \alpha(x', y') \geq 1 \).

Because \( f \) is \( \alpha \) - admissible, \( \alpha(f(x'), f(y')) \geq 1 \).

By definition of \( f \), if \( h(A, B) = d(x', y') \) then \( h(f(A), f(B)) = d(x', y') \).

Hence \( \alpha(F_i(A), F_i(B)) = \alpha(f(A), f(B)) \).

\[ = \inf\{\alpha(f(x'), f(y'))| d(x', y') = h(f(A), f(B)), x' \in A, y' \in B \} \cdot \]

\[ = \inf\{\alpha(f(x'), f(y'))| d(x', y') = h(f(A), f(B)), x' \in X, y' \in Y \} \cdot \]

For all \( x' \in A, y' \in B \) such that \( h(A, B) = d(x', y') \), \( \alpha(x', y') \geq 1 \).

So \( \inf\{\alpha(f(x'), f(y'))| d(x', y') = h(f(A), f(B)), x' \in X, y' \in Y \} \geq \alpha(f(A), f(B)) \geq 1 \).

Therefore, if \( A, B \in H(X) \), \( \alpha(A, B) \geq 1 \) then \( \alpha(F_i(A), F_i(B)) \geq 1 \).

That is, \( F_i \) is \( \alpha \) - admissible.

**Lemma 4.2**

If there exists \( X_0 \in X \) such that \( x_0' f(x_0) \geq 1 \), then there exists \( x_0 \in H(X) \) such that \( \alpha_0(A_0, F_i(A_0)) \geq 1 \).

**Proof:** Let \( A_0 := \{x_0 \in H(X) | \alpha_0(A_0, F_i(A_0)) = \alpha_0(A_0, f(A_0)) \} \)
\[ = \inf\{\alpha(x_0, f(x_0))| d(x_0, f(x_0)) = h(A_0, f(A_0)), x_0 \in A_0, f(x_0) \in f(A_0) \} \cdot \]
\[ = \alpha_0(x_0, f(x_0)) \geq 1 \cdot \]

The system \( \{X, f_i : i = 1, 2, \ldots, N\} \) consisting of a family of continuous maps \( f_i : X \to X \), will be called iterated function system, shortly IFS on \( X \).

We define \( F_{i_1 \ldots i_N} : H(X) \to H(X) \) by
\[ F_{i_1 \ldots i_N}(A) := \bigcap_{i=1}^N f_i(A) \cdot \]

for all \( A \in H(X) \). The image of \( \phi \neq A \subseteq X \) under \( f_i \) is given by \( f_i(A) := \bigcap_{x \in A} f_i(x) \).

We consider continuities that are meant in the Hausdorff sense.

Map \( F_{i_1 \ldots i_N} : H(X) \to H(X) \) is

(1) Contraction, if there exists a Lipschitz constant \( 0 \leq L < 1 \) such that
\[ h(F_{i_1 \ldots i_N}(A), F_{i_1 \ldots i_N}(B)) \leq L \cdot h(A, B) \] for all \( A, B \in H(X) \).

(2) Weak contraction, if there exists a comparison function \( \phi \in \Phi \) such that
\[ h(F_{i_1 \ldots i_N}(A), F_{i_1 \ldots i_N}(B)) \leq \phi(h(A, B)) \] for all \( A, B \in H(X) \).

(3) Continuous at \( H(X) \), if
\[ \forall A \in H(X), \forall \varepsilon > 0, \exists \delta(A, \varepsilon) > 0, \forall B \in H(X)(h(A, B) < \delta(A, \varepsilon)) \cdot \]
\[ h(F_{i_1 \ldots i_N}(A), F_{i_1 \ldots i_N}(B)) < \varepsilon \cdot \]

**Remark 4.1**

If \( f_i \) for all \( i = 1, 2, \ldots, N \) is Banach contraction then \( F_{i_1 \ldots i_N} \) is Banach contraction too [12].

If \( f_i \) for all \( i = 1, 2, \ldots, N \) is weak contraction then \( F_{i_1 \ldots i_N} \) is weak contraction too [14].

Let \( X, f_i : i = 1, 2, \ldots, N \) be a IFS consisting of continuous functions. Now, we give a counter-example involving a map \( F_{i_1 \ldots i_N} : H(X) \to H(X) \) that is not continuous.

**Counter-Example 4.1**

Consider the functions
\[ f_i(x) = \begin{cases} 0, & x \leq 0 \\ -2x, & x > 0 \end{cases} \]

and \( f_i(x) = \begin{cases} 0, & x \leq 0 \\ 2x, & x > 0 \end{cases} \)

Then, \( f_i^n(x) = f_i^2(x) = 0 \) for all \( n \geq 2 \) and for all \( x \in R \).

Moreover, since \( f_i(x) = \bigcup_{n=1}^\infty \bigcup_{i=1}^\infty f_i^n(x) = \bigcup_{i=1}^\infty \bigcup_{n=1}^\infty f_i^n(x) \)
and \( f_i(x) = \{0, 2x\} \) and \( f_i^2(x) = \{0, 4x\} \) and \( f_i^3(x) = \{0, 6x\} \)
for all \( x \neq 0 \).

\[ F_{i_1 \ldots i_N}(0) = \{0\} \]

and \( F^2_{i_1 \ldots i_N}(0) = \{0\} \) and \( F^3_{i_1 \ldots i_N}(0) = \{0\} \) and \( F^4_{i_1 \ldots i_N}(0) = \{0\} \).
Since for all fixed $x > 0$, $h(F_{n-1}^{2n-1}f_{i}(x)), F_{n-1}^{2n-1}f_{i}(0)) = (-2)^{2n-1}x \to +\infty$ and $h(F_{n-2}^{2n-2}f_{i}(x)), F_{n-2}^{2n-2}f_{i}(0)) = (-2)^{2n-2}x \to +\infty$. As $n \to +\infty$, $h(F_{n-1}^{2n-1}f_{i}(x)), F_{n-1}^{2n-1}f_{i}(0))$ is not convergence. It is clear that $F_{n-1}^{2n-1}f_{i}$ is not continuous at $[0]$. This demonstrates the discontinuity of $F_{n-1}^{2n-1}f_{i}$.

So we assume that for all $i = 1, 2, \ldots, N$,

$f : X \to X$ are continuous mappings such that $F_{n-1}^{2n-1}f : H(X) \to H(X)$ is a continuous mapping.

**Lemma 4.3**

$f : X \to X$ is an $\alpha - \phi$ - contraction mapping. Then

$\alpha_{n}(A, B) h(F_{n}(A), F_{n}(B)) \leq \phi(h(A, B))$.

That is, $F_{n} : H(X) \to H(X)$ is a $\alpha_{n} - \phi$ - contraction (with the same function $\phi$).

**Proof**

Let New let $A, B \in H(X)$. Then

$\alpha_{n}(A, B) d(F_{n}(A), F_{n}(B)) = \alpha_{n}(A, B) d(f(A), f(B))$

$= \alpha_{n}(A, B) \max_{i \in [n]} \min_{x, y \in [A]} d(f(x), f(y)) = \alpha_{n}(A, B) \max_{i \in [n]} \min_{x, y \in [A]} d(f(x), f(y))$

$\leq \alpha_{n}(A, B) \min_{x, y \in [A]} d(f(x), f(y)) \leq \phi(h(A, B))$.

Similarly $\alpha_{n}(A, B) d(F_{n}(B), F_{n}(A)) \leq \phi(h(A, B))$.

Hence

$\alpha_{n}(A, B) h(F_{n}(A), F_{n}(B)) \leq \phi(h(A, B))$.

This completes the proof.

**Theorem 4.1**

Let $(X, d)$ be a complete metric space and $f : X \to X$ be an $\alpha - \phi$ - contraction mapping satisfying the following conditions:

1. $f$ is $\alpha$ - admissible;
2. There exists $x_{0} \in X$ such that $\alpha(x_{0}, f(x_{0})) \geq 1$.

Then the mapping $F_{i} : A \to H(X)$ is an $\alpha_{n} - \phi$ - contraction (with the same function $\phi$) too from $H(X)$, into itself. That is, $F_{i}$ has fixed point and there exists $k \in H(X)$ such that $F_{i}(K) = K$.

**Proof:**

$(H(X), h)$ is a complete metric space [6,7] and by Lemma 1, Lemma 2 and Lemma 3, $F_{i} : H(X) \to H(X)$ is an $\alpha_{n} - \phi$ - contraction mapping satisfying the following conditions:

1. $F_{i}$ is $\alpha_{n}$ - admissible; (1) $f$ is $\alpha$ - admissible;
2. There exists $x_{0} \in X$ such that $\alpha(x_{0}, f(x_{0})) \geq 1$; (2) There exists $x_{0} \in X$ such that $\alpha(x_{0}, f(x_{0})) \geq 1$ for all $i = 0, 1, 2, \ldots, N$.
3. $F_{i}$ is continuous. Hence by Theorem 1.1, $F_{i}$ has a fixed point $F_{i}(K) = K \in H(X)$.

The next Theorem provides an important method for combining contraction mappings on $(H(X), h)$ to produce new contraction mappings on $(H(X), h)$.

**Theorem 4.2**

Let $(X, d)$ be a complete metric space and for $i = 1, 2, \ldots, N$,

$f_{i} : X \to X$ be an $\alpha - \phi$ - contraction mapping satisfying the following conditions:

1. $f_{i}$ is $\alpha$ - admissible for all $i = 1, 2, \ldots, N$;
2. There exists $x_{i} \in X$ such that $\alpha(x_{i}, f_{i}(x_{i})) \geq 1$ for all $i = 1, 2, \ldots, N$.

Then the mapping $F_{i} : H(X) \to H(X)$ is an $\alpha_{n} - \phi$ - contraction too from $H(X)$ into itself. That is, $F_{i}$ has fixed point $K \in H(X)$ such that $F_{i}(K) = K$.

**Proof:** We demonstrate the claim for $N = 2$. An inductive argument then completes the proof. Let $A, B \in H(X)$. By Lemma 2.3,

$\alpha_{n}(A, B) h(F_{n}(A), F_{n}(B)) \leq \phi(h(A, B))$.

Because for $A, B, C$ and $D$ in $H(X)$, $h(A \cup B, C \cup D) \leq \max\{h(A, C), h(B, D)\}$.

$\alpha_{n}(A, B) h(F_{n}(A), F_{n}(B)) = \alpha_{n}(A, B) h(F_{n}(A), F_{n}(B)) \leq \max\{\alpha_{n}(A, B) h(f_{n}(A), f_{n}(B)), \alpha_{n}(A, B) h(f_{n}(B), f_{n}(A))\} \leq \phi(h(A, B))$.

Hence $F_{i} : H(X) \to H(X)$ is an $\alpha_{n} - \phi$ - contraction mapping.

Let $A_{1} := [x_{1}] \in H(X)$. Since $\alpha_{n}(A_{1}, f_{1}(A_{1})) = \alpha_{n}(A_{1}, f_{1}(A_{1})) \geq 1$ and $\alpha_{n}(A_{1}, f_{1}(A_{1})) \geq 1$, $\alpha_{n}(A_{1}, f_{1}(A_{1})) \geq 1$.

So $F_{i}$ is $\alpha_{n}$ - admissible (cf. proof of Lemma 2.1).

Hence the result of theorem follows from the Theorem 1.1. This completes the proof.

**Corollary 4.1**

Let $(X, d)$ be a complete metric space and for all $i = 1, 2, \ldots, N$,

$f_{i} : X \to X$ be an $\alpha - \phi$ - contraction mapping satisfying the following conditions:

1. $f_{i}$ is $\alpha_{n}$ - admissible for all $i = 1, 2, \ldots, N$;
2. There exists $x_{i} \in X$ such that $\alpha(x_{i}, f_{i}(x_{i})) \geq 1$ for all $i = 1, 2, \ldots, N$.

Then the mapping $F_{i} : H(X) \to H(X)$ is an $\alpha_{n} - \phi$ - contraction too from $H(X)$ into itself. That is, $F_{i}$ has fixed point $K \in H(X)$ such that $F_{i}(K) = K$.

**Proof:** Since $\phi(t) = \max_{t \in [0, 1]} \phi(t)$ is non-decreasing continuous functions, it is obvious.

**References**