Exponential Stability of Nonlinear Nonautonomous Multivariable Discrete Systems

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Abstract

We consider a class of nonautonomous discrete-time systems governed by semilinear vector difference equations with slowly varying linear parts. Sharp exponential stability conditions are suggested. They are formulated in terms of the eigenvalues of the coefficients and constants characterizing the nonlinearities. Our approach is based on the recent norm estimates for solutions of matrix equations.

Introduction

The basic method for the stability analysis of nonlinear time-discrete systems is the Lyapunov functions one, cf. [1,2]. By this method many very strong results are obtained, but finding Lyapunov's functions is often connected with serious mathematical difficulties. Explicit stability stability conditions for nonlinear multivariable systems are established mainly in the case of autonomous linear parts, cf. [3-9].

In the present paper we investigate a class of discrete-time systems governed by semilinear vector difference equations with non-autonomous linear parts. Explicit stability conditions are suggested. They are formulated in terms of the eigenvalues of the coefficients and constants characterizing the nonlinearities. It is shown that the suggested stability conditions are sharp.

Let $\mathbb{C}^n$ be a Euclidean space of n-complex vectors endowed with a scalar product $(.,.)$ and the Euclidean norm $\|\cdot\| = \sqrt{(.,.)}$, I is the identity matrix and $\omega_r := \{h \in \mathbb{C}^n : \|h\| \le r\}$ for a positive $r \le \infty$. Our main object is the equation

$$u_{k+1} = A_k u_k + F_k(u_k) \ (k=0,1,\ldots), \tag{1}$$

Where $A_k (k=0,1,\ldots)$ are $n \times n$-matrices $F_k : \omega_r \mathbb{C}^n$ are mappings satisfying

$$\|F_k(w)\| \le v_k \|w\| \ (w \in \omega_r) \tag{2}$$

Where v_k are given non-negative constants.

The zero solution of system (1) is said to be exponentially stable if there are constants $M \ge 1, \zeta \in (0,1)$ and >0, such that $\|u_k\| \le M\zeta^k \|u_0\|$ $(k=1,2,\ldots)$ for any solution u_k of (1), provided $\|u_0\| < \delta$.

Introduce the notations. For an $n \times n$ matrix A, A^* is the adjoint one; $\lambda_k(A), k=1,\ldots,n$, are the eigenvalues of A, counted with their multiplicities; $\|A\| = \sup_{h \in \mathbb{C}^n} \|Ah\| / \|h\|$ is the spectral (operator) norm of A; $r_s(A)$ is the spectral radius. The following quantity (the departure from normality of A) plays a key role hereafter:

$$g(A) = [|A|_F^2 - \sum_{j=1}^n |\lambda_j(A)|^2]^{1/2},$$

Where $|A|_F = (Trace \ AA^*)^{1/2}$ is the Frobenius (Hilbert-Schmidt norm) of A. If A is a normal matrix: $AA^* = A^* A$, then $g(A)$.

Suppose that

$$\sup_{k \ge 1,2,\ldots} r_s(A_k) < 1 \tag{3}$$

and put

$$\mu_k := \sum_{m=0}^{n-1} \left(\sum_{j=0}^{n-1} \frac{g^j(A_k) m! r_s^{m-j}(A_k)}{(j!)^{3/2}(m-j)!} \right)^2.$$

If A_k is a normal matrix, then

$$\mu_k := \sum_{m=0} r_s^{2m}(A_k) = \frac{1}{1 - r_s^2(A_k)} \tag{4}$$

Now we are in a position to formulate our main result.

Theorem 1

Let the conditions (2), (3) and

$$\sup_k [\mu_k^2 \|A_k - A_{k-1}\| (\|A_{k-1}\| + \|A_k\|) + \mu_k(2\|A_k\| v_k + v_k^2)] < 1 \tag{5}$$

hold. Then the zero solution of (1) is exponentially stable.

The proof of this theorem is divided into a series of lemmas which are presented in the next two sections. Below we show that condition (5) is sharp.

From inequality (8) proved below it follows that

$$\mu_k \le \hat{\mu}_k \ , \ where \ \hat{\mu}_k := \left(\sum_{j=0}^{n-1} \frac{g^j(A_k)}{(j!)^{1/2}(1 - r_s(A_k))^{j+1}} \right)^2.$$

Now Theorem 1 implies

Corollary 1

Let the conditions (2), (3) and

$$\sup_k [\hat{\mu}_k^2 \|A_k - A_{k-1}\| (\|A_{k-1}\| + \|A_k\|) + \hat{\mu}_k(2\|A_k\| v_k + v_k^2)] < 1$$

hold. Then the zero solution of (1) is exponentially stable.

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The following relations are checked in [10]

$$g^2(A) \le |A|_F^2 - |Trace\ A^2| \ and \ g^2(A) \le \frac{|A - A^*|_F^2}{2} = 2|A_I|_F^2,$$

Where $A_I = (A - A^*)/2i$. If A_1 and A_2 are commuting matrices, then $g(A_1 + A_2) \le g(A_1) + g(A_2)$. By the inequality between geometric and arithmetic mean values we have

$$\left(\frac{1}{n}\sum_{k=1}^{n} |\lambda_k(A)|^2\right)^n \ge \left(\prod_{k=1}^{n} |\lambda_k(A)|\right)^2.$$

So $g^2(A) \le |A|_F^2 - n(det\ A)^{2/n}$.

Auxiliary Results

Let A and C be $n \times n$ matrices and $r_s(A) < 1$. The equation

$$Y - AYA^* = C \tag{6}$$

has a unique solution $Y(A,C)$, cf. [11]. Put

$$q(A) := \sup_{C \in \mathbb{C}^{n \times n}} \frac{\|Y(A,C)\|}{\|C\|}$$

where $\mathbb{C}^{n \times n}$ is the set of all $n \times n$ matrices.

Due to Corollary 3 from [12],

$$q(A) \le \sum_{m=0}^{n-1}\left(\sum_{j=0}^{n-1} \frac{g^j(A)m!r_s^{m-j}(A)}{(j!)^{3/2}(m-j)!}\right)^2. \tag{7}$$

Recall that $\frac{1}{(m-j)!} = 0$ for $m < j$. From (7) it follows

$$q(A) \le \left(\sum_{m=0}^{n-1}\sum_{j=0}^{n-1} \frac{g^j(A)m!r_s^{m-j}(A)}{(j!)^{3/2}(m-j)!}\right)^2.$$

Now taking into account that

$$\sum_{m=0} \frac{m!r_s^{m-j}(A)}{(m-j)!} = \frac{d^j}{dx^j}\sum_{m=0} x^m = j!(1-x)^{-j-1} \ (x = r_s(A)),$$

we get

$$q(A) \le \left(\sum_{j=0}^{n-1} \frac{g^j(A)}{(j!)^{1/2}(1-r_s(A))^{j+1}}\right)^2. \tag{8}$$

Proof of Theorem 1

Let Q_k be a solution of the equation

$$Q_k - A_k^* Q_k A_k = I. \tag{9}$$

First, let $r = \infty$. Then from (1) and (2) we have

$$(Q_k u_{k+1}, u_{k+1}) = (Q_k(A_k u_k + F_k(u_k)), A_k u_k + F_k(u_k))$$

$$= (A_k^* Q A_k u_k, u_k)) + (Q_k A_k u_k, F_k(u_k)) + (Q_k F_k(u_k), A_k u_k) + (Q_k F_k(u_k), F_k(u_k))$$

$$\le ((Q_k - I)u_k, u_k)) + \|Q_k\|(2\|A_k\|v_k + v_k^2)\|u_k\|^2$$

$$= (Q_{k-1}u_k, u_k) + ((Q_k - Q_{k-1})u_k, u_k) - (u_k, u_k)) + \|Q_k\|(2\|A_k\|v_k + v_k^2)\|u_k\|^2$$

$$\le (Q_{k-1}u_k, u_k) + (\|Q_k - Q_{k-1}\| - 1)(u_k, u_k)) + \|Q_k\|(2\|A_k\|v_k + v_k^2)\|u_k\|^2.$$

If

$$\|Q_k - Q_{k-1}\| + \|Q_k\|(2\|A_k\|v_k + v_k^2) < 1, \tag{10}$$

Then

$$(Q_k u_{k+1}, u_{k+1}) \le (Q_{k-1}u_k, u_k) \le \dots \le (Q_0 u_1, u_1).$$

But $Q_k \ge I$. So

$$(u_{k+1}, u_{k+1}) \le (Q_0 u_1, u_1) \le \|Q_0\|(A_0 u_0, A_0 u_0). \tag{11}$$

Now by a small perturbation we arrive at the following result.

Lemma 1

Let the conditions (2) with $r=$ and (10) hold. Then the zero solution to (1) is (globally) exponentially stable. According to (11), under (10) there is a constant γ independent of u_0, such that $\|u_n\| \le \gamma \|u_0\|$. So taking $\|u_0\| < r/\gamma$ we can remove the condition $r = \infty$.

Furthermore, the equation

$$Y - A_k Y A_k^* = C \tag{12}$$

has a unique solution $Y(A_k, C)$ for any C. Put

$$q(A_k) := \sup_{C \in \mathbb{C}^{n \times n}} \frac{\|Y(A_k, C)\|}{\|C\|} \tag{13}$$

To estimate $\|Q_k - Q_{k-1}\|$, note that

$$Q_k - Q_{k-1} = A_k^* Q_k A_k - A_{k-1}^* Q_{k-1} A_{k-1}.$$

Denote $Y_k = Q_k - Q_{k-1}$ and $\Delta_k = A_k - A_{k-1}$. Then

$$Y_k - A_k^* Y_k A_k = C_k, \ \text{where} \ C_k := A_{k-1}^* Q_k \Delta_k + \Delta_k^* Q_k A_k.$$

Due to (13) $\|Q_k\| \le q(A_k)$ and $Y_k \le q(A_k)\|C_k\|$. In addition,

$$\|C_k\| \le \|\Delta_k\| \|Q_k\|(\|A_{k-1}\| + \|A_k\|) \le q(A_k)\|\Delta_k\|(\|A_{k-1}\| + \|A_k\|)$$

So

$$\|Q_k - Q_{k-1}\| = \|Y_k\| \le q^2(A_k)\|A_k - A_{k-1}\|(\|A_{k-1}\| + \|A_k\|).$$

Thus, Lemma 1 implies

Corollary 2

Let $\sup_k \ (q^2(A_k)\|A_k - A_{k-1}\|(\|A_{k-1}\| + \|A_k\|) + q(A_k)(2\|A_k\|v_k + v_k^2) < 1.$

Then the zero solution to (1) is exponentially stable.

Proof of Theorem 1: From (7) it follows $q(A_k) \le \mu_k$. Now the previous corollary implies the required result.

Example

Let $A_k (k = 0, 1, \dots)$ be normal matrices. Then condition (5) takes the form

$$\sup_k \left[\frac{1}{(1 - r_s^2(A_k))^2}\|A_k - A_{k-1}\|(\|A_{k-1}\| + \|A_k\|) + \frac{1}{1 - r_s^2(A_k)}(2\|A_k\|v_k + v_k^2)\right] < 1 \tag{14}$$

In particular, if $A_k \equiv A$ is constant, then due to (4) from (14) it follows

$$\frac{1}{(1 - r_s^2(A))}(2\|A\|\hat{v} + \hat{v}^2) < 1.$$

where $\hat{v} := \sup_k v_k$. But $\|A\| = r_s(A)$. So $2r_s^2(A)\hat{v} + \hat{v}^2 < 1 - r_s^2(A)$

and therefore, the exponential stability test under the consideration is

$$r_s(A) + \hat{v} < 1 \tag{15}$$

This condition shows that Theorem 1 is sharp. Indeed, take $F_k w \equiv \hat{v}w$. Then condition (15) is necessary for the exponential stability. So Theorem 1 is really sharp.

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