Existence of Multiple Solutions for P-Laplacian Problems Involving Critical Exponents and Singular Cylindrical Potential

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Abstract
In this paper, we establish the existence of multiple solutions for p-Laplacian problems involving critical exponents and singular cylindrical potential, by using Ekeland’s variational principle and mountain pass theorem without Palais-Smale conditions.

Keywords: P-Laplacian; Critical exponents; Cylindrical potential; Dimensional

Introduction
The aim of this paper is to establish the existence and multiplicity of solutions to the following quasilinear elliptic problem

\[ (P_{\lambda\mu}) \left\{ \begin{array}{l}
-L_{\lambda\mu}u - \mu|u|^{p-2}u = h(y)|u|^{q-2}u + \lambda g(x) \quad \text{in } \mathbb{R}^N, y \neq 0 \\
\end{array} \right. \]

Where \( L_{\lambda\mu}u = \text{div}(|\nabla u|^{p-2}\nabla u) \), \( 1 < p < k, k \) and \( N \) are integers with \( N > p \), \( p \leq k \leq N \), \( \mathbb{R}^N = \mathbb{R}^N \times \mathbb{R}^d \), the point \( x \in \mathbb{R}^N \) can be written as \( x = (y,z) \in \mathbb{R}^N \times \mathbb{R}^d \). \( \lambda \) and \( \mu \) are positive parameters which we will specify later, \( g \) is a continuous function on \( \mathbb{R}^d \), and \( h \) is a bounded positive function on \( \mathbb{R}^N \).

Let \( \mathcal{H}_p = L^p(\mathbb{R}^N, |\nabla u|^p \, dx) \) be the space defined as the completion of \( C^0(\mathbb{R}^N, |\nabla u|^p \, dx) \) with respect to the norm \( \|u\|_{\mathcal{H}_p} = \left( \int_{\mathbb{R}^N} (|\nabla u|^p + \mu |u|^p) \, dx \right)^{1/p} \).

When \( \mu < \mu_{N,p} \), Hardy type inequality implies that the norm

\[ \|u\|_{\mathcal{H}_p} = \left( \int_{\mathbb{R}^N} (|\nabla u|^p + \mu |u|^p) \, dx \right)^{1/p} \]

is well defined in \( \mathcal{H}_p \) and \( \mathcal{H}_p \) is equivalent to \( \mathcal{H}_k \); since the following inequalities hold:

\[ (1 - (\max(\mu, 0)/\mu_{N,p}))^k \mathcal{H}_k \subseteq \mathcal{H}_p \subseteq (1 - (\min(\mu, 0)/\mu_{N,p}))^k \mathcal{H}_k, \]

for all \( \mu \in \mathcal{H}_p \).

We define the weighted Sobolev space \( \mathcal{D} := \mathcal{H}_k \cap L^p(\mathbb{R}^N, |\nabla u|^p \, dx) \) which is a Banach space with respect to the norm defined by \( \mathcal{N}(u) := \|u\|_{\mathcal{H}_k} + \|(1/q)u\|_q \).

Several existence results are available in the case \( p = 2 \) and \( k = N \); we quote for example [1-3], and the references therein. For more details, when \( h \equiv 1, \mu = 0 \) and \( q = 2 \), the regular problem \( (P_{\lambda\mu}) \) has been considered, on the bounded domain \( \Omega \), by Tarantello [4]. She proved that for \( g \in \mathcal{H}_k(\Omega) \), not identically zero and satisfying a suitable condition, the problem considered admits two solutions. Also, they are two nontrivial non-negative solutions when \( g \neq 0 \).


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Smale conditions with positive energy.

Our main result is given as follows

**Theorem 1:** Suppose that \( p + k < N \), \( 0 \leq s < p \), \( \mu < \tilde{\mu}_{l,p} \), hypothesis (H) holds, \( g \in H^s\cap C(\mathbb{R}^N) \) and \( g \not\equiv 0 \). Then there exists \( \lambda_0 > 0 \) such that the problem \( (P_{l,p}) \) has at least two solutions for any \( \lambda \in (0, \lambda_0) \).

This paper is organized as follows. In Section 2, we give some preliminaries.

Section 3 is devoted to the proof of Theorem 1.

**Preliminaries**

We start by recalling the following definition and properties from the paper [6].

The first inequality that we need is the Hardy inequality

\[
\int_{\mathbb{R}^N} |\nabla u|^p \, dx \geq c_0 \left( \int_{\mathbb{R}^N} (|x|^{2s} u^p) \, dx \right)^{p/2},
\]

for all \( u \in D^p_0(\mathbb{R}^N) \).

**Definition 1:** An entire solution \( u \) to \( (P_{l,p}) \) is a ground state solution if it achieves the best constant

\[
S_{l,p} = S_{l,p}(k,N) = \lim_{n \to \infty} \frac{\int_{\mathbb{R}^N} (|\nabla u|^p - \mu |u|^{p}) \, dx}{\int_{\mathbb{R}^N} |u|^p \, dx}.
\]

**Lemma 1:** Assume [6] that \( p + k < N, 0 \leq s < p \) and \( \mu < \tilde{\mu}_{l,p} \).

Then, the infimum \( S_{l,p} \) is achieved on \( H_0((\mathbb{R}^N \setminus \{0\}) \times \mathbb{R}^N) \).

**Lemma 2:** Let \( (u_n) \subset D \) be a Palais-Smale sequence \( (PS) \) of \( I_{l,p} \), i.e.,

\[
I_{l,p}(u_n) \to c \quad \text{and} \quad I'_{l,p}(u_n) \to 0 \quad \text{in} \quad D (dualo\text{f} D) \quad \text{as} \quad n \to \infty
\]

for some \( c \in \mathbb{R} \).

Then, \( u_n \to u \) in \( D \) and \( I'_{l,p}(u) = 0 \).

**Proof:** From (4.3);

We have

\[
(1/p) \left| \int_{\mathbb{R}^N} (|\nabla u|^p - (1/q) \int_{\mathbb{R}^N} b(x) |\nabla u|^q \, dx - \lambda \int_{\mathbb{R}^N} g(x) u^p \, dx \right| \to c + o_n(1)\]

and

\[
\left| \int_{\mathbb{R}^N} (\lambda |u|^p - \lambda (q - 1)/q \int_{\mathbb{R}^N} g(x) u^p \, dx \right| \to o_n(1),
\]

for \( n \) large, where \( o_n(1) \) denotes \( o_n(1) \to 0 \) as \( n \to \infty \). Then,

\[
c + o_n(1) = I_{l,p}(u_n) - (1/p) \int_{\mathbb{R}^N} b(x) |\nabla u|^q \, dx \geq \left( (q - 1)/q \right) \left| \int_{\mathbb{R}^N} g(x) u^p \, dx \right|.
\]

\( (u_n) \) is bounded in \( D \). Up to a subsequence if necessary, we obtain that

\[
u_n \to u \quad \text{in} \quad D
\]

\[
u_n \to u \quad \text{in} \quad L_0(\mathbb{R}^N; |\nu|^q)
\]

\[
u_n \to u \quad \text{a.e. in} \quad \mathbb{R}^N.
\]

Consequently, we get

\[
I'_{l,p}(u) = 0.
\]

**Lemma 3:** Let \( (u_n) \subset D \) be a Palais-Smale sequence \( (PS) \) of \( I_{l,p} \) for some \( c \in \mathbb{R} \).

Then, \( u_n \to u \) in \( D \) and either \( u_n \to u \) in \( D \) or \( c \geq I_{l,p}(u) + ((q - p)/pq)(\|h_{l,p}S_{l,p}\|^{(q-/p)})\)

for all \( p \neq (p', 0) \).

**Proof:** We know that \( (u_n) \) is bounded in \( D \). Up to a subsequence if necessary, we have that

\[
u_n \to u \quad \text{in} \quad D
\]

\[
u_n \to u \quad \text{a.e. in} \quad \mathbb{R}^N.
\]

Denote \( v_n = u_n - u \), then \( v_n \to 0 \). As in Brezis and Lieb [2]; we have

\[
\|v_n\| = \|v_n\| - \|u\| \quad \text{and}
\]

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} b(y) \left| |v_n|^q - |u|^q \right| \, dy = \int_{\mathbb{R}^N} b(y) |v|^q \, dy.
\]

On the other hand, by using the assumption \( (H) \), we obtain

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} b(y) \left| |v_n|^q - |u|^q \right| \, dy = h_0 \lim_{n \to \infty} \int_{\mathbb{R}^N} b(y) |v|^q \, dy\]

Thus, we get

\[
I_{l,p}(u_n) = I_{l,p}(u) + ((1/p) \left| \int_{\mathbb{R}^N} (|\nabla u|^q - (1/q) \int_{\mathbb{R}^N} b(x) |\nabla u|^q \, dx \right| \, dx\right)
\]

\[
\left( (q - 1)/q \right) \left| \int_{\mathbb{R}^N} g(x) u^p \, dx \right| + o_n(1)
\]

Then we can assume that

\[
\lim_{n \to \infty} \|v_n\| = h_0 \lim_{n \to \infty} \|v_n\| = l_0 \geq 0.
\]

Assume \( l_0 > 0 \), we have by definition of \( S_{l,p} \)

\[
l \geq S_{l,p}(\|h_{l,p}\|^{(q/p)})
\]

and so that

\[
l \geq (h_{l,p}S_{l,p})^{(q/p)}.
\]

Thus we get

\[
c = I_{l,p}(u) + ((q - p)/pq)l
\]

\[
\geq I_{l,p}(u) + ((q - p)/pq) (h_{l,p}S_{l,p})^{(q/p)}.
\]

**Proof of Theorem 1**

The proof of Theorem 1 is given in two parts.

**Existence of a local minimizer**

We prove that there exists \( \lambda_0 > 0 \) such that for any \( \lambda \in (0, \lambda_0) \), \( I_{l,p} \) can achieve a local minimizer. First, we establish the following result.

**Proposition 1:** Suppose that \( p + k < N, 0 \leq s < p, \mu < \tilde{\mu}_{l,p} \), hypothesis \( (H) \) holds, \( g \in H_0^s \cap C(\mathbb{R}^N) \) and \( g \not\equiv 0 \). Then there exists \( \lambda_0 \) such that for all \( \lambda \in (0, \lambda_0) \) we have

\[
I_{l,p}(u) \geq \delta > 0 \quad \text{for} \quad \|u\| = \delta
\]

**Proof:** By the Holder inequality and the definition of \( \|u\| = \delta \), we get

\[
\text{for all} \quad u \in D \setminus \{0\} \quad \text{and} \quad \delta > 0
\]
It follows that has a subsequence, still denoted by in and such that.

On the other hand, such that small enough such that (5.7) holds.

Then, we obtain a critical point such that (5.8) for all ε > 0.

Using the Ekeland’s variational principle, for the complete metric space with respect to the norm of , we can prove that there exists a (PC)$_i$ sequence such that $u_i \to u_i$ for some $u_i$ with $N(u_i) \leq \rho$. Now, we claim that $u_i \to u_i$. If not, by Lemma??; we have

where $\phi \in C^\infty_0(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} g(x) \phi dx > 0$. It follows that for $\epsilon > 0$ small,

and is a contradiction.

Then we obtain a critical point $u_i$ of $I_{\lambda,\rho}$ for all $\lambda \in (0, \lambda_i)$. On the other hand we have

which is a contradiction.

Thus $u_i$ is a nontrivial solution of our problem with negative energy.

Existence of mountain pass type solution

We use the mountain pass theorem without Palais-Smale conditions to prove the existence of a nontrivial solution with positive energy. For this, we need the following Lemma.

**Lemma 4:** Let $\lambda' > 0$ such that

Then, there exist $\Lambda \in (0, \lambda')$ and $\varphi \in D$ for $\epsilon > 0$ such that

**Proof:** Let

where $\phi \in C^\infty_0(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} g(x) \phi dx > 0$. Then, we claim that there is an $\epsilon_0$ such that

In fact, $g(x) \geq 0$ or $g(x) \leq 0$ for all $x \in \mathbb{R}^N$, and (5.7) holds obviously. If there exists $x_0 \in \mathbb{R}^N$ such that $g(x_0) > 0$, by the continuity of $g(x)$ there is an $\eta > 0$ such that $g(x) > 0$ for all $x \in B_\eta(x_0)$. Then, by the definition of $\omega_\epsilon(x-x_0)$, it is easy to see that there exists an $\epsilon_0$ small enough such that

Now, we consider the following functions

and

By the continuity of $f(t)$, there exists $T > 0$ such that $f(t) \in \mathbb{D}$ and $\epsilon < \delta \leq 0$. For all $t \in (0, \epsilon)$. On the other hand, we have

Then we obtain

Set

We deduce that

Since $\lim_{\lambda \to \lambda_0} (T\varphi_r) = -\varphi_r$, we can choose $T > 0$ large enough such that $I_{\lambda,\rho}(T\varphi_r) < 0$. From Proposition 1, we have $I_{\lambda,\rho}(\varphi_r) \geq \delta > 0$ for all $\lambda \in (0, \lambda_i)$. By mountain pass theorem without the Palais-Smale condition, there exists a $(PC)_i$ sequence $(u_n)$ in $D$ which is characterized by

with

Then, $(u_n)$ has a subsequence, still denoted by $(u_n)$ such that $u_n \to u$ in $D$.

By Lemma 3, if $u_i$ doesn't converge to $u_i$, we get
\[ c_2 \geq I_{\lambda,\mu}(u_2) + \left( (q - p) / pq \right) \left( b_0^{1/p^q} S_{\lambda,\mu} \right)^{q/p} \geq c_{\lambda,\mu}^{*}, \]

what contradicts the fact that, by Lemma 4, we have 
\[ \sup_{r \in D} I_{\lambda,\mu}(t \varphi_r) < c_{\lambda,\mu}^{*}, \]

for all \( \lambda \in (0, \Lambda) \). Then \( u_\epsilon \rightarrow u_2 \) in \( D \).

Thus, we obtain a critical point \( u_2 \) of \( I_{\lambda,\mu} \) for all \( \lambda \in (0, \lambda_*) \) with
\[ \Lambda_* = \min \{ \lambda_*, \Lambda \} \]
satisfying \( I_{\lambda,\mu}(u_2) > 0 \).

References