

Explicit Calculations of Tensor Product Coefficients for E_7

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Abstract

We propose a new method to calculate coupling coefficients of E_7 tensor products. Our method is based on explicit use of E_7 characters in the definition of a tensor product.

When applying Weyl character formula for E_7 Lie algebra, one needs to make sums over 2903040 elements of E_7 Weyl group. To implement such enormous sums, we show we have a way which makes their calculations possible. This will be accomplished by decomposing an E_7 character into 72 participating A_7 characters.

Keywords: Coupling coefficients; Lie algebra; Irreducible representations; Subdominants; Tensor coupling coefficients

Introduction

Let $G_7 = E_7, A_7$ and Λ, Λ' be two dominant weights of G_7 where $R(\Lambda)$ and $R(\Lambda')$ are corresponding irreducible representations. For general terms, we follow the book of Humphreys [1] as ever.

Tensor product of these two irreducible representations is defined by,

$$R(\Lambda) \otimes R(\Lambda') = R(\Lambda + \Lambda') + \sum_{\lambda \in S(\Lambda + \Lambda')} t(\lambda < \Lambda + \Lambda') (\lambda) \quad (I.1)$$

where $S(\Lambda + \Lambda')$ is the set of $\Lambda + \Lambda'$ subdominants and $t(\lambda < \Lambda + \Lambda')$ s are tensor coupling coefficients. Though Steinberg formula is the best known way, a natural way to calculate tensor coupling coefficients is also to solve the equation

$$Ch(\Lambda) \otimes Ch(\Lambda') = Ch(\Lambda + \Lambda') + \sum_{\lambda \in S(\Lambda + \Lambda')} t(\lambda < \Lambda + \Lambda') (\lambda) \quad (I.2)$$

for tensor coupling coefficients. $Ch(\lambda)$ here is the character of an irreducible representation $R(\lambda)$ which corresponds to a dominant weight λ and it is defined by the famous Weyl Character formula:

$$Ch(\lambda^+) = \frac{A(\lambda^{++})}{A(\rho_{G_7})} \quad (I.3)$$

where for a weight μ in general

$$A(\mu) \equiv \sum_{\sigma \in W(G_7)} \varepsilon(\sigma) e^{\sigma(\mu)} \quad (I.4)$$

$W(G_7)$ is the Weyl Group of G_7 and each and every element σ is the so-called Weyl reflection while $\varepsilon(\sigma)$ denotes its sign and $e^{\sigma(\lambda^{++})}$'s here are known as formal exponentials. Throughout this work, we assume λ^{++} λ^{++} denotes a strictly dominant weight defined for a dominant λ^+ by

$$\lambda^{++} \equiv \rho_{G_7} + \lambda^+ \quad (I.5)$$

where ρ_{G_7} is the Weyl vector of G_7 .

The crucial fact here is that

$$\|W(E_7)\| = 2903040 \quad (I.6)$$

where $\|S\|$ denotes order of set S . It is easy to see then to implement the sum in (I.4) would not be realizable explicitly. We, instead, propose 72 specifically chosen Weyl reflections which give us A_7 dominant weights participating within the same E_7 Weyl orbit $W(\Lambda^+)$ for any E_7 dominant weight Λ^+ . As it is shown in the next section, this makes the evaluation of (I.4) realizable for E_7 but in terms of 72 A_7 characters and hence easily

implementable.

A_7 Decomposition of E_7 Lie Algebra

For $i=1,2,\dots,7$, let λ_i 's and α_i 's be respectively the fundamental dominant weights and simple roots of A_7 Lie algebra with the following Dynkin diagram (Figure 1).

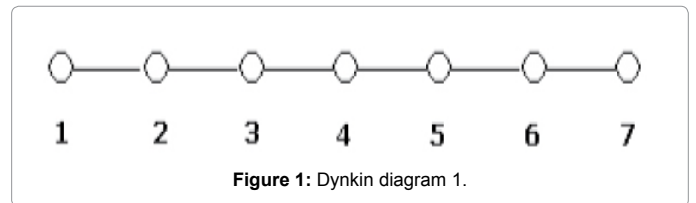


Figure 1: Dynkin diagram 1.

where $\rho_{A_7} = \lambda_1 + \dots + \lambda_7$ is A_7 Weyl vector and Λ_i 's be fundamental dominant weights of E_7 Lie algebra in according with the following Dynkin diagram, (Figure 2).

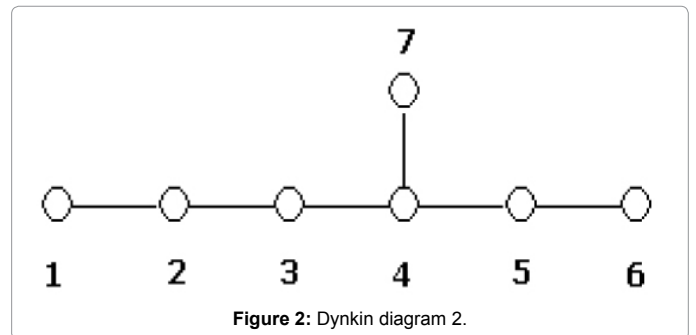


Figure 2: Dynkin diagram 2.

where $\rho_{E_7} = \Lambda_1 + \dots + \Lambda_7$ is A_7 Weyl vector. We suggest following relations allows us to embed A_7 subalgebra into E_7 algebra:

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$$\begin{aligned} \Lambda_1 &= \lambda_2 \\ \Lambda_2 &= \lambda_1 + \lambda_3 \\ \Lambda_3 &= 2 \lambda_3 \\ \Lambda_4 &= 2 \lambda_3 + \lambda_6 \\ \Lambda_5 &= \lambda_3 + \lambda_5 \\ \Lambda_6 &= \lambda_4 \\ \Lambda_7 &= \lambda_3 + \lambda_7 \end{aligned} \quad (II.1)$$

This essentially means that

$$\frac{\|W(E_7)\|}{\|W(A_7)\|} = 72 \quad (II.2)$$

which tells us that there are at most 72 A_7 dominant weights inside a Weyl orbit $W(\Lambda^+)$. Note here that it is exactly 72 when Λ^+ is a strictly dominant weight. From the now on, $W(\mu)$ will always denotes the Weyl orbit of a weight μ .

As the main point of view of this work, we present in appendix, 72 Weyl reflections to give 72 A_7 dominant weights participating in the same E_7 Weyl orbit $W(\Lambda^+)$ when they are exerted on the dominant weight Λ^+ . To this end, the Weyl reflections with respect to simple roots α_i will be called simple reflections σ_i . We extend multiple products of simple reflections trivially by

$$\sigma_{i_1 i_2}(\lambda) \equiv \sigma_{i_1}(\sigma_{i_2}(\lambda)).$$

For $s=1, \dots, 72$, $\Sigma(s)$'s are 72 Weyl reflections mentioned above. As will also be seen by their definitions that,

- 1) $\varepsilon(\sigma(s))=+1$ $s=1,2,\dots,36$
- 2) $\varepsilon(\sigma(s))=-1$ $s=37,38,\dots,72$

Calculating Tensor Coupling Coefficients

Let us proceed in the instructive example

$$R(\Lambda_3) \otimes R(\Lambda_4) = R(\Lambda_3 + \Lambda_4) + \sum_{j=1}^{39} m(j)(\theta_j) \quad (III.1)$$

of (I.1). One can see that there are 39 sub-dominant weights θ_j of $\Lambda_3 + \Lambda_4$:

$$\begin{aligned} \theta_1 &= \Lambda_3 + \Lambda_4 \\ \theta_2 &= \Lambda_1 \theta_3 = \Lambda_7 \\ \theta_4 &= 3 \Lambda_1 \\ \theta_5 &= \Lambda_1 + 2 \Lambda_2 \\ \theta_6 &= 2 \Lambda_1 + \Lambda_3 \\ \theta_7 &= \Lambda_1 + 2 \Lambda_5 \\ \theta_8 &= \Lambda_1 + \Lambda_4 + \Lambda_6 \\ \theta_9 &= 2 \Lambda_2 + \Lambda_7 \\ \theta_{10} &= \Lambda_2 + \Lambda_3 + \Lambda_6 \\ \theta_{11} &= \Lambda_2 + \Lambda_5 + \Lambda_7 \end{aligned}$$

$$\begin{aligned} \theta_{12} &= \Lambda_3 + 2 \Lambda_6 \\ \theta_{13} &= 3 \Lambda_7 \\ \theta_{14} &= 3 \Lambda_1 + \Lambda_6 \\ \theta_{15} &= \Lambda_1 + \Lambda_2 + 2 \Lambda_6 \\ \theta_{16} &= \Lambda_1 + 3 \Lambda_6 \\ \theta_{17} &= \Lambda_1 + \Lambda_6 + 2 \Lambda_7 \\ \theta_{18} &= \Lambda_1 + \Lambda_2 + \Lambda_5 \\ \theta_{19} &= \Lambda_1 + \Lambda_3 + \Lambda_7 \\ \theta_{20} &= 2 \Lambda_1 + \Lambda_6 + \Lambda_7 \\ \theta_{21} &= \Lambda_4 + \Lambda_7 \\ \theta_{22} &= \Lambda_5 + \Lambda_6 + \Lambda_7 \\ \theta_{23} &= \Lambda_1 + \Lambda_2 \\ \theta_{24} &= \Lambda_1 + \Lambda_6 \\ \theta_{25} &= 2 \Lambda_1 + \Lambda_7 \\ \theta_{26} &= \Lambda_1 + 2 \Lambda_7 \\ \theta_{27} &= \Lambda_2 + \Lambda_3 \\ \theta_{28} &= \Lambda_3 \\ \theta_{29} &= \Lambda_3 + \Lambda_5 \\ \theta_{30} &= 2 \Lambda_6 + \Lambda_7 \\ \theta_{31} &= \Lambda_1 + \Lambda_5 + \Lambda_6 \\ \theta_{32} &= \Lambda_1 + 2 \Lambda_6 \\ \theta_{33} &= \Lambda_6 + \Lambda_7 \\ \theta_{34} &= \Lambda_1 + \Lambda_4 \\ \theta_{35} &= \Lambda_1 + \Lambda_5 \\ \theta_{36} &= \Lambda_1 + \Lambda_2 + \Lambda_6 \\ \theta_{37} &= \Lambda_2 + \Lambda_6 + \Lambda_7 \\ \theta_{38} &= \Lambda_2 + \Lambda_7 \\ \theta_{39} &= \Lambda_3 + \Lambda_6 \\ \theta_{40} &= \Lambda_5 + \Lambda_7 \end{aligned}$$

To this end, we should care about specialization of formal exponentials [2]. Let us consider the so-called Fundamental Weights μ_i which are defined for ($i=1, \dots, 8$) as in the following [3]:

$$\alpha_i \equiv \mu_i - \mu_{i+1}, \quad (i=1, \dots, 7.) \quad (III.2)$$

α_i 's here are A_7 simple roots mentioned above and the best way to calculate A_7 and hence E_7 characters is to use the specialization in terms of parameters $u_i \equiv e^{\mu_i}$ which are subjects of the condition $\mu_1 + \mu_2 + \dots + \mu_8 = 0$ or

$$u_1 u_2 \dots u_8 = 1.$$

To exemplify (I.3) for E_7 , we would like to give detailed calculation of $Ch(\Lambda_3 + \Lambda_4)$. By applying 72 specifically chosen Weyl reflections on strictly dominant weight $\rho_{E_7} + \Lambda_3 + \Lambda_4$, one can see we have the following decompositions:

$$A(\rho_{E_7} + \Lambda_3 + \Lambda_4) = \sum_{k=1}^{36} Ch(v_k) - \sum_{k=37}^{72} Ch(v_k) \quad (III.3)$$

where

$$\begin{aligned}
 v_1 &= \lambda_1 + \lambda_2 + 11\lambda_3 + \lambda_4 + \lambda_5 + 2\lambda_6 + \lambda_7 \\
 v_2 &= 2\lambda_1 + 2\lambda_2 + 8\lambda_3 + \lambda_4 + \lambda_5 + 5\lambda_6 + \lambda_7 \\
 v_3 &= 5\lambda_1 + \lambda_2 + 7\lambda_3 + \lambda_4 + 3\lambda_5 + 2\lambda_6 + 3\lambda_7 \\
 v_4 &= 4\lambda_1 + \lambda_2 + 6\lambda_3 + \lambda_4 + 3\lambda_5 + 4\lambda_6 + 3\lambda_7 \\
 v_5 &= 2\lambda_1 + 4\lambda_2 + 4\lambda_3 + \lambda_4 + 5\lambda_5 + \lambda_6 + 5\lambda_7 \\
 v_6 &= 5\lambda_1 + 2\lambda_2 + 5\lambda_3 + 2\lambda_4 + 2\lambda_5 + 3\lambda_6 + 4\lambda_7 \\
 v_7 &= 5\lambda_1 + 2\lambda_2 + 5\lambda_3 + \lambda_4 + 4\lambda_5 + 3\lambda_6 + 2\lambda_7 \\
 v_8 &= 7\lambda_1 + \lambda_2 + 6\lambda_3 + \lambda_4 + 2\lambda_5 + 2\lambda_6 + 5\lambda_7 \\
 v_9 &= 7\lambda_1 + \lambda_2 + 5\lambda_3 + 2\lambda_4 + 3\lambda_5 + 2\lambda_6 + 3\lambda_7 \\
 v_{10} &= 2\lambda_1 + 2\lambda_2 + 4\lambda_3 + \lambda_4 + 6\lambda_5 + \lambda_6 + 6\lambda_7 \\
 v_{11} &= 3\lambda_1 + 3\lambda_2 + 3\lambda_3 + 2\lambda_4 + 4\lambda_5 + 2\lambda_6 + 6\lambda_7 \\
 v_{12} &= 3\lambda_1 + 3\lambda_2 + 3\lambda_3 + \lambda_4 + 6\lambda_5 + 2\lambda_6 + 4\lambda_7 \\
 v_{13} &= 6\lambda_1 + \lambda_2 + 5\lambda_3 + \lambda_4 + 2\lambda_5 + 4\lambda_6 + 5\lambda_7 \\
 v_{14} &= 6\lambda_1 + \lambda_2 + 4\lambda_3 + 2\lambda_4 + 3\lambda_5 + 4\lambda_6 + 3\lambda_7 \\
 v_{15} &= \lambda_1 + 6\lambda_2 + \lambda_3 + 4\lambda_4 + 2\lambda_5 + \lambda_6 + 8\lambda_7 \\
 v_{16} &= 4\lambda_1 + 4\lambda_2 + 3\lambda_3 + \lambda_4 + 4\lambda_5 + \lambda_6 + 7\lambda_7 \\
 v_{17} &= 4\lambda_1 + 4\lambda_2 + 2\lambda_3 + 2\lambda_4 + 5\lambda_5 + \lambda_6 + 5\lambda_7 \\
 v_{18} &= \lambda_1 + 6\lambda_2 + \lambda_3 + \lambda_4 + 8\lambda_5 + \lambda_6 + 2\lambda_7 \\
 v_{19} &= 7\lambda_1 + 2\lambda_2 + 4\lambda_3 + \lambda_4 + 3\lambda_5 + 3\lambda_6 + 4\lambda_7 \\
 v_{20} &= 8\lambda_1 + 2\lambda_2 + 2\lambda_3 + 4\lambda_4 + \lambda_5 + 5\lambda_6 + \lambda_7 \\
 v_{21} &= 10\lambda_1 + \lambda_2 + 3\lambda_3 + 3\lambda_4 + \lambda_5 + 4\lambda_6 + 2\lambda_7 \\
 v_{22} &= 11\lambda_1 + \lambda_2 + \lambda_3 + 6\lambda_4 + \lambda_5 + 2\lambda_6 + \lambda_7 \\
 v_{23} &= \lambda_1 + 4\lambda_2 + \lambda_3 + 4\lambda_4 + 3\lambda_5 + \lambda_6 + 9\lambda_7 \\
 v_{24} &= 4\lambda_1 + 2\lambda_2 + 3\lambda_3 + \lambda_4 + 5\lambda_5 + \lambda_6 + 8\lambda_7 \\
 v_{25} &= 4\lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4 + 6\lambda_5 + \lambda_6 + 6\lambda_7 \\
 v_{26} &= \lambda_1 + 4\lambda_2 + \lambda_3 + \lambda_4 + 9\lambda_5 + \lambda_6 + 3\lambda_7 \\
 v_{27} &= 2\lambda_1 + 5\lambda_2 + \lambda_3 + 3\lambda_4 + 2\lambda_5 + 2\lambda_6 + 9\lambda_7 \\
 v_{28} &= 5\lambda_1 + 3\lambda_2 + 2\lambda_3 + \lambda_4 + 5\lambda_5 + 2\lambda_6 + 6\lambda_7 \\
 v_{29} &= 9\lambda_1 + \lambda_2 + 2\lambda_3 + 3\lambda_4 + \lambda_5 + 6\lambda_6 + 2\lambda_7 \\
 v_{30} &= \lambda_1 + 7\lambda_2 + \lambda_3 + 2\lambda_4 + 2\lambda_5 + \lambda_6 + 10\lambda_7 \\
 v_{31} &= 10\lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4 + \lambda_5 + 6\lambda_6 + \lambda_7 \\
 v_{32} &= 13\lambda_1 + \lambda_2 + \lambda_3 + 4\lambda_4 + \lambda_5 + 3\lambda_6 + \lambda_7 \\
 v_{33} &= 2\lambda_1 + 2\lambda_2 + \lambda_3 + 5\lambda_4 + \lambda_5 + \lambda_6 + 12\lambda_7 \\
 v_{34} &= \lambda_1 + 5\lambda_2 + \lambda_3 + 2\lambda_4 + 3\lambda_5 + \lambda_6 + 11\lambda_7 \\
 v_{35} &= 16\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 + 2\lambda_5 + \lambda_6 + \lambda_7 \\
 v_{36} &= \lambda_1 + \lambda_2 + 3\lambda_3 + 2\lambda_4 + \lambda_5 + \lambda_6 + 15\lambda_7 \\
 v_{37} &= 3\lambda_1 + \lambda_2 + 9\lambda_3 + \lambda_4 + \lambda_5 + 4\lambda_6 + \lambda_7 \\
 v_{38} &= 2\lambda_1 + \lambda_2 + 8\lambda_3 + \lambda_4 + \lambda_5 + 6\lambda_6 + \lambda_7 \\
 v_{39} &= 4\lambda_1 + 2\lambda_2 + 6\lambda_3 + \lambda_4 + 3\lambda_5 + 3\lambda_6 + 3\lambda_7 \\
 v_{40} &= 6\lambda_1 + \lambda_2 + 6\lambda_3 + 2\lambda_4 + 2\lambda_5 + 2\lambda_6 + 4\lambda_7 \\
 v_{41} &= 6\lambda_1 + \lambda_2 + 6\lambda_3 + \lambda_4 + 4\lambda_5 + 2\lambda_6 + 2\lambda_7 \\
 v_{42} &= 2\lambda_1 + 3\lambda_2 + 4\lambda_3 + \lambda_4 + 5\lambda_5 + 2\lambda_6 + 5\lambda_7 \\
 v_{43} &= 5\lambda_1 + \lambda_2 + 5\lambda_3 + 2\lambda_4 + 2\lambda_5 + 4\lambda_6 + 4\lambda_7 \\
 v_{44} &= 5\lambda_1 + \lambda_2 + 5\lambda_3 + \lambda_4 + 4\lambda_5 + 4\lambda_6 + 2\lambda_7 \\
 v_{45} &= 3\lambda_1 + 4\lambda_2 + 3\lambda_3 + 2\lambda_4 + 4\lambda_5 + \lambda_6 + 6\lambda_7 \\
 v_{46} &= 3\lambda_1 + 4\lambda_2 + 3\lambda_3 + \lambda_4 + 6\lambda_5 + \lambda_6 + 4\lambda_7 \\
 v_{47} &= 6\lambda_1 + 2\lambda_2 + 5\lambda_3 + \lambda_4 + 2\lambda_5 + 3\lambda_6 + 5\lambda_7 \\
 v_{48} &= 6\lambda_1 + 2\lambda_2 + 4\lambda_3 + 2\lambda_4 + 3\lambda_5 + 3\lambda_6 + 3\lambda_7 \\
 v_{49} &= 8\lambda_1 + \lambda_2 + 5\lambda_3 + \lambda_4 + 3\lambda_5 + 2\lambda_6 + 4\lambda_7 \\
 v_{50} &= 9\lambda_1 + \lambda_2 + 3\lambda_3 + 4\lambda_4 + \lambda_5 + 4\lambda_6 + \lambda_7
 \end{aligned}$$

$$\begin{aligned}
 v_{51} &= 3\lambda_1 + 2\lambda_2 + 3\lambda_3 + 2\lambda_4 + 5\lambda_5 + \lambda_6 + 7\lambda_7 \\
 v_{52} &= 3\lambda_1 + 2\lambda_2 + 3\lambda_3 + \lambda_4 + 7\lambda_5 + \lambda_6 + 5\lambda_7 \\
 v_{53} &= \lambda_1 + 5\lambda_2 + \lambda_3 + 4\lambda_4 + 2\lambda_5 + 2\lambda_6 + 8\lambda_7 \\
 v_{54} &= 4\lambda_1 + 3\lambda_2 + 3\lambda_3 + \lambda_4 + 4\lambda_5 + 2\lambda_6 + 7\lambda_7 \\
 v_{55} &= 4\lambda_1 + 3\lambda_2 + 2\lambda_3 + 2\lambda_4 + 5\lambda_5 + 2\lambda_6 + 5\lambda_7 \\
 v_{56} &= \lambda_1 + 5\lambda_2 + \lambda_3 + \lambda_4 + 8\lambda_5 + 2\lambda_6 + 2\lambda_7 \\
 v_{57} &= 7\lambda_1 + \lambda_2 + 4\lambda_3 + \lambda_4 + 3\lambda_5 + 4\lambda_6 + 4\lambda_7 \\
 v_{58} &= 8\lambda_1 + \lambda_2 + 2\lambda_3 + 4\lambda_4 + \lambda_5 + 6\lambda_6 + \lambda_7 \\
 v_{59} &= 2\lambda_1 + 6\lambda_2 + \lambda_3 + 3\lambda_4 + 2\lambda_5 + \lambda_6 + 9\lambda_7 \\
 v_{60} &= 5\lambda_1 + 4\lambda_2 + 2\lambda_3 + \lambda_4 + 5\lambda_5 + \lambda_6 + 6\lambda_7 \\
 v_{61} &= 9\lambda_1 + 2\lambda_2 + 2\lambda_3 + 3\lambda_4 + \lambda_5 + 5\lambda_6 + 2\lambda_7 \\
 v_{62} &= 12\lambda_1 + \lambda_2 + \lambda_3 + 5\lambda_4 + \lambda_5 + 2\lambda_6 + 2\lambda_7 \\
 v_{63} &= 11\lambda_1 + \lambda_2 + 3\lambda_3 + 2\lambda_4 + \lambda_5 + 5\lambda_6 + \lambda_7 \\
 v_{64} &= \lambda_1 + 2\lambda_2 + \lambda_3 + 6\lambda_4 + \lambda_5 + \lambda_6 + 11\lambda_7 \\
 v_{65} &= 2\lambda_1 + 4\lambda_2 + \lambda_3 + 3\lambda_4 + 3\lambda_5 + \lambda_6 + 10\lambda_7 \\
 v_{66} &= 5\lambda_1 + 2\lambda_2 + 2\lambda_3 + \lambda_4 + 6\lambda_5 + \lambda_6 + 7\lambda_7 \\
 v_{67} &= \lambda_1 + 2\lambda_2 + \lambda_3 + \lambda_4 + 11\lambda_5 + \lambda_6 + \lambda_7 \\
 v_{68} &= \lambda_1 + 6\lambda_2 + \lambda_3 + 2\lambda_4 + 2\lambda_5 + 2\lambda_6 + 10\lambda_7 \\
 v_{69} &= 10\lambda_1 + \lambda_2 + 2\lambda_3 + 2\lambda_4 + \lambda_5 + 7\lambda_6 + \lambda_7 \\
 v_{70} &= 15\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 + 3\lambda_5 + \lambda_6 + \lambda_7 \\
 v_{71} &= \lambda_1 + 3\lambda_2 + \lambda_3 + 4\lambda_4 + \lambda_5 + \lambda_6 + 13\lambda_7 \\
 v_{72} &= \lambda_1 + \lambda_2 + 2\lambda_3 + 2\lambda_4 + \lambda_5 + \lambda_6 + 16\lambda_7
 \end{aligned}$$

A_7 characters $Ch(v_k)$'s are defined by

$$A(\rho_{A_7}) Ch(v_k) = \sum_{\sigma \in W(A_7)} \varepsilon(\sigma) e^{\sigma(\rho_{A_7} + v_k)} \quad (III.4)$$

Note here that $W(A_7)$ is the permutation group of 8 objects.

To display our result here, we use the following specialization of formal exponentials with only one free parameter x:

$$\begin{aligned}
 u_1 &= 1 \\
 u_2 &= 2 \\
 u_3 &= 3 \\
 u_4 &= 4 \\
 u_5 &= 5 \\
 u_6 &= 6 \\
 u_7 &= x \\
 u_8 &= 1 / (720 \cdot x) \quad (III.5)
 \end{aligned}$$

In this specialization, one obtains the following one-parameter characters:

$$\begin{aligned}
 A(\rho_{A_7}) &= -\frac{1}{2^{20} \times 3^{11} \times 5^6 \times x^7} \times \\
 &(-6+x) \times (-5+x) \times (-4+x) \times (-3+x) \times (-2+x) \times \\
 &(-1+x) \times (-1+720x) \times (-1+1440x) \times (-1+2160x) \times \\
 &(-1+2880x) \times (-1+3600x) \times (-1+4320x) \times (-1+720x^2)
 \end{aligned}$$

$$A(\rho_{E_7}) = -\frac{7^3 \times 11 \times 13^2 \times 17^2 \times 23 \times 29 \times 47 \times 59^2 \times 71 \times 89 \times 179 \times 239 \times 359}{2^{40} \times 3^{19} \times 5^9 \times x^{10}} \times$$

$$(-1+6x) \times (-1+8x) \times (-1+10x) \times (-1+12x)^2 \times (-1+15x)$$

$$(-1+18x) \times (-1+20x) \times (-1+24x)^2 \times (-1+30x)^2 \times (-1+36x)$$

$$(-1+40x) \times (-1+48x) \times (-1+60x)^2 \times (-1+72x) \times (-1+90x) \times (-1+120x)$$

$$ch(\Lambda_4) = \frac{1}{2^{16} \times 3^8 \times 5^4 \times x^4} ($$

$$2^2 \times 7 \times 29 \times 31 \times 113 \times 25849 +$$

$$3 \times 5^3 \times 7^2 \times 41 \times 13469 \times 25841 x +$$

$$2^2 \times 7^2 \times 227 \times 1997 \times 1004276389 x^2 +$$

$$2^2 \times 3^4 \times 5^2 \times 7^4 \times 17 \times 41 \times 13469 \times 45307 x^3 +$$

$$2^7 \times 3^2 \times 266944787316406807 x^4 +$$

$$2^6 \times 3^6 \times 5^3 \times 7^4 \times 17 \times 41 \times 13469 \times 45307 x^5 +$$

$$2^{10} \times 3^4 \times 5^2 \times 7^2 \times 227 \times 1997 \times 1004276389 x^6 +$$

$$2^{12} \times 3^7 \times 5^6 \times 7^2 \times 41 \times 13469 \times 25841 x^7 +$$

$$2^{18} \times 3^8 \times 5^4 \times 7 \times 29 \times 31 \times 113 \times 25849 x^8)$$

$$ch(\Lambda_3 + \Lambda_4) = \frac{7}{2^{28} \times 3^{13} \times 5^6 \times x^7} ($$

$$2^5 \times 7^2 \times 139 \times 40819 \times 22523219 +$$

$$3 \times 296955329011336071883 x +$$

$$7 \times 93629 \times 104327 \times 20612147800357 x^2 +$$

$$2^2 \times 7 \times 89 \times 509 \times 407193532921684756441 x^3 +$$

$$2^2 \times 7 \times 19 \times 1447 \times 73091587 \times 1489316745532201 x^4 +$$

$$2^4 \times 3 \times 11^2 \times 37 \times 1117 \times 18045889 \times 1661840436868789 x^5 +$$

$$2^4 \times 3^2 \times 7 \times 11^2 \times 5237 \times 130069 \times 351401 \times 12440163841487 x^6 +$$

$$2^9 \times 3^3 \times 11 \times 83 \times 757 \times 4830390973 \times 258355213888973 x^7 +$$

$$2^8 \times 3^4 \times 5 \times 7 \times 11^2 \times 5237 \times 130069 \times 351401 \times 12440163841487 x^8 +$$

$$2^{12} \times 3^5 \times 5^2 \times 11^2 \times 37 \times 1117 \times 18045889 \times 1661840436868789 x^9 +$$

$$2^{14} \times 3^6 \times 5^3 \times 7 \times 19 \times 1447 \times 73091587 \times 1489316745532201 x^{10} +$$

$$2^{18} \times 3^8 \times 5^4 \times 7 \times 89 \times 509 \times 407193532921684756441 x^{11} +$$

$$2^{20} \times 3^{10} \times 5^5 \times 7 \times 9362 \times 104327 \times 20612147800357 x^{12} +$$

$$2^{24} \times 3^{13} \times 5^6 \times 296955329011336071883 x^{13} +$$

$$2^{33} \times 3^{14} \times 5^7 \times 7^2 \times 139 \times 40819 \times 22523219 x^{14})$$

$$ch(\theta_2) = \frac{1}{144 x} (3 \times 7 \times 17 + 5 \times 25841 x + 2^4 \times 3^3 \times 5 \times 7 \times 17 x^2)$$

$$ch(\theta_3) = \frac{1}{2^8 \times 3^4 \times 5 \times x^2} ($$

$$5 \times 25841 + 3 \times 7 \times 19 \times 83 \times 2459 x + 2^2 \times 3^3 \times 7^2 \times 17 \times 45307 x^2 +$$

$$2^4 \times 3^3 \times 5 \times 7 \times 19 \times 83 \times 2459 x^3 + 2^8 \times 3^4 \times 5^3 \times 25841 x^4)$$

$$ch(\theta_4) = \frac{1}{2^{12} \times 3^6 \times 5^2 \times x^3} ($$

$$2 \times 3^2 \times 7 \times 251 \times 11287 +$$

$$2 \times 7 \times 11 \times 1634509733 x +$$

$$3^2 \times 7^2 \times 23 \times 73 \times 101 \times 281 \times 5783 x^2 +$$

$$41 \times 953 \times 5987 \times 153547507 x^3 +$$

$$2^4 \times 3^4 \times 5 \times 7^2 \times 23 \times 73 \times 101 \times 281 \times 5783 x^4 +$$

$$2^9 \times 3^4 \times 5^2 \times 7 \times 11 \times 1634509733 x^5 +$$

$$2^{13} \times 3^8 \times 5^3 \times 7 \times 251 \times 11287 x^6)$$

$$ch(\theta_5) = \frac{7}{2^{20} \times 3^{10} \times 5^4 \times x^5} ($$

$$3^4 \times 389 \times 2621 \times 326611 +$$

$$44372305108670731 x +$$

$$2 \times 3 \times 7 \times 888351682544432651 x^2 +$$

$$2 \times 6835359680937443668841 x^3 +$$

$$2^3 \times 3 \times 17 \times 313 \times 2029 \times 4912124122975679 x^4 +$$

$$2^2 \times 7 \times 17 \times 87942983 \times 1353499222770007 x^5 +$$

$$2^7 \times 3^3 \times 5 \times 17 \times 313 \times 2029 \times 4912124122975679 x^6 +$$

$$2^9 \times 3^4 \times 5^2 \times 6835359680937443668841 x^7 +$$

$$2^{13} \times 3^7 \times 5^3 \times 7 \times 888351682544432651 x^8 +$$

$$2^{16} \times 3^8 \times 5^4 \times 44372305108670731 x^9 +$$

$$2^{20} \times 3^{14} \times 5^5 \times 389 \times 2621 \times 326611 x^{10})$$

$$ch(\theta_6) = \frac{1}{2^{20} \times 3^{10} \times 5^4 \times x^5} ($$

$$2 \times 3^5 \times 5 \times 7 \times 67 \times 118953227 +$$

$$2 \times 29 \times 191 \times 1259 \times 18041 \times 1205779 x +$$

$$3 \times 7 \times 419 \times 953 \times 23122934503549 x^2 +$$

$$7 \times 11 \times 13 \times 3881 \times 9624859 \times 1896757637 x^3 +$$

$$2^2 \times 3^2 \times 7 \times 111767 \times 14341763 \times 22885698919 x^4 +$$

$$2^2 \times 3^2 \times 5 \times 31 \times 114376828127 \times 589217525459 x^5 +$$

$$2^6 \times 3^4 \times 5 \times 7 \times 111767 \times 14341763 \times 22885698919 x^6 +$$

$$2^8 \times 3^4 \times 5^2 \times 7 \times 11 \times 13 \times 3881 \times 9624859 \times 1896757637 x^7 +$$

$$2^{12} \times 3^7 \times 5^3 \times 7 \times 419 \times 953 \times 23122934503549 x^8 +$$

$$2^{17} \times 3^8 \times 5^4 \times 29 \times 191 \times 1259 \times 18041 \times 1205779 x^9 +$$

$$2^{21} \times 3^{15} \times 5^6 \times 7 \times 67 \times 118953227 x^{10})$$

$$ch(\theta_7) = \frac{7}{2^{26} \times 3^{14} \times 5^6 \times x^7} ($$

$$2^2 \times 3^3 \times 71 \times 137 \times 897339719$$

$$+ 2^3 \times 3^2 \times 9427181 \times 12037823647 x +$$

$$3 \times 5 \times 7^3 \times 6472827206445606859 x^2 +$$

$$7 \times 167 \times 1217 \times 3345931489 \times 3986554903 x^3 +$$

$$2^3 \times 3^2 \times 1913 \times 28712521177048302440611 x^4 +$$

$$2^3 \times 3^2 \times 5 \times 7 \times 31 \times 5504034331596437227418843 x^5 +$$

$$2^4 \times 3^5 \times 5^3 \times 157 \times 311 \times 6473 \times 6273427 \times 25424142019 x^6 +$$

$$2^{11} \times 3^5 \times 5 \times 11 \times 13 \times 193 \times 38358559 \times 321202718587369 x^7 +$$

$$2^8 \times 3^7 \times 5^4 \times 157 \times 311 \times 6473 \times 6273427 \times 25424142019 x^8 +$$

$$2^{11} \times 3^6 \times 5^3 \times 7 \times 31 \times 5504034331596437227418843 x^9 +$$

$$2^{15} \times 3^8 \times 5^3 \times 1913 \times 28712521177048302440611 x^{10} +$$

$$2^{16} \times 3^8 \times 5^4 \times 7 \times 167 \times 1217 \times 3345931489 \times 3986554903 x^{11} +$$

$$2^{20} \times 3^{11} \times 5^6 \times 7^3 \times 6472827206445606859 x^{12} +$$

$$2^{27} \times 3^{14} \times 5^6 \times 9427181 \times 12037823647 x^{13} +$$

$$2^{30} \times 3^{17} \times 5^7 \times 71 \times 137 \times 897339719 x^{14})$$

$$ch(\theta_8) = \frac{7}{2^{26} \times 3^{12} \times 5^6 \times x^7} ($$

$$70782069982080 +$$

$$2^5 \times 3 \times 7 \times 19 \times 101 \times 2437 \times 7883 \times 45307 x +$$

$$3^1 \times 839 \times 2591 \times 566506158976003 x^2 +$$

$$5^2 \times 127 \times 743526604818728544311 x^3 +$$

$$2^2 \times 5 \times 83 \times 263 \times 6791 \times 13901 \times 522761 \times 27245893 x^4 +$$

$$2^2 \times 1439 \times 6833 \times 57110377 \times 29440768222739 x^5 +$$

$$2^8 \times 3^2 \times 79 \times 2833 \times 5435185631 \times 1421253019763 x^6 +$$

$$2^6 \times 3^2 \times 5 \times 7 \times 37577587 \times 1652664511 \times 109347145543 x^7 +$$

$$2^{12} \times 3^4 \times 5 \times 79 \times 2833 \times 5435185631 \times 1421253019763 x^8 +$$

$$2^{10} \times 3^4 \times 5^2 \times 1439 \times 6833 \times 57110377 \times 29440768222739 x^9 +$$

$$2^{14} \times 3^6 \times 5^4 \times 83 \times 263 \times 6791 \times 13901 \times 522761 \times 27245893 x^{10} +$$

$$2^{16} \times 3^8 \times 5^6 \times 127 \times 743526604818728544311 x^{11} +$$

$$2^{20} \times 3^{11} \times 5^5 \times 839 \times 2591 \times 566506158976003 x^{12} +$$

$$2^{29} \times 3^{13} \times 5^6 \times 7 \times 19 \times 101 \times 2437 \times 7883 \times 45307 x^{13} +$$

$$2^{35} \times 3^{15} \times 5^8 \times 19 \times 101 \times 2437 \times 7883 x^{14})$$

$$ch(\theta_9) = \frac{1}{2^{23} \times 3^{12} \times 5^5 \times x^6} ($$

$$5 \times 11 \times 1730263 \times 475374719 +$$

$$3 \times 5 \times 7 \times 11 \times 4933 \times 15316021986509 x +$$

$$5 \times 7 \times 13 \times 172831287782587727803 x^2 +$$

$$3 \times 7^2 \times 205820307753559239132761 x^3 +$$

$$2^2 \times 3^3 \times 41 \times 707606359 \times 1135814759505821 x^4 +$$

$$2 \times 3 \times 5 \times 7 \times 41 \times 1252496251 \times 20514266507074489 x^5 +$$

$$2^3 \times 3^3 \times 5 \times 67 \times 16903 \times 2819023 \times 2271886853005801 x^6 +$$

$$2^5 \times 3^3 \times 5^2 \times 7 \times 41 \times 1252496251 \times 20514266507074489 x^7 +$$

$$2^{10} \times 3^7 \times 5^2 \times 41 \times 707606359 \times 1135814759505821 x^8 +$$

$$2^{12} \times 3^7 \times 5^3 \times 7^2 \times 205820307753559239132761 x^9 +$$

$$2^{16} \times 3^8 \times 5^5 \times 7 \times 13 \times 172831287782587727803 x^{10} +$$

$$2^{20} \times 3^{11} \times 5^6 \times 7 \times 11 \times 4933 \times 15316021986509 x^{11} +$$

$$2^{24} \times 3^{12} \times 5^7 \times 11 \times 1730263 \times 475374719 x^{12})$$

$$ch(\theta_{10}) = \frac{1}{2^{26} \times 3^{12} \times 5^6 \times x^7} ($$

$$2^2 \times 3 \times 5 \times 7^2 \times 63113 \times 7700299 +$$

$$3 \times 7^3 \times 45307 \times 63113 \times 7700299 x +$$

$$3 \times 5^2 \times 7 \times 17 \times 191 \times 30444055104860819 x^2 +$$

$$2^2 \times 13 \times 854020492656685314145399 x^3 +$$

$$2^2 \times 3 \times 7 \times 13 \times 137 \times 1619903 \times 40109150434199081 x^4 +$$

$$2^4 \times 3 \times 7 \times 13 \times 948542029 \times 239506095510065339 x^5 +$$

$$2^4 \times 3 \times 7 \times 11485393 \times 14627879671585167600299 x^6 +$$

$$2^6 \times 3^7 \times 5^2 \times 19 \times 61 \times 883 \times 8053 \times 44101 \times 244021 \times 6224627 x^7 +$$

$$2^8 \times 3^3 \times 5 \times 7 \times 11485393 \times 14627879671585167600299 x^8 +$$

$$2^{12} \times 3^5 \times 5^2 \times 7 \times 13 \times 948542029 \times 239506095510065339 x^9 +$$

$$2^{14} \times 3^7 \times 5^3 \times 7 \times 13 \times 137 \times 1619903 \times 40109150434199081 x^{10} +$$

$$2^{18} \times 3^8 \times 5^4 \times 13 \times 854020492656685314145399 x^{11} +$$

$$2^{20} \times 3^{11} \times 5^7 \times 7 \times 17 \times 191 \times 30444055104860819 x^{12} +$$

$$2^{24} \times 3^{13} \times 5^6 \times 7^3 \times 45307 \times 63113 \times 7700299 x^{13} +$$

$$2^{30} \times 3^{15} \times 5^8 \times 7^2 \times 63113 \times 7700299 x^{14})$$

$$ch(\theta_{11}) = \frac{11}{2^{26} \times 3^{14} \times 5^6 \times x^7} ($$

$$3^2 \times 7 \times 233 \times 2439503305003 + 251580659156174514557 x +$$

$$2 \times 3 \times 7 \times 197 \times 479 \times 83919415489238839 x^2 +$$

$$2 \times 3^5 \times 7 \times 257 \times 13163 \times 561199 \times 23985981091 x^3 +$$

$$2^4 \times 3 \times 5^2 \times 7 \times 23 \times 31 \times 4297 \times 5779 \times 1802909 \times 106751927 x^4 +$$

$$2^3 \times 3^3 \times 207797403557 \times 58222225954444073 x^5 +$$

$$2^5 \times 3^3 \times 7 \times 22924263480995135201368307741 x^6 +$$

$$2^6 \times 3^5 \times 29 \times 10268423472246027440877336011 x^7 +$$

$$2^9 \times 3^5 \times 5 \times 7 \times 22924263480995135201368307741 x^8 +$$

$$2^{11} \times 3^7 \times 5^2 \times 207797403557 \times 58222225954444073 x^9 +$$

$$2^{16} \times 3^7 \times 5^5 \times 7 \times 23 \times 31 \times 4297 \times 5779 \times 1802909 \times 106751927 x^{10} +$$

$$2^{17} \times 3^{13} \times 5^4 \times 7 \times 257 \times 13163 \times 561199 \times 23985981091 x^{11} +$$

$$2^{21} \times 3^{11} \times 5^5 \times 7 \times 197 \times 479 \times 83919415489238839 x^{12} +$$

$$2^{24} \times 3^{12} \times 5^6 \times 251580659156174514557 x^{13} +$$

$$2^{28} \times 3^{16} \times 5^7 \times 7 \times 233 \times 2439503305003 x^{14})$$

$$ch(\theta_{12}) = \frac{1}{2^{24} \times 3^{12} \times 5^6 \times x^7} ($$

$$2^4 \times 3^3 \times 5^2 \times 7^2 \times 41 \times 13469 +$$

$$2^2 \times 3^3 \times 5 \times 7^3 \times 41 \times 13469 \times 45307 x +$$

$$3 \times 7^3 \times 41 \times 13469 \times 115259 \times 666353 x^2 +$$

$$5 \times 2999 \times 75703 \times 67759950443299 x^3 +$$

$$2^2 \times 3 \times 7 \times 41 \times 73^2 \times 367 \times 2736080070054181 x^4 +$$

$$2^2 \times 3^2 \times 7 \times 7685969351914603034407849 x^5 +$$

$$2^4 \times 3^3 \times 5^2 \times 7^2 \times 373 \times 571 \times 973787 \times 1054751382911 x^6 +$$

$$2^6 \times 3^4 \times 5 \times 37 \times 223 \times 311 \times 3767 \times 12319481 \times 1298488019 x^7 +$$

$$2^8 \times 3^5 \times 5^3 \times 7^2 \times 373 \times 571 \times 973787 \times 1054751382911 x^8 +$$

$$2^{10} \times 3^6 \times 5^2 \times 7 \times 7685969351914603034407849 x^9 +$$

$$2^{14} \times 3^7 \times 5^3 \times 7 \times 41 \times 73^2 \times 367 \times 2736080070054181 x^{10} +$$

$$2^{16} \times 3^8 \times 5^5 \times 2999 \times 75703 \times 67759950443299 x^{11} +$$

$$2^{20} \times 3^{11} \times 5^5 \times 7^3 \times 41 \times 13469 \times 115259 \times 666353 x^{12} +$$

$$2^{26} \times 3^{15} \times 5^7 \times 7^3 \times 41 \times 13469 \times 45307 x^{13} +$$

$$2^{32} \times 3^{17} \times 5^9 \times 7^2 \times 41 \times 13469 x^{14})$$

$$ch(\theta_{13}) = \frac{11}{2^{24} \times 3^{12} \times 5^5 \times x^6} ($$

$$3238985337918907 +$$

$$3^4 \times 5 \times 7 \times 841610704258279 x +$$

$$7^2 \times 13 \times 1482820942905285871 x^2 +$$

$$3^3 \times 7 \times 193 \times 467 \times 5690719 \times 2565345073 x^3 +$$

$$2^2 \times 3^2 \times 626862641769013044421771 x^4 +$$

$$2^4 \times 3^3 \times 5 \times 7 \times 79134994435751528887903 x^5 +$$

$$2^6 \times 3^4 \times 31 \times 149 \times 181 \times 9313661572349696183 x^6 +$$

$$2^8 \times 3^5 \times 5^2 \times 7 \times 79134994435751528887903 x^7 +$$

$$2^{10} \times 3^6 \times 5^2 \times 626862641769013044421771 x^8 +$$

$$2^{12} \times 3^9 \times 5^3 \times 7 \times 193 \times 467 \times 5690719 \times 2565345073 x^9 +$$

$$2^{16} \times 3^8 \times 5^4 \times 7^2 \times 13 \times 1482820942905285871 x^{10} +$$

$$2^{20} \times 3^{14} \times 5^6 \times 7 \times 841610704258279 x^{11} +$$

$$2^{24} \times 3^{12} \times 5^6 \times 3238985337918907 x^{12})$$

$$ch(\theta_{14}) = \frac{1}{2^{18} \times 3^8 \times 5^4 \times x^5} ($$

$$2^3 \times 3^2 \times 5 \times 7 \times 251 \times 11287 +$$

$$2 \times 3^2 \times 7^2 \times 251 \times 11287 \times 45307 x +$$

$$2 \times 7^2 \times 13 \times 47 \times 5077 \times 280370003 x^2 +$$

$$1305047 \times 32776442887619 x^3 +$$

$$7 \times 29 \times 67792850454653521231 x^4 +$$

$$2^2 \times 5 \times 59 \times 557 \times 739 \times 1838570856221483 x^5 +$$

$$2^4 \times 3^2 \times 5 \times 7 \times 29 \times 67792850454653521231 x^6 +$$

$$2^8 \times 3^4 \times 5^2 \times 1305047 \times 32776442887619 x^7 +$$

$$2^{13} \times 3^6 \times 5^3 \times 7^2 \times 13 \times 47 \times 5077 \times 280370003 x^8 +$$

$$2^{17} \times 3^{10} \times 5^4 \times 7^2 \times 251 \times 11287 \times 45307 x^9 +$$

$$2^{23} \times 3^{12} \times 5^6 \times 7 \times 251 \times 11287 x^{10})$$

$$ch(\theta_{15}) = \frac{7}{2^{24} \times 3^{12} \times 5^6 \times x^7} ($$

$$2^{10} \times 3^3 \times 5^3 \times 51977 +$$

$$2^8 \times 3^3 \times 5^2 \times 7 \times 45307 \times 51977 x +$$

$$2^6 \times 3 \times 5 \times 7 \times 51977 \times 115259 \times 666353 x^2 +$$

$$3^2 \times 861317 \times 4060866137260903 x^3 +$$

$$3 \times 5 \times 41 \times 53 \times 16481 \times 24852500017071257 x^4 +$$

$$2^3 \times 223 \times 1035354314616125752647209 x^5 +$$

$$2^6 \times 3^2 \times 5 \times 495553698751 \times 90995390639929 x^6 +$$

$$2^7 \times 3^2 \times 5 \times 7 \times 163 \times 227 \times 99661 \times 33171175875735791 x^7 +$$

$$2^{10} \times 3^4 \times 5^2 \times 495553698751 \times 90995390639929 x^8 +$$

$$2^{11} \times 3^4 \times 5^2 \times 223 \times 1035354314616125752647209 x^9 +$$

$$2^{12} \times 3^7 \times 5^4 \times 41 \times 53 \times 16481 \times 24852500017071257 x^{10} +$$

$$2^{16} \times 3^{10} \times 5^4 \times 861317 \times 4060866137260903 x^{11} +$$

$$2^{26} \times 3^{11} \times 5^6 \times 7 \times 51977 \times 115259 \times 666353 x^{12} +$$

$$2^{32} \times 3^{15} \times 5^8 \times 7 \times 45307 \times 51977 x^{13} +$$

$$2^{38} \times 3^{17} \times 5^{10} \times 51977 x^{14})$$

$$ch(\theta_{16}) = \frac{1}{2^{22} \times 3^{10} \times 5^6 \times x^7} ($$

$$2^6 \times 3^3 \times 5^3 \times 7 \times 17 +$$

$$2^4 \times 3^3 \times 5^2 \times 7^2 \times 17 \times 45307 x +$$

$$2^2 \times 3 \times 5 \times 7^2 \times 17 \times 115259 \times 666353 x^2 +$$

$$3 \times 7^3 \times 17 \times 37 \times 221303 \times 196292779 x^3 +$$

$$5 \times 7 \times 29 \times 147145853 \times 90252382399 x^4 +$$

$$2^2 \times 5 \times 86214774415429023276727 x^5 +$$

$$2^4 \times 3^4 \times 5^2 \times 7 \times 863 \times 4003 \times 33749 \times 4576169137 x^6 +$$

$$2^6 \times 3^3 \times 5^2 \times 23 \times 43 \times 7907 \times 14463096839915333 x^7 +$$

$$2^8 \times 3^6 \times 5^3 \times 7 \times 863 \times 4003 \times 33749 \times 4576169137 x^8 +$$

$$2^{10} \times 3^4 \times 5^3 \times 86214774415429023276727 x^9 +$$

$$2^{12} \times 3^6 \times 5^4 \times 7 \times 29 \times 147145853 \times 90252382399 x^{10} +$$

$$2^{16} \times 3^9 \times 5^4 \times 7^3 \times 17 \times 37 \times 221303 \times 196292779 x^{11} +$$

$$2^{22} \times 3^{11} \times 5^6 \times 7^2 \times 17 \times 115259 \times 666353 x^{12} +$$

$$2^{28} \times 3^{15} \times 5^8 \times 7^2 \times 17 \times 45307 x^{13} +$$

$$2^{34} \times 3^{17} \times 5^{10} \times 7 \times 17 x^{14})$$

$$ch(\theta_{17}) = \frac{7}{2^{26} \times 3^{12} \times 5^6 \times x^7} ($$

$$2^2 \times 3^2 \times 5 \times 1920917020339 +$$

$$3^2 \times 7 \times 45307 \times 1920917020339 x +$$

$$2^2 \times 11 \times 1699 \times 20848837 \times 3497751463 x^2 +$$

$$2^3 \times 3 \times 13 \times 31 \times 47 \times 2517821 \times 2229533970737 x^3 +$$

$$5 \times 23 \times 239 \times 22442053352210346516283 x^4 +$$

$$2^3 \times 3 \times 11 \times 244186487395311739881589177 x^5 +$$

$$2^8 \times 3^3 \times 11^2 \times 3229 \times 1352409142907613715103 x^6 +$$

$$2^8 \times 3^2 \times 5 \times 11 \times 993682140710634056756709367 x^7 +$$

$$2^{12} \times 3^5 \times 5 \times 11^2 \times 3229 \times 1352409142907613715103 x^8 +$$

$$2^{11} \times 3^5 \times 5^2 \times 11 \times 244186487395311739881589177 x^9 +$$

$$2^{12} \times 3^6 \times 5^4 \times 23 \times 239 \times 22442053352210346516283 x^{10} +$$

$$2^{19} \times 3^9 \times 5^4 \times 13 \times 31 \times 47 \times 2517821 \times 2229533970737 x^{11} +$$

$$2^{22} \times 3^{10} \times 5^5 \times 11 \times 1699 \times 20848837 \times 3497751463 x^{12} +$$

$$2^{24} \times 3^{14} \times 5^6 \times 7 \times 45307 \times 1920917020339 x^{13} +$$

$$2^{30} \times 3^{16} \times 5^8 \times 1920917020339 x^{14})$$

$$ch(\theta_{18}) = \frac{1}{2^{22} \times 3^{12} \times 5^5 \times x^6} ($$

$$2^2 \times 3^3 \times 5 \times 11 \times 36319 \times 4281313 +$$

$$2^4 \times 3 \times 5 \times 7 \times 11^2 \times 46549 \times 679298969 x +$$

$$3^2 \times 5^2 \times 7 \times 11 \times 41 \times 10712225499148973 x^2 +$$

$$3 \times 5^2 \times 7 \times 569 \times 476209471 \times 24269673079 x^3 +$$

$$2^7 \times 5^2 \times 7 \times 717343331 \times 35775113699503 x^4 +$$

$$2^8 \times 3^2 \times 7 \times 20327 \times 3472171 \times 37024507616807 x^5 +$$

$$2^4 \times 3^2 \times 3301 \times 3253233143634211446037993 x^6 +$$

$$2^{12} \times 3^4 \times 5 \times 7 \times 20327 \times 3472171 \times 37024507616807 x^7 +$$

$$2^{15} \times 3^4 \times 5^4 \times 7 \times 717343331 \times 35775113699503 x^8 +$$

$$2^{12} \times 3^7 \times 5^5 \times 7 \times 569 \times 476209471 \times 24269673079 x^9 +$$

$$2^{16} \times 3^{10} \times 5^6 \times 7 \times 11 \times 41 \times 10712225499148973 x^{10} +$$

$$2^{24} \times 3^{11} \times 5^6 \times 7 \times 11^2 \times 46549 \times 679298969 x^{11} +$$

$$2^{26} \times 3^{15} \times 5^7 \times 11 \times 36319 \times 4281313 x^{12})$$

$$ch(\theta_{19}) = \frac{7}{2^{24} \times 3^{12} \times 5^5 \times x^6} ($$

$$11 \times 19 \times 54132425994601 +$$

$$2^2 \times 3 \times 1873 \times 801641 \times 1539041989 x +$$

$$2^3 \times 5 \times 3391 \times 157884472828805717 x^2 +$$

$$3 \times 5 \times 41 \times 421 \times 26894683525927701469 x^3 +$$

$$2^3 \times 5296547 \times 40807969 \times 561290628359 x^4 +$$

$$2^2 \times 3^2 \times 5 \times 7 \times 18542831 \times 2641233933522645623 x^5 +$$

$$2^4 \times 3^4 \times 11699291 \times 71109193 \times 2011720082303 x^6 +$$

$$2^6 \times 3^4 \times 5^2 \times 7 \times 18542831 \times 2641233933522645623 x^7 +$$

$$2^{11} \times 3^4 \times 5^2 \times 5296547 \times 40807969 \times 561290628359 x^8 +$$

$$2^{12} \times 3^7 \times 5^4 \times 41 \times 421 \times 26894683525927701469 x^9 +$$

$$2^{19} \times 3^8 \times 5^5 \times 3391 \times 157884472828805717 x^{10} +$$

$$2^{22} \times 3^{11} \times 5^5 \times 1873 \times 801641 \times 1539041989 x^{11} +$$

$$2^{24} \times 3^{12} \times 5^6 \times 11 \times 19 \times 54132425994601 x^{12})$$

$$ch(\theta_{20}) = \frac{7}{2^{22} \times 3^{10} \times 5^5 \times x^6} ($$

$$2^3 \times 5 \times 11 \times 1634509733 +$$

$$2 \times 7 \times 11 \times 45307 \times 1634509733 x +$$

$$3^1 \times 5^2 \times 188673113 \times 883791547 x^2 +$$

$$2^5 \times 821 \times 5573 \times 41867209636027 x^3 +$$

$$3^4 \times 269 \times 27680957 \times 2625874917649 x^4 +$$

$$2^3 \times 3^2 \times 13 \times 29 \times 71 \times 74073603909992156759 x^5 +$$

$$2^5 \times 3^4 \times 5 \times 1489 \times 291506349305759397989 x^6 +$$

$$2^7 \times 3^4 \times 5 \times 13 \times 29 \times 71 \times 74073603909992156759 x^7 +$$

$$2^8 \times 3^8 \times 5^2 \times 269 \times 27680957 \times 2625874917649 x^8 +$$

$$2^{17} \times 3^6 \times 5^3 \times 821 \times 5573 \times 41867209636027 x^9 +$$

$$2^{16} \times 3^9 \times 5^6 \times 188673113 \times 883791547 x^{10} +$$

$$2^{21} \times 3^{10} \times 5^5 \times 7 \times 11 \times 45307 \times 1634509733 x^{11} +$$

$$2^{27} \times 3^{12} \times 5^7 \times 11 \times 1634509733 x^{12})$$

$$ch(\theta_{21}) = \frac{7}{2^{24} \times 3^{12} \times 5^5 \times x^6} ($$

$$2^3 \times 11 \times 1091 \times 12756480773 +$$

$$3 \times 5 \times 7 \times 4954321 \times 8800968989 x +$$

$$3^2 \times 7 \times 106275811 \times 430270161227 x^2 +$$

$$2^3 \times 3 \times 5 \times 7^2 \times 13 \times 3137 \times 25147 \times 246251 \times 502781 \times x^3 +$$

$$2^2 \times 3^3 \times 82567 \times 255832037 \times 34487228383 x^4 +$$

$$2^5 \times 3^3 \times 7 \times 23 \times 28753 \times 45439 \times 24160314328237 x^5 +$$

$$2^7 \times 3^5 \times 5 \times 7 \times 23 \times 31 \times 1867 \times 102273597906161063 x^6 +$$

$$2^9 \times 3^5 \times 5 \times 7 \times 23 \times 28753 \times 45439 \times 24160314328237 x^7 +$$

$$2^{10} \times 3^7 \times 5^2 \times 82567 \times 255832037 \times 34487228383 x^8 +$$

$$2^{15} \times 3^7 \times 5^4 \times 7^2 \times 13 \times 3137 \times 25147 \times 246251 \times 502781 x^9 +$$

$$2^{16} \times 3^{10} \times 5^4 \times 7 \times 106275811 \times 430270161227 x^{10} +$$

$$2^{20} \times 3^{11} \times 5^6 \times 7 \times 4954321 \times 8800968989 x^{11} +$$

$$2^{27} \times 3^{12} \times 5^6 \times 11 \times 1091 \times 12756480773 x^{12})$$

$$ch(\theta_{22}) = \frac{1}{2^{24} \times 3^{12} \times 5^4 \times x^7} ($$

$$2^2 \times 3^2 \times 5^2 \times 7 \times 23 \times 443 \times 110339 +$$

$$3^2 \times 5 \times 7^2 \times 23 \times 443 \times 45307 \times 110339 x +$$

$$2 \times 5 \times 7 \times 79 \times 107166946015042819 x^2 +$$

$$2 \times 3 \times 7^2 \times 36713 \times 34478412780970831 x^3 +$$

$$3^3 \times 7 \times 8069 \times 48408629149359661453 x^4 +$$

$$2^3 \times 3^2 \times 13 \times 17 \times 4099 \times 26525879 \times 4080242157239 x^5 +$$

$$2^4 \times 3^3 \times 7 \times 127701402659006223221510827 x^6 +$$

$$2^6 \times 3^4 \times 5 \times 401 \times 1257177934278466121552321 x^7 +$$

$$2^8 \times 3^5 \times 5 \times 7 \times 127701402659006223221510827 x^8 +$$

$$2^{11} \times 3^4 \times 5^2 \times 13 \times 17 \times 4099 \times 26525879 \times 4080242157239 x^9 +$$

$$2^{12} \times 3^9 \times 5^3 \times 7 \times 8069 \times 48408629149359661453 x^{10} +$$

$$2^{17} \times 3^9 \times 5^4 \times 7^2 \times 36713 \times 34478412780970831 x^{11} +$$

$$2^{21} \times 3^{10} \times 5^4 \times 7 \times 79 \times 107166946015042819 x^{12} +$$

$$2^{24} \times 3^{14} \times 5^7 \times 7^2 \times 23 \times 443 \times 45307 \times 110339 x^{13} +$$

$$2^{30} \times 3^{16} \times 5^9 \times 7 \times 23 \times 443 \times 110339 x^{14})$$

$$ch(\theta_{23}) = \frac{1}{2^{12} \times 3^6 \times 5^2 \times x^3} ($$

$$2^6 \times 3 \times 5 \times 7 \times 51977 +$$

$$3^2 \times 37 \times 967 \times 991 \times 1277 x +$$

$$3 \times 5 \times 7^3 \times 37 \times 67 \times 199 \times 58321 x^2 +$$

$$2^3 \times 11 \times 104090304018661 x^3 +$$

$$2^4 \times 3^3 \times 5^2 \times 7^3 \times 37 \times 67 \times 199 \times 58321 x^4 +$$

$$2^8 \times 3^6 \times 5^2 \times 37 \times 967 \times 991 \times 1277 x^5 +$$

$$2^{18} \times 3^7 \times 5^4 \times 7 \times 51977 x^6)$$

$$ch(\theta_{24}) = \frac{7}{2^{10} \times 3^4 \times 5^2 \times x^3} ($$

$$2^2 \times 3 \times 5 \times 17 + 3 \times 7 \times 17 \times 45307 x +$$

$$5 \times 7103 \times 192949 x^2 +$$

$$2^2 \times 5 \times 29 \times 541 \times 1456541 x^3 +$$

$$2^4 \times 3^2 \times 5^2 \times 7103 \times 192949 x^4 +$$

$$2^8 \times 3^5 \times 5^2 \times 7 \times 17 \times 45307 x^5 +$$

$$2^{14} \times 3^7 \times 5^4 \times 17 x^6)$$

$$ch(\theta_{25}) = \frac{1}{2^{16} \times 3^8 \times 5^3 \times x^4} ($$

$$2 \times 7 \times 11 \times 1634509733 +$$

$$3 \times 5 \times 7 \times 11149 \times 240779587 x +$$

$$2^3 \times 97 \times 5019793 \times 33822727 x^2 +$$

$$3^3 \times 7^2 \times 19 \times 1429 \times 862136910847 x^3 +$$

$$2^2 \times 3^5 \times 7 \times 11 \times 127 \times 190523 \times 803051407 x^4 +$$

$$2^4 \times 3^5 \times 5 \times 7^2 \times 19 \times 1429 \times 862136910847 x^5 +$$

$$2^{11} \times 3^4 \times 5^2 \times 97 \times 5019793 \times 33822727 x^6 +$$

$$2^{12} \times 3^7 \times 5^4 \times 7 \times 11149 \times 240779587 x^7 +$$

$$2^{17} \times 3^8 \times 5^4 \times 7 \times 11 \times 1634509733 x^8)$$

$$ch(\theta_{26}) = \frac{1}{2^{20} \times 3^{10} \times 5^4 \times x^5} ($$

$$3^2 \times 7 \times 1920917020339 +$$

$$2^6 \times 1932627887748647 x +$$

$$2^2 \times 3 \times 7 \times 687598205349751411 x^2 +$$

$$5 \times 7^2 \times 673 \times 77746343167071049 x^3 +$$

$$2^2 \times 3^2 \times 7 \times 41 \times 4513 \times 8209 \times 397543 \times 6033427 x^4 +$$

$$2^4 \times 3^2 \times 42256259 \times 5662271823471397 x^5 +$$

$$2^6 \times 3^4 \times 5 \times 7 \times 41 \times 4513 \times 8209 \times 397543 \times 6033427 x^6 +$$

$$2^8 \times 3^4 \times 5^3 \times 7^2 \times 673 \times 77746343167071049 x^7 +$$

$$2^{14} \times 3^7 \times 5^3 \times 7 \times 687598205349751411 x^8 +$$

$$2^{22} \times 3^8 \times 5^4 \times 1932627887748647 x^9 +$$

$$2^{20} \times 3^{12} \times 5^5 \times 7 \times 1920917020339 x^{10})$$

$$ch(\theta_{27}) = \frac{1}{2^{20} \times 3^9 \times 5^4 \times x^5} ($$

$$7^2 \times 63113 \times 7700299 +$$

$$5^2 \times 7 \times 1699 \times 195399775663 x +$$

$$2^3 \times 7 \times 875675732223561979 x^2 +$$

$$2^3 \times 3 \times 103 \times 367 \times 5791 \times 125219 \times 13800671 x^3 +$$

$$2^4 \times 7 \times 233 \times 95531 \times 6117073 \times 43889063 x^4 +$$

$$2^4 \times 3^2 \times 7 \times 31 \times 790500401078791217977 x^5 +$$

$$2^8 \times 3^2 \times 5 \times 7 \times 233 \times 95531 \times 6117073 \times 43889063 x^6 +$$

$$2^{11} \times 3^5 \times 5^2 \times 103 \times 367 \times 5791 \times 125219 \times 13800671 x^7 +$$

$$2^{15} \times 3^6 \times 5^3 \times 7 \times 875675732223561979 x^8 +$$

$$2^{16} \times 3^8 \times 5^6 \times 7 \times 1699 \times 195399775663 x^9 +$$

$$2^{20} \times 3^{10} \times 5^5 \times 7^2 \times 63113 \times 7700299 x^{10})$$

$$ch(\theta_{28}) = \frac{7}{2^{12} \times 3^6 \times 5^2 \times x^3} ($$

$$3 \times 7 \times 41 \times 13469 + 5 \times 25841 \times 172357 x +$$

$$2^2 \times 3 \times 7 \times 17 \times 151 \times 229 \times 76837 x^2 +$$

$$2^2 \times 3^2 \times 13 \times 41 \times 8271569177 x^3 +$$

$$2^6 \times 3^3 \times 5 \times 7 \times 17 \times 151 \times 229 \times 76837 x^4 +$$

$$2^8 \times 3^4 \times 5^3 \times 25841 \times 172357 x^5 +$$

$$2^{12} \times 3^7 \times 5^3 \times 7 \times 41 \times 13469 x^6)$$

$$ch(\theta_{29}) = \frac{1}{2^{22} \times 3^{12} \times 5^5 \times x^6} ($$

$$3^2 \times 7 \times 37 \times 87365639293 +$$

$$3 \times 7 \times 17 \times 25158739 \times 163675601 x +$$

$$2 \times 7^2 \times 293207 \times 571321 \times 153212833 x^2 +$$

$$2 \times 3 \times 5^2 \times 7^2 \times 19 \times 4966016564031243349 x^3 +$$

$$2^2 \times 3^2 \times 79 \times 383 \times 1049 \times 29339 \times 2479950110851 x^4 +$$

$$2^2 \times 3^3 \times 7 \times 259650710821 \times 25089495454453 x^5 +$$

$$2^7 \times 3^4 \times 7^2 \times 333504308440846597412509 x^6 +$$

$$2^6 \times 3^5 \times 5 \times 7 \times 259650710821 \times 25089495454453 x^7 +$$

$$2^{10} \times 3^6 \times 5^2 \times 79 \times 383 \times 1049 \times 29339 \times 2479950110851 x^8 +$$

$$2^{13} \times 3^7 \times 5^3 \times 7^2 \times 19 \times 4966016564031243349 x^9 +$$

$$2^{17} \times 3^8 \times 5^4 \times 7^2 \times 293207 \times 571321 \times 153212833 x^{10} +$$

$$2^{20} \times 3^{11} \times 5^5 \times 7 \times 17 \times 25158739 \times 163675601 x^{11} +$$

$$2^{24} \times 3^{14} \times 5^6 \times 7 \times 37 \times 87365639293 x^{12})$$

$$ch(\theta_{30}) = \frac{1}{2^{20} \times 3^{10} \times 5^5 \times x^6} ($$

$$2^4 \times 3^2 \times 5^3 \times 25841 +$$

$$2^2 \times 3^2 \times 5^2 \times 7 \times 25841 \times 45307 x +$$

$$5 \times 7 \times 25841 \times 115259 \times 666353 x^2 +$$

$$3 \times 7 \times 55717 \times 2804293 \times 13315567 x^3 +$$

$$2^3 \times 3^2 \times 86399 \times 959723737277659 x^4 +$$

$$2^6 \times 3^3 \times 5 \times 7 \times 19 \times 372825737 \times 955373927 x^5 +$$

$$2^8 \times 3^4 \times 5 \times 11 \times 59 \times 1627 \times 137055035733221 x^6 +$$

$$2^{10} \times 3^5 \times 5^2 \times 7 \times 19 \times 372825737 \times 955373927 x^7 +$$

$$2^{11} \times 3^6 \times 5^2 \times 86399 \times 959723737277659 x^8 +$$

$$2^{12} \times 3^7 \times 5^3 \times 7 \times 55717 \times 2804293 \times 13315567 x^9 +$$

$$2^{16} \times 3^8 \times 5^5 \times 7 \times 25841 \times 115259 \times 666353 x^{10} +$$

$$2^{22} \times 3^{12} \times 5^7 \times 7 \times 25841 \times 45307 x^{11} +$$

$$2^{28} \times 3^{14} \times 5^9 \times 25841 x^{12})$$

$$ch(\theta_{31}) = \frac{1}{2^{20} \times 3^{10} \times 5^5 \times x^6} ($$

$$2^2 \times 3^2 \times 5 \times 71 \times 1558283 +$$

$$3^2 \times 7 \times 71 \times 45307 \times 1558283 x +$$

$$2 \times 3 \times 7^2 \times 1499 \times 3758088575183 x^2 +$$

$$2 \times 5^2 \times 7^2 \times 11261 \times 185753 \times 172860353 x^3 +$$

$$3^2 \times 5^2 \times 13 \times 29 \times 241069 \times 7378760342027 x^4 +$$

$$2^3 \times 3^3 \times 5 \times 7 \times 11 \times 136666703950440834769 x^5 +$$

$$2^5 \times 3^4 \times 5^2 \times 7^2 \times 1216883197 \times 110293780367 x^6 +$$

$$2^7 \times 3^5 \times 5^2 \times 7 \times 11 \times 136666703950440834769 x^7 +$$

$$2^8 \times 3^6 \times 5^4 \times 13 \times 29 \times 241069 \times 7378760342027 x^8 +$$

$$2^{13} \times 3^6 \times 5^5 \times 7^2 \times 11261 \times 185753 \times 172860353 x^9 +$$

$$2^{17} \times 3^9 \times 5^4 \times 7^2 \times 1499 \times 3758088575183 x^{10} +$$

$$2^{20} \times 3^{12} \times 5^5 \times 7 \times 71 \times 45307 \times 1558283 x^{11} +$$

$$2^{26} \times 3^{14} \times 5^7 \times 71 \times 1558283 x^{12})$$

$$ch(\theta_{32}) = \frac{7}{2^{16} \times 3^8 \times 5^4 \times x^5} ($$

$$2^4 \times 3^3 \times 5^2 \times 17 +$$

$$2^2 \times 3^3 \times 5 \times 7 \times 17 \times 45307 x +$$

$$3 \times 7 \times 17 \times 115259 \times 666353 x^2 +$$

$$5 \times 2512868354279147 x^3 +$$

$$2^2 \times 3^2 \times 5 \times 1061 \times 218249 \times 31482709 x^4 +$$

$$2^4 \times 3^2 \times 5^2 \times 7 \times 2069 \times 634759 \times 1884341 x^5 +$$

$$2^6 \times 3^4 \times 5^2 \times 1061 \times 218249 \times 31482709 x^6 +$$

$$2^8 \times 3^4 \times 5^3 \times 2512868354279147 x^7 +$$

$$2^{12} \times 3^7 \times 5^3 \times 7 \times 17 \times 115259 \times 666353 x^8 +$$

$$2^{18} \times 3^{11} \times 5^5 \times 7 \times 17 \times 45307 x^9 +$$

$$2^{24} \times 3^{13} \times 5^7 \times 17 x^{10})$$

$$ch(\theta_{33}) = \frac{1}{2^{14} \times 3^6 \times 5^3 \times x^4} ($$

$$2^2 \times 5^2 \times 25841 + 5 \times 7 \times 25841 \times 45307 x +$$

$$3 \times 7^2 \times 19 \times 83 \times 2459 \times 45307 x^2 +$$

$$2^3 \times 3 \times 7^2 \times 43 \times 53841866849 x^3 +$$

$$2^5 \times 3^4 \times 5 \times 71 \times 109 \times 401 \times 1543 \times 1789 x^4 +$$

$$2^7 \times 3^3 \times 5 \times 7^2 \times 43 \times 53841866849 x^5 +$$

$$2^8 \times 3^5 \times 5^2 \times 7^2 \times 19 \times 83 \times 2459 \times 45307 x^6 +$$

$$2^{12} \times 3^6 \times 5^4 \times 7 \times 25841 \times 45307 x^7 +$$

$$2^{18} \times 3^8 \times 5^6 \times 25841 x^8)$$

$$ch(\theta_{34}) = \frac{1}{2^{20} \times 3^{10} \times 5^4 \times x^5} ($$

$$2^5 \times 3 \times 7 \times 19 \times 101 \times 2437 \times 7883 +$$

$$3^3 \times 7 \times 17 \times 43 \times 670034705677 x +$$

$$3 \times 5 \times 7^2 \times 1093 \times 72068616146713 x^2 +$$

$$2^2 \times 5 \times 7^2 \times 144593 \times 92161130410753 x^3 +$$

$$2^2 \times 3^3 \times 7 \times 4127 \times 8389 \times 31543 \times 1323118663 x^4 +$$

$$2^7 \times 3^2 \times 11 \times 11943616469 \times 264210902899 x^5 +$$

$$2^6 \times 3^5 \times 5 \times 7 \times 4127 \times 8389 \times 31543 \times 1323118663 x^6 +$$

$$2^{10} \times 3^4 \times 5^3 \times 7^2 \times 144593 \times 92161130410753 x^7 +$$

$$2^{12} \times 3^7 \times 5^4 \times 7^2 \times 1093 \times 72068616146713 x^8 +$$

$$2^{16} \times 3^{11} \times 5^4 \times 7 \times 17 \times 43 \times 670034705677 x^9 +$$

$$2^{25} \times 3^{11} \times 5^5 \times 7 \times 19 \times 101 \times 2437 \times 7883 x^{10})$$

$$ch(\theta_{35}) = \frac{1}{2^{14} \times 3^8 \times 5^3 \times x^4} ($$

$$3^2 \times 71 \times 1558283 +$$

$$3 \times 7^2 \times 11 \times 71 \times 57706391 x +$$

$$5 \times 7 \times 71 \times 569 \times 15083 \times 158143 x^2 +$$

$$3^2 \times 5 \times 7^2 \times 11 \times 76091 \times 256751687 x^3 +$$

$$2^6 \times 3^2 \times 5 \times 238247 \times 29004591751 x^4 +$$

$$2^4 \times 3^4 \times 5^2 \times 7^2 \times 11 \times 76091 \times 256751687 x^5 +$$

$$2^8 \times 3^4 \times 5^3 \times 7 \times 71 \times 569 \times 15083 \times 158143 x^6 +$$

$$2^{12} \times 3^7 \times 5^3 \times 7^2 \times 11 \times 71 \times 57706391 x^7 +$$

$$2^{16} \times 3^{10} \times 5^4 \times 71 \times 1558283 x^8)$$

$$ch(\theta_{36}) = \frac{1}{2^{18} \times 3^8 \times 5^4 \times x^5} ($$

$$2^8 \times 3 \times 5^2 \times 7 \times 51977 +$$

$$2^6 \times 3 \times 5 \times 7^2 \times 45307 \times 51977 x +$$

$$3 \times 7 \times 13 \times 577 \times 696629 \times 1185337 x^2 +$$

$$5 \times 509 \times 547 \times 37532952421247 x^3 +$$

$$2^3 \times 7 \times 255247 \times 401008447405121 x^4 +$$

$$2^4 \times 5 \times 53 \times 2861 \times 1461797 \times 14111392523 x^5 +$$

$$2^7 \times 3^2 \times 5 \times 7 \times 255247 \times 401008447405121 x^6 +$$

$$2^8 \times 3^4 \times 5^3 \times 509 \times 547 \times 37532952421247 x^7 +$$

$$2^{12} \times 3^7 \times 5^3 \times 7 \times 13 \times 577 \times 696629 \times 1185337 x^8 +$$

$$2^{22} \times 3^9 \times 5^5 \times 7^2 \times 45307 \times 51977 x^9 +$$

$$2^{28} \times 3^{11} \times 5^7 \times 7 \times 51977^1 x^{10})$$

$$ch(\theta_{37}) = \frac{1}{2^{22} \times 3^{10} \times 5^5 \times x^6} ($$

$$2^2 \times 5 \times 7 \times 31 \times 193 \times 3624787 +$$

$$7^2 \times 31 \times 193 \times 45307 \times 3624787 x +$$

$$2 \times 31 \times 79 \times 6043 \times 672439 \times 3360061 x^2 +$$

$$3 \times 5 \times 7^2 \times 40277 \times 1031399 \times 973746491 x^3 +$$

$$2^2 \times 3 \times 5 \times 7 \times 29 \times 43 \times 97 \times 1375243 \times 57552822853 x^4 +$$

$$2^4 \times 3 \times 7^2 \times 239 \times 45007 \times 10354439038651493 x^5 +$$

$$2^6 \times 3^6 \times 5 \times 11 \times 107 \times 163 \times 208977919833364987 x^6 +$$

$$2^8 \times 3^3 \times 5 \times 7^2 \times 239 \times 45007 \times 10354439038651493 x^7 +$$

$$2^{10} \times 3^5 \times 5^3 \times 7 \times 29 \times 43 \times 97 \times 1375243 \times 57552822853 x^8 +$$

$$2^{12} \times 3^7 \times 5^4 \times 7^2 \times 40277 \times 1031399 \times 973746491 x^9 +$$

$$2^{17} \times 3^8 \times 5^4 \times 31 \times 79 \times 6043 \times 672439 \times 3360061 x^{10} +$$

$$2^{20} \times 3^{10} \times 5^5 \times 7^2 \times 31 \times 193 \times 45307 \times 3624787 x^{11} +$$

$$2^{26} \times 3^{12} \times 5^7 \times 7 \times 31 \times 193 \times 3624787 x^{12})$$

$$ch(\theta_{38}) = \frac{7}{2^{16} \times 3^8 \times 5^3 \times x^4} ($$

$$31 \times 193 \times 3624787 +$$

$$2 \times 3 \times 661 \times 39827 \times 198031 x +$$

$$3^4 \times 5 \times 9371 \times 3530299103 x^2 +$$

$$2^2 \times 3 \times 5 \times 91297 \times 232912445897 x^3 +$$

$$2^4 \times 3^3 \times 370597 \times 315220711859 x^4 +$$

$$2^6 \times 3^3 \times 5^2 \times 91297 \times 232912445897 x^5 +$$

$$2^8 \times 3^8 \times 5^3 \times 9371 \times 3530299103 x^6 +$$

$$2^{13} \times 3^7 \times 5^3 \times 661 \times 39827 \times 198031 x^7 +$$

$$2^{16} \times 3^8 \times 5^4 \times 31 \times 193 \times 3624787 x^8)$$

$$ch(\theta_{39}) = \frac{1}{2^{18} \times 3^8 \times 5^4 \times x^5} ($$

$$2^2 \times 3 \times 5 \times 7^2 \times 41 \times 13469 +$$

$$3 \times 7^3 \times 41 \times 13469 \times 45307 x +$$

$$5 \times 7 \times 19813 \times 26321 \times 2574193 x^2 +$$

$$2^2 \times 3 \times 7 \times 41946929 \times 2853739787 x^3 +$$

$$2^2 \times 3 \times 7 \times 11 \times 875056131284689657 x^4 +$$

$$2^4 \times 3^2 \times 5^2 \times 8352802228682522537 x^5 +$$

$$2^6 \times 3^3 \times 5 \times 7 \times 11 \times 875056131284689657 x^6 +$$

$$2^{10} \times 3^5 \times 5^2 \times 7 \times 41946929 \times 2853739787 x^7 +$$

$$2^{12} \times 3^6 \times 5^4 \times 7 \times 19813 \times 26321 \times 2574193 x^8 +$$

$$2^{16} \times 3^9 \times 5^4 \times 7^3 \times 41 \times 13469 \times 45307 x^9 +$$

$$2^{22} \times 3^{11} \times 5^6 \times 7^2 \times 41 \times 13469 x^{10})$$

$$ch(\theta_{40}) = \frac{7}{2^{18} \times 3^{10} \times 5^4 \times x^5} ($$

$$3^2 \times 5 \times 23 \times 443 \times 110339 +$$

$$5 \times 29 \times 31 \times 113 \times 25841 \times 25849 x +$$

$$3 \times 7 \times 19 \times 29 \times 31 \times 83 \times 113 \times 2459 \times 25849 x^2 +$$

$$3^2 \times 7 \times 419 \times 2364953 \times 600815491 x^3 +$$

$$2^{10} \times 3^3 \times 11 \times 23 \times 8912381 \times 41333989 x^4 +$$

$$2^4 \times 3^4 \times 61 \times 67 \times 17529680553917933 x^5 +$$

$$2^{14} \times 3^5 \times 5 \times 11 \times 23 \times 8912381 \times 41333989 x^6 +$$

$$2^8 \times 3^6 \times 5^2 \times 7 \times 419 \times 2364953 \times 600815491 x^7 +$$

$$2^{12} \times 3^7 \times 5^3 \times 7 \times 19 \times 29 \times 31 \times 83 \times 113 \times 2459 \times 25849 x^8 +$$

$$2^{16} \times 3^8 \times 5^5 \times 29 \times 31 \times 113 \times 25841 \times 25849 x^9 +$$

$$2^{20} \times 3^{12} \times 5^6 \times 23 \times 443 \times 110339 x^{10})$$

Now, one can see that the characters above fulfill the following equation:

$$ch(\Lambda_3) \times ch(\Lambda_4) = ch(\Lambda_3 + \Lambda_4) +$$

$$ch(\theta_2) + ch(\theta_3) + ch(\theta_4) + ch(\theta_5) +$$

$$ch(\theta_6) + ch(\theta_7) + ch(\theta_8) + ch(\theta_9) +$$

$$ch(\theta_{10}) + ch(\theta_{11}) + ch(\theta_{12}) + ch(\theta_{13}) +$$

$$ch(\theta_{14}) + ch(\theta_{15}) + ch(\theta_{16}) + ch(\theta_{17}) +$$

$$2 ch(\theta_{18}) + 2 ch(\theta_{19}) + 2 ch(\theta_{20}) + 2 ch(\theta_{21}) + 2 ch(\theta_{22}) +$$

$$3 ch(\theta_{23}) + 3 ch(\theta_{24}) + 3 ch(\theta_{25}) + 3 ch(\theta_{26}) +$$

$$3 ch(\theta_{27}) + 3 ch(\theta_{28}) + 3 ch(\theta_{29}) + 3 ch(\theta_{30}) +$$

$$4 ch(\theta_{31}) + 4 ch(\theta_{32}) + 4 ch(\theta_{33}) +$$

$$5 ch(\theta_{34}) + 5 ch(\theta_{35}) + 5 ch(\theta_{36}) + 5 ch(\theta_{37}) +$$

$$+ 5 ch(\theta_{38}) + 5 ch(\theta_{39}) + 5 ch(\theta_{40}) \tag{III.6}$$

Conclusion

One should note however that, the 1-parameter specialization (III.5) above is not enough to find all the tensor coupling coefficients completely so we saw that at least 3-parameters specializations will be sufficient, which we used the following one.

$$u_1 = 1$$

$$u_2 = 2$$

$$u_3 = 3$$

$$u_4 = 4$$

$$u_5 = x$$

$$u_6 = y$$

$$u_7 = z$$

$$u_8 = 1 / (24 x y z)$$

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