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Existence of Positive Solutions to Periodic Boundary Value Problems for Nonlinear Ordinary Differential Systems with Sign-changing Green Function

Wanjun Li[1]* and Yukun An[2]

[1]Department of Mathematics, Longdong University, Qingyang 745000, Gansu, P. R. China
[2]Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing, 210016, P. R. China

Abstract

This paper deals with the periodic boundary value problems

$$\begin{cases} u'' + \rho_1^2 u = f_1(t,u(t),v(t)), & t \in (0,T), \\ v'' + \rho_2^2 v = f_2(t,u(t),v(t)), & t \in (0,T), \\ u(0) = u(T), u'(0) = u'(T), \\ v(0) = v(T), v'(0) = v'(T), \end{cases}$$

where $0 < \rho_i \le \frac{3\pi}{2T}(i=1,2)$ is a constant and in which case the associated Greens function may changes sign. The existence result of positive solutions is established by using the fixed point index theory of cone mapping.

Keywords: Positive solution; Nonlinear ordinary differential systems; Periodic boundary value problem; Sign-changing Greens function; Fixed point theorem

Introduction

The purpose of this paper is to establish the existence of positive solutions to a class of periodic boundary value problems for systems of second order nonlinear differential equations

$$\begin{cases} u'' + \rho_1^2 u = f_1(t,u(t),v(t)), & t \in (0,T), \\ v'' + \rho_2^2 v = f_2(t,u(t),v(t)), & t \in (0,T), \\ u(0) = u(T), u'(0) = u'(T), v(0) = v(T), v'(0) = v'(T), \end{cases} \quad (3.1)$$

where $0 < \rho_i \le \frac{3\pi}{2T}(i=1,2)$ are constants and in which case the associated Greens function may changes sign.

In recent years, because of wide interests in physics and engineering, the periodic boundary value problems have been investigated by many authors. For example, the periodic boundary value problems

$$\begin{cases} u'' + a(t)u = f(t,u), & t \in (0,T), \\ u(0) = u(T), u'(0) = u'(T), \end{cases} \quad (3.2)$$

where f is a continuous or $L^1 - Caratheodory$ type function have been extensively studied; see, for example, Atici and Guseinov [1], Nkashama and Santannilla et al. [2] Rachunkoväa and Tvrdäy [3,4] Kiguradze and Stanck [5],Torres [6], Jiang et al. [7,8,14], ORegan et al. [9], Wang [10], Graef et al. [11], Zhang et al. [12], and the references contained therein. In these papers, the major assumption is that their associated Greens functions are of one sign.

Recently, in [13], the hypothesis is weakened as the Green function G(t, s) associated with problem (3.2) is non-negative. In [15], the author improve the result of (3.2) and prove the existence results of at least two positive solutions under conditions weaker than sub- and super-linearity.

More recently, In [16,17], the authors consider the boundary value problem (3.2) and establish the existence of positive solutions in the case where the associated Greens function may changes sign.

Inspired by the work of the above mentioned papers, we investigate the periodic boundary value problems (3.1) in this paper, and the associated Greens function may changes sign. The aim is to prove the existence of positive solutions to the problem by using the fixed point index theory of cone mapping. Our ideas mainly come from references [7-10].

Preliminary Results

In this section, we present some notation and lemmas that will be used in the paper. We shall consider the Banach space $E = C[0,T] \times C[0,T]$ equipped with the standard norm $\|(u,v)\| = \|u\| + \|v\| = \max_{0\le t\le T}|u(t)| + \max_{0\le t\le T}|v(t)|$, $(u,v) \in E$.

To prove our result, we need the following fixed point index theorem of cone mapping.

Lemma 4.1 (see [18]) Let E be a Banach space and $K \subset E$ be a closed convex cone in E. Let $A: K \to K$ be a completely continuous operator and let $i(A, K_r, K)$ denote the fixed point index of operator A.

(i) If $\mu Au \ne u$ for every $u \in \partial K_r$ and $0 < \mu \le 1$, then $i(A, K_r, k) = 1$.

(ii) If $\inf_{u \in \partial K_r} \|Au\| > 0$ and $\mu Au \ne u$ for every $u \in \partial K_r$ and $\mu \ge 1$, then $i(A, K_r, k) = 0$.

***Corresponding author:** Wanjun Li, Department of Mathematics, Longdong University, Qingyang 745000, Gansu, P. R. China, E-mail: lwj1965@163.com

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Let $G_i(t,s)(i=1,2)$ denote the Green's function of homogeneous periodic boundary value problem:

$$\begin{cases} \omega'' + \rho_i^2 \omega = e(t), & t \in (0,T), \\ \omega(0) = \omega(T), \omega'(0) = \omega'(T), & \end{cases} \tag{4.1}$$

where $0 < \rho_i \le \frac{3\pi}{2T}(i=1,2)$ and $e(t)$ is a continuous function on $[0,T]$. It is well known that the solutions of (4.1) can be expressed in the following forms

$$\omega(t) = \int_0^T G_i(t,s)e(s)ds, \tag{4.2}$$

and $G_i(t,s)$ can be expressed as

$$G_i(t,s) = \begin{cases} \dfrac{\sin\rho_i(t-s) + \sin\rho_i(T-t+s)}{2\rho_i(1-\cos\rho_i T)}, & 0 \le t \le s \le T, \\ \dfrac{\sin\rho_i(s-t) + \sin\rho_i(T-s+t)}{2\rho_i(1-\cos\rho_i T)}, & 0 \le s \le t \le T. \end{cases} \tag{4.3}$$

By direct computation, we get

$$\frac{\sin\rho_i T}{2\rho_i(1-\cos\rho_i T)} \le G_i(t,s) \le \frac{\sin\frac{\rho_i T}{2}}{\rho_i(1-\cos\rho_i T)} = \max_{t,s\in[0,T]} G_i(t,s), \tag{4.4}$$

and

$$G_i(t,s) < 0$$

for $|t-s| < \frac{T}{2} - \frac{\pi T}{2\rho_i}$ when $\frac{\pi}{T} \le \rho_i \le \frac{3\pi}{2T}$, and

$$g_i(t) = \int_0^T G_i(t,s)ds = \frac{1}{\rho_i^2}, t \in [0,T], \tag{4.5}$$

$$\min_{t\in[0,T]} \frac{\int_0^T G_i^+(t,s)ds}{\int_0^T G_i^-(t,s)ds} = \frac{1}{1-\sin\frac{\rho_i T}{2}} \tag{4.6}$$

where G_i^+ and G_i^- are the positive and negative parts of G_i.

We denote

$$\sigma_i = \frac{1}{\rho_i^2 \max_{t,s\in[0,T]} G_i(t,s)} = \frac{2\sin\frac{\rho_i T}{2}}{\rho_i}, \tag{4.7}$$

$$\gamma_i = \begin{cases} +\infty, & 0 \le \rho_i \le \frac{\pi}{T}, \\ \dfrac{1}{1-\sin\frac{\rho_i T}{2}}, & \frac{\pi}{T} < \rho_i \le \frac{3\pi}{2T}, \end{cases} \tag{4.8}$$

and

$$\sigma = \min\{\sigma_1, \sigma_2\}, \gamma = \max\{\gamma_1, \gamma_2\}. \tag{4.9}$$

Define the cone K in E by

$$K = \{(u,v) \in E : u(t) \ge 0, v(t) \ge 0, \int_0^T (u(s)+v(s))ds \ge \sigma \|(u,v)\|\}. \tag{4.10}$$

It is well known that the system (1.1) is equivalent to the equation

$$(u(t),v(t)) = (\int_0^T G_1(t,s)f_1(s,u(s),v(s))ds, \int_0^T G_2(t,s)f_2(s,u(s),v(s))ds). \tag{4.11}$$

Define two operators $A_i : E \to C[0,T]$ as

$$A_i(u,v)(t) = \int_0^T G_i(t,s)f_i(s,u(s),v(s))ds, i=1,2, \tag{4.12}$$

and define an operator $A : E \to E$ as

$$A(u,v) = (A_1(u,v), \ A_2(u,v)). \ (u,v) \in E. \tag{4.13}$$

It is clear that the existence of a positive solution to problem (1.1) is equivalent to the existence of a fixed point of A in K.

We also make the following assumptions:

$(H1)$ $f_i(i=1,2) : [0,T]\times[0,+\infty)\times[0,+\infty) \to [0,+\infty)$ is continuous;

$(H2)$ $0 \le m_i = \inf_{(t,u,v)\in[0,T]\times[0,+\infty)\times[0,+\infty)} f_i(t,u,v)$ and $M_i = \sup_{(t,u,v)\in[0,T]\times[0,+\infty)\times[0,+\infty)} f_i(t,u,v) \le +\infty$, $i=1,2$;

$(H3)$ $\frac{M}{m} \le \gamma$, when $m=0$ we define $\frac{M}{m} = +\infty$, here $m = \min_i\{m_i\}, M = \max_i\{M_i\}$.

Lemma 4.3 Assume that $(H1)$, $(H2)$ and $(H3)$ holds, then $A : E \to E$ is completely continuous and $A(K) \subset K$.

Proof. Let $(u,v) \in K$, then in case of $\gamma_i = +\infty$, since $G_i(t,s) \ge 0$, we have $A_i(u,v)(t) \ge 0$ on $[0,T], i=1,2$; in case of $\gamma_i < +\infty, i=1,2$, for any $(t,s) \in [0,T]\times[0,T]$, we have

$$\begin{aligned} A_i(u,v)(t) &= \int_0^T G_i(t,s)f_i(s,u(s),v(s))ds \\ &= \int_0^T (G_i^+(t,s) - G_i^-(t,s))f_i(s,u(s),v(s))ds \\ &\ge \int_0^T (G_i^+(t,s)m - G_i^-(t,s)M)ds \\ &= m\int_0^T (G_i^+(t,s) - \frac{M}{m}G_i^-(t,s))ds \\ &\ge m\int_0^T (G_i^+(t,s) - \gamma_i G_i^-(t,s))ds \\ &\ge 0, \ i=1,2. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\int_0^T (A_1(u,v)(t) + A_2(u,v)(t))dt \\ &= \int_0^T (\int_0^T G_1(t,s)f_1(s,u(s),v(s))ds + \int_0^T G_2(t,s)f_2(s,u(s),v(s))ds)dt \\ &= \int_0^T\int_0^T G_1(t,s)f_1(s,u(s),v(s))dsdt + \int_0^T\int_0^T G_2(t,s)f_2(s,u(s),v(s))dsdt \\ &= \int_0^T f_1(s,u(s),v(s))\int_0^T G_1(t,s)dtds + \int_0^T f_2(s,u(s),v(s))\int_0^T G_2(t,s)dtds \\ &\ge \frac{1}{\rho_1^2}\int_0^T f_1(s,u(s),v(s))ds + \frac{1}{\rho_2^2}\int_0^T f_2(s,u(s),v(s))ds, \end{aligned}$$

and

$$A_i(u,v)(t) = \int_0^T G_i(t,s)f_i(s,u(s),v(s))ds \le \max_{t,s\in[0,T]} G_i(t,s)\int_0^T f_i(s,u(s),v(s))ds$$

for $t \in [0,T]$. Thus,

$$\int_0^T (A_1(u,v)(t)+A_2(u,v)(t))dt \geq \frac{A_1(u,v)(t)}{\rho_1^2 \max_{t,s\in[0,T]} G_1(t,s)} + \frac{A_2(u,v)(t)}{\rho_2^2 \max_{t,s\in[0,T]} G_2(t,s)}$$

$$\geq \sigma_1 A_1(u,v)(t) + \sigma_2 A_2(u,v)(t)$$

$$\geq \sigma(A_1(u,v)(t) + A_2(u,v)(t)),$$

hence, $\int_0^T (A_1(u,v)(t)+A_2(u,v)(t))dt \geq \sigma \| A(u,v)\|$. Therefore, $A(K)\subset K$. A standard argument can be used to show that $A: E \to E$ is completely continuous.

Existence Results

In the section we discuss the existence of at least one solution to the system (1.1). To be convenience, we introduce the notations:

$$f_{i0} = \lim_{u+v\to 0}\frac{f_i(t,u,v)}{u+v}, f_{i\infty} = \lim_{u+v\to\infty}\frac{f_i(t,u,v)}{u+v},$$

and suppose that $f_{i0}, f_{i\infty} \in [0,+\infty]$, $i=1,2$.

Theorem 5.1 Assume that $(H1)$, $(H2)$ and $(H3)$ hold. Furthermore, suppose that $f_{i0} > \rho = \max\{\rho_1^2, \rho_2^2\}$ and $f_{i\infty} < \omega = \min\{\frac{\rho_1^2}{2}, \frac{\rho_2^2}{2}\}$ in case of $\gamma = +\infty$. Then problem (1.1) has at least one positive solution.

Proof. By $f_{i0} > \rho$, we can choose $\varepsilon > 0$ such that $f_{i0} \geq \rho + \varepsilon$. Then there exists $r_0 > 0$ such that

$$f_i(t,u,v) \geq (\rho+\varepsilon)(u+v), \quad forall\ u+v\in[0,r_0]. \tag{5.1}$$

Let $r\in(0,r_0)$ and $\Omega_r = \{(u,v)\in K \,|\, \|(u,v)\| \leq r\}$, then for every $(u,v)\in\partial\Omega_r$, we have $\|(u,v)\| = r$, and so

$$\| A_i(u,v)\| \geq \frac{1}{T}\int_0^T A_i(u,v)(t)dt$$

$$= \frac{1}{T}\int_0^T\int_0^T G_i(t,s) f_i(s,u(s),v(s))dsdt$$

$$= \frac{1}{T}\int_0^T f_i(s,u(s),v(s))\int_0^T G_i(t,s)dtds$$

$$\geq \frac{1}{T\rho_i^2}\int_0^T f_i(s,u(s),v(s))ds$$

$$\geq \frac{\rho+\varepsilon}{T\rho_i^2}\int_0^T (u(s)+v(s))ds$$

$$\geq \frac{\rho+\varepsilon}{T\rho}\sigma \|(u,v)\|$$

$$= \frac{(\rho+\varepsilon)\sigma r}{T\rho} > 0, i=1,2.$$

Hence, $\inf_{(u,v)\in\partial\Omega_r} \| A_i(u,v)\| > 0$, $i=1,2$, and so

$$\inf_{(u,v)\in\partial\Omega_r} \| A(u,v)\| = \inf_{(u,v)\in\partial\Omega_r} (\| A_1(u,v)\| + \| A_2(u,v)\|)$$

$$\geq \inf_{(u,v)\in\partial\Omega_r} \| A_1(u,v)\| + \inf_{(u,v)\in\partial\Omega_r} \| A_2(u,v)\| > 0.$$

Next, we show that $\mu A(u,v) \neq (u,v)$ for any $(u,v)\in\partial\Omega_r$ and $\mu \geq 1$. In fact, if there exist $(u_0,v_0)\in\partial\Omega_r$, and $\mu_0 \geq 1$ such that $\mu_0 A(u_0,v_0) = (u_0,v_0)$, i.e., $\mu_0 A_1(u_0,v_0) = u_0$ and $\mu_0 A_2(u_0,v_0) = v_0$, then (u_0,v_0) satisfies

$$\begin{cases} u_0'' + \rho_1^2 u_0 = \mu_0 f_1(t,u_0(t),v_0(t)), & t\in(0,T), \\ v_0'' + \rho_2^2 v_0 = \mu_0 f_2(t,u_0(t),v_0(t)), & t\in(0,T), \\ u_0(0) = u_0(T), u_0'(0) = u_0'(T), v_0(0) = v_0(T), v_0'(0) = v_0'(T), & \end{cases} \tag{5.2}$$

Integrating the first equation in (5.2) from 0 to T and using the periodicity of $u_0(t)$ and (5.1), we get

$$\rho\int_0^T u_0(t)dt \geq \rho_1^2\int_0^T u_0(t)dt$$

$$= \mu_0\int_0^T f_1(t,u_0(t),v_0(t))dt$$

$$\geq (\rho+\varepsilon)\int_0^T (u_0(t)+v_0(t))dt$$

$$\geq (\rho+\varepsilon)\int_0^T u_0(t)dt.$$

Similarly, we have $\rho\int_0^T v_0(t)dt \geq (\rho+\varepsilon)\int_0^T v_0(t)dt$, thus we get

$$\rho\int_0^T (u_0(t)+v_0(t))dt \geq (\rho+\varepsilon)\int_0^T (u_0(t)dt + v_0(t))dt.$$

Since $\int_0^T (u_0(t)+v_0(t))dt \geq \sigma \|(u_0,v_0)\| = \sigma r > 0$ and we see that $\rho \geq \rho+\varepsilon$, which is a contradiction. Hence, by Lemma 4.1, we have

$$i(A,\Omega_r,K) = 0. \tag{5.3}$$

On the other hand, because $f_{i\infty} < \omega$, there exist $\varepsilon\in(0,\omega)$ and $R_0 > 0$ such that

$$f_i(t,u,v) \leq (\omega-\varepsilon)(u+v), \quad forall\ u+v \geq R_0. \tag{5.4}$$

Let $M_{R_0} = \max_{0\leq u+v\leq R_0} f(t,u,v)$, then it is clear that

$$f_i(t,u,v) \leq (\omega-\varepsilon)(u+v) + M_{R_0},\ forall\ u+v \geq 0. \tag{5.5}$$

If there exist $(u_0,v_0)\in\partial\Omega_r$, and $0<\mu_0\leq 1$ such that $\mu_0 A(u_0,v_0) = (u_0,v_0)$, then (5.2) is valid. Integrating again the first equation in (5.2) from 0 to T, and from (5.5), we have

$$\omega\int_0^T u_0(t)dt \leq \frac{\rho_1^2}{2}\int_0^T u_0(t)dt$$

$$= \frac{\mu_0}{2}\int_0^T f_1(t,u_0(t),v_0(t))dt$$

$$= \frac{1}{2}\int_0^T ((\omega-\varepsilon)(u_0(t)+v_0(t)) + M_{R_0})dt$$

$$\leq \frac{\omega-\varepsilon}{2}\int_0^T (u_0(t)+v_0(t))dt + \frac{TM_{R_0}}{2} \tag{5.6}$$

Similarly, we have

$$\omega\int_0^T v_0(t)dt \leq \frac{\omega-\varepsilon}{2}\int_0^T (u_0(t)+v_0(t))dt + \frac{TM_{R_0}}{2} \tag{5.7}$$

From (5.6) and (5.7), we get

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$$\omega\int_0^T (u_0(t)+v_0(t))dt \le (\omega-\varepsilon)\int_0^T (u_0(t)+v_0(t))dt + TM_{R_0}.$$

Therefore, we obtain that

$$\sigma \| (u_0,v_0) \| \le \int_0^T (u_0(t)+v_0(t))dt \le \frac{TM_{R_0}}{\varepsilon}, i.e., \| (u_0,v_0) \| \le \frac{TM_{R_0}}{\sigma\varepsilon}.$$

We choose $R > \max\{r_0, \frac{TM_{R_0}}{\sigma\varepsilon}\}$ and $\Omega_R = \{(u,v)\in K \mid \| (u,v) \| < R\}$, then $\mu A(u,v) \ne (u,v)$ for any $(u,v)\in\partial\Omega_R$ and $0<\mu\le 1$. Therefore, by Lemma 4.1, we get

$$i(A,\Omega_R,K)=1. \tag{5.8}$$

From (5.3) and (5.8) it follows that

$$i(A,\Omega_R \setminus \overline{\Omega}_r, K) = i(A,\Omega_R,K) - i(A,\Omega_r,K) = 1.$$

Therefore, A has a fixed poind in $\Omega_R \setminus \overline{\Omega}_r$, which is the positive solution of BVP (1.1). The proof is completed.

An Example

As an example, we consider the existence of positive solutions for the following system

$$\begin{cases} u'' + \rho_1^2 u = 1 + h(\frac{\pi}{T}-\rho_1)(u+v)^p + (1-h(\frac{\pi}{T}-\rho_i))\dfrac{2\sin\frac{\rho_1 T}{2}}{\pi(1-\sin\frac{\rho_i T}{2})}\arctan(u+v), \quad t\in(0,T), \\ v'' + \rho_2^2 v = 1 + h(\frac{\pi}{T}-\rho_2)(u+v)^p + (1-h(\frac{\pi}{T}-\rho_2))\dfrac{2\sin\frac{\rho_i T}{2}}{\pi(1-\sin\frac{\rho_i T}{2})}\arctan(u+v), \quad t\in(0,T), \\ u(0)=u(T), u'(0)=u'(T), v(0)=v(T), v'(0)=v'(T), \end{cases} \tag{6.1}$$

Let $p\in(0,1)$, $0<\rho_i\le\frac{3\pi}{2T}(i=1,2)$, h be the function:

$$h(x)=\begin{cases} 1, & x>0, \\ 0, & x<0, \end{cases}$$

By the direct calculation, we get $m_i=1$ and $M_i=\gamma_i$, and $f_{i0}=\infty$ and $f_{i\infty}=0$ in case of $\gamma=+\infty$. It is clear the conditions of Theorem 3.1 hold for the problem (6.1) and therefore, (6.1) have at least one positive solution from Theorem 3.1.

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