Journal of
Applied & Computational Mathematics

Chamekh, J Appl Computat Math 2016, 5:6
DOI: 10.4172/2168-9679.1000335

Research Article Open Access

Existence of Solutions for a Fractional and Non-Local Elliptic Operator

Mourad Chamekh*

Mathematics Department, Faculty of Sciences and Arts-Alkamel, University of Jeddah, Jeddah, Saudi Arabia

Abstract

In this paper, we consider a fractional and p-laplacian elliptic equation. In order to study this problem, we apply the technique of Nehari manifold and fibering map, which permit treating the existence of nontrivial solutions of a fractional and nonlocal equation, satisfies the homogeneous Dirichlet boundary conditions.

Keywords: Nontrivial solutions; Fractional p-laplacian equation; Nehari manifold

Classification: 35J35, 35J50, 35J60

Introduction

Consider the fractional and p-laplacian elliptic problem

$$\begin{cases}(-\Delta)_p^{\gamma} u-\alpha\phi(t)\,|u|^{q-2}\,u-\psi(t)\,|u|^{r-2}\,u=0 \quad in \quad \Omega \\ u=0 \quad in \quad \mathbb{R}^n\setminus\Omega\end{cases}, \tag{1}$$

We assume that the Ω is a bounded domain in $\mathbb{R}^n$ and $\partial\Omega$ its smooth boundary, $p\geq 2$, $\gamma\in(0,1), 1<q<p<r<p^*=\dfrac{np}{n-\gamma p}$ if $n>p$ and $p^*=\infty$ else, $\varphi(t),\Psi(t)\in C(\Omega)$. $\alpha>0$

and the fractional p-laplacian operator may be defined for $p\in(1,\infty)$ as

$$(-\Delta)_p^{\gamma}u(t)=2\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^n\setminus B_\varepsilon(t)}\frac{|u(t)-u(z)|^{p-2}\,(u(t)-u(z))}{|t-z|^{n+p\gamma}}dz,\ t\in\mathbb{R}^n. \tag{2}$$

Over the recent years, numerous scientists have been attracted by the fractional and or p-laplacian equations. In fact, a few great models have been upgraded considerably for satisfactory answers to the modelling issues. We mention as examples the fractional Navies Stokes equations [1], fractional transport equations [2] and fractional Schrödinger equations [3], integral equations of fractional order [4,5]. Generally, a large variety of applications leads to these types of equations in ecology, elasticity and finance [6-8]. Despite significant progress in the field, and because of the difficulty to find an exact solution, research projects are still ongoing.

In this paper, we will think about the partial and p-laplacian elliptic equation (1). A considerable measure has been given for to explore this type of problems as of late. We can discover comparative equations in the many works where the issue of the existence of solutions has been dealt with. For instance, in [9], a local operator issue has been treated with $\varphi(t)=\Psi(t)=1$. In addition, in [10] we have comes a class of Kirchhoff sort having a similar right-hand-side term that in the problem (1). See likewise [11] for a late consideration of the fractional and p-laplacian elliptic issue with $\varphi=\Psi=0$. In this case, the solution u called a γ-p-harmonic function. Partial Laplacian equations satisfy the homogeneous Dirichlet boundary has been as of late considered in [9,11-13], using variational techniques. The existence of solutions has been considered at Ghanmi [14] utilizing a right-hand-side term of the treated condition comprises a homogeneous map, yet at the same time positive. Moreover, Xiang et al. in [15], use non-negative weight functions with the same issue. Here, we have treated the issue with sign-changing weight functions, and we proposed another proof for the existence of solutions. In view of the disintegration of the Nehari manifold is by all accounts less demanding. The remainder of this paper is organized as follows. In section 2, a few preliminaries are presented, in section 3 we explore the principle comes about.

Preliminaries

We start with some preliminaries on the notation we will use in this report. See Ghanmi A, Nezza ED, Brown KJ, [16-19] for further detail.

For all $h\in C(\Omega)$, we consider the following properties

- $||h||_\infty=1$;
- $h^{\pm}:=max(\pm h,0)\neq 0$.

For $r\in[1\infty]$, we consider $||.||_r$ the norm of $L^r(\Omega)$. For all measurable functions $u:\mathbb{R}^n\to\mathbb{R}$, we define the Gagliardo seminorm, by

$$|u|_{\gamma,p}:=\left(\int_{\mathbb{R}^{2n}}\frac{|u(t)-u(z)|^p}{|t-z|^{n+p\gamma}}dtdz\right)^{\frac{1}{p}}.$$

Following Di Nezza [16], we consider the fractional Sobolev space

$$W^{\gamma,p}(\mathbb{R}^n):=\left\{u\in L^p(\mathbb{R}^n): u\ \ measurable\ ,|u|_{\gamma,p}<\infty\right\},$$

with the norm defined by

$$\|u\|_{\gamma,p}:=\left(\|u\|_p^p+|u|_{\gamma,p}^p\right)^{\frac{1}{p}}.$$

We consider, thereafter, the closed subspace

$$S:=\left\{u\in W^{\gamma,p}(\mathbb{R}^n): u(t)=0\ a.e.\ in\ \mathbb{R}^n\setminus\Omega\right\},$$

with the norm $||.||=|.|_{\gamma,p}$. It is easy to verify that $(S,||.||)$ is a uniformly convex Banach space and that the embedding $S\hookrightarrow L^r(\Omega)$ is continuous for all $1\leq r\leq p_\gamma^*$, and compact for all $1\leq r\leq p_\gamma^*$. The dual space of $(S,||.||)$ is denoted by $(S^*,||.||_*)$, and $\prec .,.\succ$ denotes the usual duality between S and S^*

We define a weak solutions by,

Definition 2.1: *A function u is a weak solution of* (1) *in S; if for every v∈S we have:*

***Corresponding author:** Chamekh M, Mathematics Department, Faculty of Sciences and Arts-Alkamel, University of Jeddah, Jeddah, Saudi Arabia, Tel: +216 98562268; E-mail: chamekhmourad1@gmail.com

Received October 12, 2016; **Accepted** December 26, 2016; **Published** December 30, 2016

Citation: Chamekh M (2016) Existence of Solutions for a Fractional and Non-Local Elliptic Operator. J Appl Computat Math 5: 335. doi: 10.4172/2168-9679.1000335

$$\int_{\mathbb{R}^{2n}} \frac{|u(t)-u(z)|^{p-2}(u(t)-u(z))(v(t)-v(z))}{|t-z|^{n+p\gamma}} dtdz =$$

$$\alpha\int_{\Omega}\phi(t)|u(t)|^{q-2}u(t)v(t)dt+\int_{\Omega}\psi(t)|u(t)|^{r-2}u(t)v(t)dt.$$

The energy functional associated to the problem (1) is given by

$$\mathcal{E}_\alpha(u)=\frac{1}{p}\|u\|^p-\frac{\alpha}{q}\int_\Omega\phi(t)|u(t)|^q\,dt-\frac{1}{r}\int_\Omega\psi(t)|u(t)|^r\,dt.$$

The functional ε_α is frechet differentiable. We have $\prec \mathcal{E}'_\alpha(u),u\succ=0$, if u is a weak solution in S of (1). Then, the weak solutions of (1) are critical points of the functional $\mathcal{E}_\alpha$. The energy functional $\mathcal{E}_\alpha$ is unbounded below on the space S. Besides, this will certainly require the construction of an additional subset $\mathcal{F}_\alpha$ of S, where the functional $\mathcal{E}$ is bounded. To accomplish this end, we will study the following Nehari manifold to ensure that a solution exists

$$\mathcal{F}_\alpha:=\{u\in S:\prec\mathcal{E}'_\alpha(u),u\succ=0\}.$$

Then, u ∈ $\mathcal{F}_\alpha$ if and only if

$$\|u\|^p-\alpha\int_\Omega\phi(t)|u(t)|^q\,dt-\int_\Omega\psi(t)|u(t)|^r\,dt=0. \quad (3)$$

The aim in the following to provide an existence result.

Theorem 2.2: *If f and g satisfying ($\mathcal{P}_1$ – $\mathcal{P}_2$). Then, there exists α_0 > 0 such that for all 0 < α < α_0, problem (1) has at least two nontrivial solutions.*

The proof of the last theorem comprises basically of a simple few stages.

Lemma 2.3: *$\mathcal{E}_\alpha$ is coercive and bounded bellow on $\mathcal{F}_\alpha$.*

Proof: Let u ∈ $\mathcal{F}_\alpha$, then, we have

$$\mathcal{E}_\alpha(u)=(\frac{1}{p}-\frac{1}{r})\|u\|^p-\frac{\alpha}{q}\int_\Omega\phi(t)|u(t)|^q\,dt-\frac{1}{r})\int_\Omega\psi(t)|u(t)|^r\,dt$$

$$=(\frac{1}{p}-\frac{1}{r})\|u\|^p-\alpha(\frac{1}{q}-\frac{1}{r})\int_\Omega\phi(t)|u(t)|^q\,dt$$

$$\geq c_1\|u\|^p-c_2\|u\|^q.$$

Hence, $\mathcal{E}_\alpha$ is bounded bellow and coercive on $\mathcal{F}_\alpha$.

We define fiber maps F_u:[0,∞)→R according Drabek P and Brown [17,20] by,

$F_u(s)=\mathcal{E}_\alpha(su)$.

These fiber maps F_u Act as an important use in the proof because the Nehari manifold is closely linked to the behavior for them.

For u∈S, we can denote that tu∈$\mathcal{F}_\alpha$ if and only if $F'_u(s)=0$. Thus, we consider the follow parts $\mathcal{F}_\alpha$ into three parts corresponding to relative minima, relative maxima and points of inflection.

$\mathcal{F}_\alpha^+=\{su\in S:F'_u=0,F''_u>0\}$, $\mathcal{F}_\alpha^-=\{su\in S:F'_u=0,F''_u<0\}$, and $\mathcal{F}_\alpha^0=\{su\in S:F'_u=F''_u=0\}$.

We need to define m_u:[0,∞)→R by

$$\mu_u(s)=s^{p-q}\|u\|^p-s^{r-q}\int_\Omega\psi(s)|u|^r\,dt.$$

Clearly, for $s>0$, s_u∈$\mathcal{F}_\alpha$ if and only if s is a solution of

$$\mu_u(s)=\alpha\int_\Omega\phi(s)|u(t)|^q\,dt.$$

We consider the following subsets

$$\mathcal{V}^+=\{u\in S:I_\phi>0\};\mathcal{V}^-=\{u\in S:I_\phi<0\};\mathcal{W}^+=\{u\in S:I_\psi>0\}$$

and $\mathcal{W}^-=\{u\in S:I_\psi<0\}$.

with $I_\phi=\int_\Omega\phi(t)|u(t)|^q\,dt$ and $I_\psi=\int_\Omega\psi(t)|u(t)|^r\,dt$.

For studying the fiber map F_u correspond to the sign of I_φ and I_ψ, then, four possible cases can occur:

- If u∈$\mathcal{W}^-$∩$\mathcal{V}^-$, then, $F_u(0)=0$ and $F'_u(t)>0,\forall t>0$ which implies that F_u is strictly increasing, this resulting the absence of critical points.
- u∈$\mathcal{W}^+$∩$\mathcal{V}^-$ (or u∈$\mathcal{W}^-$∩$\mathcal{V}^+$). As we have $m_u(s_1)=0$. Here the only critical point of F_u is s_1, which is a absolute minimum point. Hence $s_1u\in\mathcal{F}_\alpha^+$.
- u∈$\mathcal{W}^+$∩$\mathcal{V}^+$, it exists μ_0 > 0 such that for α ∈ (0, μ_0), F_u has exactly one relative minimum s_1 and one relative maxima s_2. Thus $s_1u\in\mathcal{F}_\alpha^+$ and $s_2u\in\mathcal{F}_\alpha^-$.

We have the following result:

Corollary 2.4: *If α < μ_0, then, there exists δ_1 > 0 such that $\mathcal{E}_\alpha(u)>\delta_1$ for all $u\in\mathcal{F}_\alpha^-$.*

Proof. Let $u\in\mathcal{F}_\alpha^-$, then F_u has a positive absolute maximum at $T=1$ and $\int_\Omega\phi(t)|u(t)|^q\,dt>0$. Thus, if α < μ_0, then we have

$$\mathcal{E}_\alpha(u)=F_u(1)=F_u(T)$$

$$\geq\delta^{\frac{q}{p}}\left(\delta^{\frac{p-q}{p}}-\alpha c\right)>0,$$

the value of δ is given in Lemma 2.5.

Lemma 2.5: *There exists $_0$ > 0 such that for α∈(0,μ_0), F_u take positive value for all non-zero u∈S. Moreover, if*

u∈$\mathcal{F}^+$ $\mathcal{V}^+$, then, F_u has exactly two critical points.

Proof. Let u∈S, define

$$M_u(s)=\frac{s^p}{p}\|u\|^p-\frac{s^r}{r}\int_\Omega\psi(t)|u|^r\,dt.$$

Then,

$$M'_u(s)=s^{p-1}\|u\|^p-s^{r-1}\int_\Omega\psi(t)|u|^r\,dt.$$

If $\int_\Omega\psi(t)|u|^r\,dt$, M_u reaches its maximum value at $T=\left(\frac{\|u\|^p}{\int_\Omega\psi(t)|u|^r\,dt}\right)^{\frac{1}{r-p}}$. Moreover,

$$M_u(T)=(\frac{1}{p}-\frac{1}{r})\left(\frac{\|u\|^r}{\int_\Omega\psi(t)|u|^r\,dtdt}\right)^{\frac{p}{r-p}}$$

and

$$M''_u(T)=(p-r)\frac{\|u\|^{\frac{p(r-2)}{r-p}}}{\left(\int_\Omega\psi(t)|u|^r\,dt\right)^{\frac{p-2}{r-p}}}<0.$$

For $\leq\nu<p_\alpha^*$ we denoted by S_ν be the Sobolev constant of embedding S↣Lv(Ω), then, by 3 we have

Citation: Chamekh M (2016) Existence of Solutions for a Fractional and Non-Local Elliptic Operator. J Appl Computat Math 5: 335. doi: 10.4172/2168-9679.1000335

$$M_u(T) \geq \frac{r-p}{rp(KS_r^r)^{\frac{p}{r-p}}} = \delta, \tag{4}$$

which is independent of u. We now show that there exists $\mu_0 > 0$ such that $F_u(T) > 0$. Using condition g satisfying $(\mathcal{P}_1 - \mathcal{P}_2)$ and the Sobolev imbedding, we get

$$\frac{T^q}{q}\int_\Omega \phi(t)\,|u(t)|^q\, dt \leq \frac{S_q^q}{q}\|u\|^q\, T^q$$

$$= \frac{S_q^q}{q}\|u\|^q\left(\frac{\|u\|^p}{\int_\Omega \psi(t)\,|u|^r\, dt}\right)^{\frac{q}{r-p}}$$

$$= \frac{S_q^q}{q}\left(\frac{\|u\|^r}{\int_\Omega \psi(t)\,|u|^r\, dt}\right)^{\frac{q}{r-p}}$$

$$= \frac{S_q^q}{q}(M_u(T))^{\frac{q}{p}},$$

Thus

$$F_u(T) = M_u(T) - \alpha\frac{T^q}{q}\int_\Omega \phi(t)\,|u(t)|^q\, dt$$

$$\leq M_u(T) - \alpha c M_u(T)^{\frac{q}{p}}$$

$$= \delta^{\frac{q}{p}}\left(\delta^{\frac{p-q}{p}} - \alpha c\right),$$

where δ is the constant given in (4).

Let

$$\mu_0 = \frac{q\delta^{\frac{p-q}{p}}}{S_q^q}. \tag{5}$$

Then, choice of such μ_0 completes the proof.

Lemma 2.6: *There exists* $_1$ *such that if* $0 < \alpha < \mu_1$, *then* $\mathcal{F}_\alpha^0 = \varnothing$.

Proof: Let

$$\mu_1 = \frac{r-p}{S_q^q(r-q)}\left(\frac{p-q}{KS_r^r(r-q)}\right)^{\frac{p-q}{r-p}},$$

where K is given by (3).

Suppose otherwise, that $0 < \alpha < \mu_1$ such that $\mathcal{F}_\alpha^0 \neq \varnothing$. Then, for $u \in \mathcal{F}_\alpha^0$, we have

$$0 = F_u''(1) = (p-1)\|u\|^p - (r-1)\int_\Omega \psi(t)\,|u|^r\, dt - \alpha(q-1)\int_\Omega \phi(t)\,|u(t)|^q\, dt.$$

So, it follows from (3) that

$$(r-p)\|u\|^p = \alpha(r-q)\int_\Omega \phi(t)\,|u|^q\, dt$$

$$\leq \alpha(r-q)S_q^q\|u\|^q,$$

and so

$$\|u\| \leq \left(S\ \frac{r \quad q}{r \quad p}\right)^{\overline{p \quad q}}. \tag{6}$$

On the other hand, by (3) we get

$$(p-q)\|u\|^p = \alpha(r-q)\int_\Omega \psi(t)\,|u|^r\, dt$$

$$\leq K(r-q)S_r^r\|u\|^r,$$

then

$$\|u\| \geq \left(\frac{p-q}{KS_r^r(r-q)}\right)^{\frac{1}{r-p}}. \tag{7}$$

Combining (6) and (7) we obtain $\alpha \geq_1$, which is a contradiction.

Lemma 2.7: *Let u be a relative minimizer for* $\mathcal{E}_\alpha$ *on subsets* $\mathcal{F}_\alpha^+$ *or* $\mathcal{F}_\alpha^-$ *of* $\mathcal{F}_\alpha$, *then u is a critical point of* $\mathcal{E}_\alpha$.

Proof: Since u is a minimizer for $\mathcal{E}_\alpha$ under the constraint $I_\alpha(u) := \prec \mathcal{E}_\alpha(u), u \succ = 0$, by the theory of Lagrange multipliers, there exists μ∈R such that $\mathcal{E}_\alpha(u) = \mu I'_\alpha(u)$. Thus:

$$\prec \mathcal{E}_\alpha(u), u \succ = \mu \prec I'_\alpha(u), u \succ = \mu F_u''(1) = 0,$$

but $u \not\in \mathcal{F}_\alpha^0$ and so $F_u''(1) \neq 0$. Hence $\mu = 0$ completes the proof.

In the remain of this section, we assume that the parameter α satisfies $0 < \alpha < \alpha_0$, where α_0 is constant. That leads us consequently to the following results on the existence of minimizers in $\mathcal{F}_\alpha^+$ and $\mathcal{F}_\alpha^+$ for $\alpha \in (0, \alpha_0)$.

Theorem 2.8: *We have the following results*

$\mathcal{E}_\alpha$ has reached its minimum on $\mathcal{F}_\alpha^+$ and its maxima on $\mathcal{F}_\alpha^-$

Proof: To prove the theorem we proceed in two steps

Step 1: Since $\mathcal{E}_\alpha$ is bounded below on $\mathcal{F}_\alpha$ and so on $\mathcal{F}_\alpha^+$, there exists a minimizing sequence $\{u_k\} \subset \mathcal{F}_\alpha^+$ such that

$$\lim_{k\to\infty}\mathcal{E}_\alpha(u_k) = \inf_{u\in\mathcal{F}_\alpha^+}\mathcal{E}_\alpha(u).$$

As $\mathcal{E}_\alpha$ is coercive on $\mathcal{F}_\alpha$, $\{u_k\}$ is a bounded sequence in S. Therefore, for all $1 \leq \nu < p_s^*$ we have

$$\begin{cases} u_k \rightharpoonup u_\alpha & \textit{weakly in } S \\ u_k \to u_\alpha & \textit{strongly in } L^\nu(\mathbb{R}^n). \end{cases}$$

If we choose $u \in S$ such that $\int_\Omega \phi(t)\,|u(t)|^q\, dt > 0$, then, there exists $s_1 > 0$ such that $s_1 u \in \mathcal{F}_\alpha^+$ and $\mathcal{E}_\alpha(s_1 u) < 0$, Hence, $\inf_{u\in\mathcal{F}_\alpha^+}\mathcal{E}_\alpha(u) < 0$.

On the other hand, since $\{u_k\} \subset \mathcal{F}_\alpha$, then we have

$$\alpha\left(\frac{1}{q} - \frac{1}{r}\right)\int_\Omega \phi(t)\,|u_k(t)|^q\, dt = \left(\frac{1}{p} - \frac{1}{r}\right)\|u_k\|^p - \mathcal{E}_\alpha(u_k).$$

and so

$$\alpha\left(\frac{1}{q} - \frac{1}{r}\right)\int_\Omega \phi(t)\,|u_k(t)|^q\, dt = \left(\frac{1}{p} - \frac{1}{r}\right)\|u_k\|^p - \mathcal{E}_\alpha(u_k).$$

Letting k to infinity, we get

$$\int_\Omega \phi(t)\,|u_\alpha(t)|^q\, dt > 0. \tag{8}$$

Next we claim that $u_k \to u_\alpha$. Suppose this is not true, then

$$\|u_\alpha\|^p < \lim\inf_{k\to\infty}\|u_k\|^p.$$

Since $F'_{u_\alpha}(s_1) = 0$ it follows that $F'_{u_k}(s_1) > 0$ for sufficiently large k. So, we must have $s_1 > 1$ but $s_1 u_\alpha \in \mathcal{F}_\alpha^+$ and so

$$\mathcal{E}_\alpha(s_1 u_\alpha) < \mathcal{E}_\alpha(u_\alpha) \leq \lim_{k\to\infty}\mathcal{E}_\alpha(u_k) = \inf_{u\in\mathcal{F}_\alpha^+}\mathcal{E}_\alpha(u),$$

Citation: Chamekh M (2016) Existence of Solutions for a Fractional and Non-Local Elliptic Operator. J Appl Computat Math 5: 335. doi: 10.4172/2168-9679.1000335

which is a contradiction. It leads to $u_k \to u_\alpha$ and so $u_\alpha \in \mathcal{F}_\alpha^+$, since $\mathcal{F}_\alpha^0 = \emptyset$. Finally, u_α is a minimizer for $\mathcal{E}_\alpha$ on $\mathcal{F}_\alpha^+$.

Step 2: Let $u \in \mathcal{F}_\alpha^-$, then from corollary 2.4, there exists $\delta_1 > 0$ such that $\mathcal{E}_\alpha(u) \geq \delta_1$. So, there exists a minimizing sequence $\{u_k\} \subset \mathcal{F}_\alpha^-$ such that

$$\lim_{k\to\infty} \mathcal{E}_\alpha(u_k) = \inf_{u\in\mathcal{F}_\alpha^-} \mathcal{E}_\alpha(u) > 0. \tag{9}$$

On the other hand, since $\mathcal{E}_\alpha$ is coercive, $\{u_k\}$ is a bounded sequence in S. Therefore, for all $1 \le \nu < p_s^*$ we have

$$\begin{cases} u_k \rightharpoonup v_\alpha & \text{weakly in } S \\ u_k \to v_\alpha & \text{strongly in } L^\nu(\mathbb{R}^n). \end{cases}$$

Since $u \in \mathcal{F}_\alpha$, then we have

$$\mathcal{E}_\alpha(u_k) = \left(\frac{1}{p} - \frac{1}{q}\right) \| u_k \|^p + \left(\frac{1}{q} - \frac{1}{r}\right) \int_\Omega \psi(t) | u_k |^r \, dt. \tag{10}$$

Letting k to infinity, it follows from (9) and (10) that

$$\int_\Omega \psi(t) | v_\alpha |^r \, dt > 0. \tag{11}$$

Conclusion

Hence, $v_\alpha \in \mathcal{F}^+$ and so F_{v_α} has a absolute maximum at some point T and consequently, $Tv_\alpha \in \mathcal{F}_\alpha^-$. on the other hand, $u_k \in \mathcal{F}_\alpha^-$ implies that 1 is a absolute maximum point for F_{u_k} i.e.

$$\mathcal{E}_\alpha(su_k) = F_{u_k}(s) \le F_{u_k}(1) = \mathcal{E}_\alpha(u_k). \tag{12}$$

Next we claim that $u_k \to u_\alpha$. Suppose it is not true, then

$$\| u_\alpha \|^p < \lim \inf_{k\to\infty} \| u_k \|^p,$$

it follows from (12) that

$$\mathcal{E}_\alpha(Tv_\alpha) = \frac{T^p}{p} \| v_\alpha \|^p - \frac{T^r}{r} \int_\Omega \psi(t) | v_\alpha |^r \, dt - \alpha \frac{T^q}{q} \int_\Omega \phi(t) | v_\alpha |^q \, dt$$

$$< \inf_{k\to\infty} \left(\frac{T^p}{p} \| u_k \|^p - \frac{T^r}{r} \int_\Omega \psi(t) | u_k |^r \, dt - \alpha \frac{T^q}{q} \int_\Omega \phi(t) | u_k |^q \, dt \right)$$

$$\le \lim_{k\to\infty} \mathcal{E}_\alpha(Tu_k) \le \lim_{k\to\infty} \mathcal{E}_\alpha(u_k) = \inf_{u\in\mathcal{F}_\alpha^-} \mathcal{E}_\alpha(u),$$

which is a contradiction. Hence, $u_k \to v_\alpha$ and so $v_\alpha \in \mathcal{F}_\alpha^-$, since $\mathcal{F}_\alpha^0 = \emptyset$.

Now, Let us proof Theorem 1.1: By the Lemmas 2.5, 2.6, 2.7 and the theorem 2.8 the problem (1) has two weak solution $u_\alpha \in \mathcal{F}_\alpha^+$ and $v_\alpha \in \mathcal{F}_\alpha^+$. On the other hand, from (8) and (11), this solutions are nontrivial. Since $\mathcal{F}_\alpha^- \cap \mathcal{F}_\alpha^+ = \emptyset$, then, u_α and v_α are distinct.

References

1. Momani S, Odibat Z (2006) Analytical solution of a time-fractional Navier-Stokes equation by Adomian decomposition method. Appl Math Comput 177: 488-494.
2. Benson DA, Schumer R, Meerschaert MM, Wheatcraft SW (2001) Fractional dispersion, Lévy motions, and the MADE tracer tests. Transport in Porous Media 42: 211-240.
3. Laskin N (2002) Fractional Schrödinger equations. Phys Rev E 66: 056-108.
4. Mishra LN, Sen M (2016) On the concept of existence and local attractivity of solutions for some quadratic Volterra integral equation of fractional order. Applied Mathematics and Computation Vol. 285: 174-183.
5. Mishra LN, Agarwal RP, Sen M (2016) Solvability and asymptotic behavior for some nonlinear quadratic integral equation involving Erd lyi-Kober fractional integrals on the unbounded interval. Progress in Fractional Differentiation and Applications 2: 153-168.
6. Meerschaert M, Sikorskii A (2012) Stochastic Models for Fractional Calculus. Walter de Gruyter GmbH & Co. KG, Berlin/Boston.
7. Baeumer B, Kovacs M, Meerschaert MM (2008) Numerical solutions for fractional reaction diffusion equations. Computers and Mathematics with Applications 55: 2212-2226.
8. Waurick M (2014) Homogenization in fractional elasticity. SIAM Journal on Mathematical Analysis 46: 1551-1576.
9. Ambrosetti A, Azorero GJ, Peral I (1996) Multiplicity results for some nonlinear elliptic equations. J Funct Anal 137: 219-242.
10. Chen C, Kuo Y, Wu T (2011) The Nehari manifold for a Kirchhoff type problem involving sign-changing weight functions. J Differential Equations 250: 1876-1908.
11. Schikorra A (2015) Nonlinear commutators for the fractional p-Laplacian and applications. Mathematische Annalen pp: 1-26.
12. Servadei R, Valdinoci E (2013) Variational methods for non-local operators of elliptic type, Discrete Contin Dyn Syst 33: 2105-2137.
13. Xifeng Su, Yuanhong W (2012) Multiplicity of solutions for non-local elliptic equations driven by fractional Laplacian.
14. Ghanmi A (2014) Multiplicity of Nontrivial Solutions of a Class of Fractional p-Laplacian Problem. Journal of Analysis and its Applications 34: 309-319.
15. Xiang M, Zhang BL, Radulescu (2016) Existence of solutions for perturbed fractional p-Laplacian equations. J Differ Equ 260: 1392-1413.
16. Di Nezza E, Palatucci G, Valdinoci E (2012) Hitchhiker's guide to the fractional Sobolev spaces. Bull Sci Math 136: 521-573.
17. Brown KJ, Wu TF (2007) A fibering map approach to a semi-linear elliptic boundary value problem Electron. J Differential Equations 69: 1-9.
18. Ghanmi A (2014) Existence of Nontrivial Solutions of p-Laplacian Equation with Sign-Changing Weight Functions. ISRN Mathematical Analysis 2014: 965: 967.
19. Drabek P, Pohozaev SI (1997) Positive solutions for the p-Laplacian: Application of the fibering method. Proc Royal Soc Edinburgh Sect A 127: 703-726.
20. Brown KJ, Zhang Y (2003) The Nehari manifold for a semilinear elliptic problem with a sign changing weight function. J Differential Equations 193: 481-499.