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Existence and Stability of Triangular Points in the Relativistic R3bp When the Primaries are Triaxial Rigid Bodies and Sources of Radiation

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Abstract

This paper deals with the triangular points and their linear stability in the relativistic R3BP when the primaries are triaxial rigid bodies and sources of radiation. It is observed that the locations of the triangular points are affected by the relativistic terms, radiation pressure forces and the triaxiality of the primaries. It is also seen that for these points the range of stability region increases or decreases according as p >0 or p<0 where p depends upon the relativistic terms, the radiation and triaxiality coefficients.

Keywords: Celestial mechanics; Radiation; Triaxiality; Relativity; R3BP

Introduction

The restricted three-body problem possesses five stationary solutions called Lagrangian points, three of which called collinear equilibria lie on the line joining the primaries and the other two called equilateral equilibria make equilateral triangles with primaries.

In general, the collinear equilibria are unstable while equilateral ones are stable, in the Lyapunov sense, only in a certain region for the mass parameter. Various authors have made studies on Lagrangian points in the restricted three-body problem by considering the more massive primary or both primaries as source of radiation.

Some of the important contributions are by Radzievskii [1,2] Simmons et al. [3] Kunitsyn and Tureshbaev [4] Singh and Ishwar [5] Singh [6]. Some of the significant studies, by considering the triaxiality of one or both primaries, are Khanna and Bhatnagar [7] and Singh [8]. Sharma et al. [9,10] have studied the stationary solutions of the planar restricted three-body problem when one or both primaries are sources of radiation and triaxial rigid bodies with one of the axes as the axis of symmetry and the equatorial plane coinciding with the plane of motion.

The theory of the general relativity is currently the most successful gravitational theory describing the nature of space and time and well confirmed by observations [11]. Brumberg [12,13] studied the problem in more details and collected most of the important results on relativistic celestial mechanics. He did not obtain only the equations of motion for the general problem of the three bodies but also deduced the equation of the motion for the restricted problem of three bodies. Bhatnagar and Hallan [14] investigated the existence and linear stability of the triangular point $L_{4,5}$ in the relativistic R3BP. They concluded that $L_{4,5}$ are always unstable in the whole range $0 \le \mu \le \frac{1}{2}$ in contrast to the classical R3BP where they are stable for $\mu < \mu_0$, μ being the mass ratio and $\mu_0 = 0.03852...$ is the Routh's value.

Douskos and Perdios [15] investigated the stability of the triangular points in the relativistic R3BP and contrary to the results of Bhatnagar and Hallan [14], they obtained a region of linear stability in the parameter space $0 \le \mu < \mu_0 - \frac{17\sqrt{69}}{486c^2}$ where $\mu_0 = 0.03852...$ is Routh's value.

Katour et al. [16] obtained new locations of the triangular points in the framework of relativistic R3BP with oblateness and photogravitional corrections to triangular points.

Singh and Bello [17,18] studied the motion of a test particle in the vicinity of the triangular points $L_{4,5}$ by considering one or both primaries as the sources of radiation in the framework of the relativistic restricted three-body problem (R3BP).

In the present work, we study the existence of the triangular points and their linear stability by considering both primaries as triaxial rigid bodies and sources of radiation.

This paper is organized as follows: In Sect. 2, the equations governing the motion are presented; Sect. 3 describes the positions of triangular points, while their linear stability is analyzed in Sect.4; The discussion of the results is given in Sect. 5. In Sect. 6, we conclude our work and highlight the differences and similarities between our work and previous works.

Equations of Motion

In our recent papers Singh and Bello [17,18] we have studied the effect of radiation pressure of the primaries in the relativistic R3BP and found that the positions of triangular points and their stability are affected by both relativistic and radiation factors. In this paper we extend this work by considering both primaries as triaxial rigid bodies as well as sources of radiation.

The pertinent equations of motion of an infinitesimal mass in the relativistic R3BP in a barycentric synodic coordinate system (ξ, η) and dimensionless variables can be written as Singh and Bello [17,18]

$$\ddot{\xi} - 2n\dot{\eta} = \frac{\partial W}{\partial \xi} - \frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\xi}}\right)$$
$$\ddot{\eta} + 2n\dot{\xi} = \frac{\partial W}{\partial \eta} - \frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\eta}}\right) \quad (1)$$

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with

$$W=\frac{1}{2}\left\{1+\frac{3}{2}\left(2\sigma_1-\sigma_2\right)+\frac{3}{2}\left(2\sigma_1'-\sigma_2'\right)\right\}(\xi^2+\eta^2)+\frac{q_1(1-\mu)}{\rho_1}+\frac{q_2\mu}{\rho_2}+\frac{q_1(1-\mu)\left(2\sigma_1-\sigma_2\right)}{2\rho_1^3}$$
$$+\frac{3q_1(1-\mu)\left(2\sigma_1-\sigma_2\right)\eta^2}{2\rho_1^5}+\frac{q_2\mu\left(2\sigma_1'-\sigma_2'\right)}{2\rho_2^3}+\frac{3q_2\mu\left(2\sigma_1'-\sigma_2'\right)\eta^2}{2\rho_2^5}+\frac{1}{c^2}\left[-\frac{3}{2}\left(1-\frac{1}{3}\mu(1-\mu)\right)(\xi^2+\eta^2)\right.$$
$$+\frac{1}{8}\left\{\dot\xi^2+\dot\eta^2+2(\xi\dot\eta-\eta\dot\xi)+(\xi^2+\eta^2)\right\}^2+\frac{3}{2}\left(\frac{q_1(1-\mu)}{\rho_1}+\frac{q_2\mu}{\rho_2}\right)\left(\dot\xi^2+\dot\eta^2+2(\xi\dot\eta-\eta\dot\xi)+(\xi^2+\eta^2)\right) \tag{2}$$
$$-\frac{1}{2}\left(\frac{q_1^2(1-\mu)^2}{\rho_1^2}+\frac{q_2^2\mu^2}{\rho_2^2}\right)+q_1q_2(1-\mu)\mu\left\{\left(4\dot\eta+\frac{7}{2}\xi\right)\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)-\frac{\eta^2}{2}\left(\frac{q_2\mu}{\rho_1^3}+\frac{q_1(1-\mu)}{\rho_2^3}\right)+\left(\frac{-1}{\rho_1\rho_2}\right.\right.$$
$$\left.\left.\left.+\frac{q_2\mu-2q_1(1-\mu)}{2\rho_1}+\frac{q_1(1-\mu)-2q_2\mu}{2\rho_2}\right)\right\}\right]$$

$$n=1+\frac{3}{4}\left(2\sigma_1-\sigma_2\right)+\frac{3}{4}\left(2\sigma_1'-\sigma_2'\right)-\frac{3}{2c^2}\left(1-\frac{1}{3}\mu(1-\mu)\right) \tag{3}$$

$$\rho_1^2=(\xi+\mu)^2+\eta^2$$
$$\rho_2^2=(\xi+\mu-1)^2+\eta^2 \tag{4}$$

where $0<\mu\le\frac{1}{2}$ is the ratio of the mass of the smaller primary to the total mass of the primaries, ρ_1 and ρ_2 are distances of the infinitesimal mass from the bigger and smaller primary, respectively; n is the mean motion of the primaries; $c$is the velocity of light.

$$\sigma_1=\frac{a^2-c^2}{5R^2},\qquad \sigma_2=\frac{b^2-c^2}{5R^2},\qquad \sigma_1'=\frac{a'^2-c'^2}{5R^2},$$ [19] and

$\sigma_i<<1,\ \sigma_i'<<1\ (i=1,2)$ characterize the triaxiality of the bigger and smaller primary with a, b, c as lengths of the semi-axes of the bigger primary and a',b',c' as those of the smaller primary. The radiation factor $q_i(i=1,2)$ is given by $F_{pi}=F_{gi}(1-q_i)$ such that $0\le(1-q_i)<<1$ Radzievskii [1] where F_{gi} and F_{pi} are respectively the gravitational and radiation pressure.

Here as Katour et al. [16] we do not include the parameters $\sigma_i,\sigma_1'\ \ (i=1,2)$ in the relativistic part of W since the magnitude of these terms is so small due to c^{-2} where c is the speed of light.

Location of Triangular Points

The libration points are obtained from equation (1) after putting $\dot\xi=\dot\eta=\ddot\xi=\ddot\eta=0$.

These points are the solutions of the equations

$$\frac{\partial W}{\partial\xi}=0=\frac{\partial W}{\partial\eta}\ \text{with}\ \dot\xi=\dot\eta=0.$$

That is

$$\xi-\frac{q_1(1-\mu)(\xi+\mu)}{\rho_1^3}-\frac{q_2\mu(\xi-1+\mu)}{\rho_2^3}+\left\{3\left(\sigma_1+\sigma_1'\right)-\frac{3}{2}\left(\sigma_2+\sigma_2'\right)\right\}\xi-\frac{3q_1(1-\mu)(\xi+\mu)(2\sigma_1-\sigma_2)}{2\rho_1^5}$$
$$-\frac{15q_1(1-\mu)(\xi+\mu)(\sigma_2-\sigma_1)\eta^2}{2\rho_1^7}-\frac{3q_2\mu(\xi-1+\mu)(2\sigma_1'-\sigma_2')}{2\rho_2^5}-\frac{15q_2\mu(\xi-1+\mu)(\sigma_2'-\sigma_1')\eta^2}{2\rho_2^7}$$
$$+\frac{1}{c^2}\left[-3\xi\left\{1-\frac{\mu(1-\mu)}{3}\right\}+\frac{1}{2}\xi(\xi^2+\eta^2)-\frac{3}{2}(\xi^2+\eta^2)\left\{\frac{q_1(1-\mu)(\xi+\mu)}{\rho_1^3}+\frac{q_2\mu(\xi-1+\mu)}{\rho_2^3}\right\}\right.$$
$$+3\left(\frac{q_1(1-\mu)}{\rho_1}+\frac{q_2\mu}{\rho_2}\right)\xi+\frac{q_1^2(1-\mu)^2(\xi+\mu)}{\rho_1^4}+\frac{q_2^2\mu^2(\xi-1+\mu)}{\rho_2^4}+q_1q_2\mu(1-\mu)\left\{\frac{7}{2}\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)\right. \tag{5}$$
$$+\frac{7}{2}\xi\left(-\frac{(\xi+\mu)}{\rho_1^3}+\frac{(\xi-1+\mu)}{\rho_2^3}\right)+\frac{3}{2}\eta^2\left(\frac{q_2\mu(\xi+\mu)}{\rho_1^5}+\frac{q_1(1-\mu)(\xi-1+\mu)}{\rho_2^5}\right)+\frac{(\xi+\mu)}{\rho_1^3\rho_2}+\frac{(\xi-1+\mu)}{\rho_1\rho_2^3}$$
$$\left.\left.-\frac{(q_2\mu-2q_1(1-\mu))(\xi+\mu)}{2\rho_1^3}-\frac{(q_1(1-\mu)-2q_2\mu)(\xi-1+\mu)}{2\rho_2^3}\right\}\right]=0$$

and

$$\eta F=0,$$

with

$$F=\left(1-\frac{q_1(1-\mu)}{\rho_1^3}-\frac{q_2\mu}{\rho_2^3}\right)+3\left(\sigma_1+\sigma_1'\right)-\frac{3}{2}\left(\sigma_2+\sigma_2'\right)+\frac{3q_1(1-\mu)}{\rho_1^5}\left(\frac{3}{2}\sigma_2-2\sigma_1\right)-\frac{15q_1(1-\mu)(\sigma_2-\sigma_1)\eta^2}{2\rho_1^7}$$
$$+\frac{3q_2\mu}{\rho_2^5}\left(\frac{3}{2}\sigma_2'-2\sigma_1'\right)-\frac{15q_2\mu\left(\sigma_2'-\sigma_1'\right)\eta^2}{2\rho_2^7}+\frac{1}{c^2}\left[-3\left(1-\frac{\mu(1-\mu)}{3}\right)+\frac{1}{2}(\xi^2+\eta^2)+3\left(\frac{q_1(1-\mu)}{\rho_1}+\frac{q_2\mu}{\rho_2}\right)\right.$$
$$-\frac{3}{2}(\xi^2+\eta^2)\left(\frac{q_1(1-\mu)}{\rho_1^3}+\frac{q_2\mu}{\rho_2^3}\right)+\frac{q_1^2(1-\mu)^2}{\rho_1^4}+\frac{q_2^2\mu^2}{\rho_2^4}+q_1q_2\mu(1-\mu)\left\{\frac{7}{2}\xi\left(-\frac{1}{\rho_1^3}+\frac{1}{\rho_2^3}\right)-\left(\frac{q_2\mu}{\rho_1^3}+\frac{q_1(1-\mu)}{\rho_2^3}\right)\right.$$
$$\left.\left.+\frac{3}{2}\eta^2\left(\frac{q_2\mu}{\rho_1^5}+\frac{q_1(1-\mu)}{\rho_2^5}\right)+\left(\frac{1}{\rho_1^3\rho_2}+\frac{1}{\rho_1\rho_2^3}-\frac{(q_2\mu-2q_1(1-\mu))}{2\rho_1^3}-\frac{(q_1(1-\mu)-2q_2\mu)}{2\rho_2^3}\right)\right\}\right]$$

For simplicity, putting $q_i=1-\left(1-q_i\right)=1-\delta_i\ (i=1,2),\ 0\le1-q_i=\delta_i<<1$ and neglecting second and higher order terms in $x,y,\frac{1}{c^2},\delta_1,\delta_2$ and their products. Following as our papers [17,18] we have obtained from the system (5) with $\eta\ne0$, the coordinates of the triangular points $(\xi,\pm\eta)$ as:

$$\xi=\frac{1-2\mu}{2}\left(1+\frac{5}{4c^2}\right)+\left(\frac{1}{8}-\frac{1}{2\mu}\right)\sigma_1-\left(\frac{1}{2\mu}+\frac{3}{8}\right)\sigma_2+\left(\frac{3}{8}-\frac{\mu}{2(1-\mu)}\right)\sigma_1'-\left(\frac{\mu}{2(1-\mu)}+\frac{7}{8}\right)\sigma_2'-\frac{1}{3}(\delta_2-\delta_1) \tag{6}$$

$$\eta=\pm\frac{\sqrt3}{2}\left[1+\frac{1}{12c^2}(-5+6\mu-6\mu^2)+\frac{2}{3}\left\{\left(-\frac{23}{8}+\frac{1}{2\mu}\right)\sigma_1+\left(\frac{19}{8}-\frac{1}{2\mu}\right)\sigma_2\right\}+\frac{2}{3}\left\{\left(-\frac{19}{8}+\frac{\mu}{2(1-\mu)}\right)\sigma_1'\right.\right.$$
$$\left.\left.+\left(\frac{15}{8}-\frac{\mu}{2(1-\mu)}\right)\sigma_2'\right\}-\frac{2}{9}(\delta_2+\delta_1)\right]$$

Stability of $L_{4,5}$

Since the nature of the linear stability about the point L_5 will be similar to that about L_4, it will be sufficient to consider here the stability only near L_4.

Let (a,b) be the coordinates of the triangular point L_4

We set $\xi=a+\alpha,\eta=b+\beta,(\alpha,\beta<<1)$, in the equations (1) of motion.

First, we compute the terms of their R.H.S, neglecting second and higher order terms, we get

$$\left(\frac{\partial W}{\partial\xi}\right)_{\xi=a+\alpha,\eta=b+\beta}=A\alpha+B\beta+C\dot\alpha+D\dot\beta$$

where,

$$A=\frac{3}{4}\left\{1+\frac{1}{2c^2}\left(2-19\mu+19\mu^2\right)\right\}+\frac{3\left(15\mu^2+19\mu-8\right)}{16\mu}\sigma_1-\frac{3\left(31\mu^2+\mu-8\right)}{16\mu}\sigma_2+\frac{3\left(15\mu^2-49\mu+26\right)}{16(1-\mu)}\sigma_1'$$
$$-\frac{3\left(31\mu^2-63\mu+24\right)}{16(1-\mu)}\sigma_2'+\frac{1}{2}(3\mu-1)\delta_1-\left(\frac{3\mu}{2}-1\right)\delta_2,$$

$$B=\frac{3\sqrt3}{4}(1-2\mu)\left(1-\frac{2}{3c^2}\right)-\frac{\sqrt3\left(89\mu^2-47\mu+8\right)}{16\mu}\sigma_1+\frac{\sqrt3\left(37\mu^2-9\mu+8\right)}{16\mu}\sigma_2+\frac{\sqrt3\left(89\mu^2-131\mu+50\right)}{16(1-\mu)}\sigma_1'$$
$$-\frac{\sqrt3\left(37\mu^2-65\mu+36\right)}{16(1-\mu)}\sigma_2'-\frac{\sqrt3}{6}(1+\mu)\delta_1+\frac{\sqrt3}{6}(2-\mu)\delta_2,$$

$$C=\frac{\sqrt3}{2c^2}(1-2\mu),$$

$$D=\frac{6-5\mu+5\mu^2}{2c^2}.$$

Similarly, we obtain

$$\left(\frac{\partial W}{\partial\eta}\right)_{\xi=a+\alpha,\eta=b+\beta}=A_1\alpha+B_1\beta+C_1\dot\alpha+D_1\dot\beta$$

where,

$$A_1=\frac{3\sqrt3}{4}(1-2\mu)\left(1-\frac{2}{3c^2}\right)-\frac{\sqrt3\left(89\mu^2-47\mu+8\right)}{16\mu}\sigma_1+\frac{\sqrt3\left(37\mu^2-9\mu+8\right)}{16\mu}\sigma_2+\frac{\sqrt3\left(89\mu^2-131\mu+50\right)}{16(1-\mu)}\sigma_1'$$
$$+\frac{\sqrt3\left(37\mu^2-65\mu+36\right)}{16(1-\mu)}\sigma_2'-\frac{\sqrt3}{6}(1+\mu)\delta_1+\frac{\sqrt3}{6}(2-\mu)\delta_2,$$

$$B_1=\frac{9}{4}\left\{1+\frac{7}{6c^2}\left(-2+3\mu-3\mu^2\right)\right\}-\frac{3\left(15\mu^2-29\mu-8\right)}{16\mu}\sigma_1+\frac{3\left(15\mu^2-7\mu-8\right)}{16\mu}\sigma_2-\frac{3\left(15\mu^2-\mu-22\right)}{16(1-\mu)}\sigma_1'$$
$$+\frac{3\mu(15\mu-23)}{16\mu}\sigma_2'+\frac{1}{2}(1-3\mu)\delta_1+\left(\frac{3\mu}{2}-1\right)\delta_2,$$

$$C_1=\frac{1}{2c^2}\left(-4+\mu-\mu^2\right),$$

$$D_1=-\frac{\sqrt{3}(1-2\mu)}{2c^2}.$$

$$\frac{d}{dt}\left(\frac{\partial W}{\partial\dot{\xi}}\right)_{\xi=a+\alpha,\,\eta=b+\beta}=A_2\dot{\alpha}+B_2\dot{\beta}+C_2\ddot{\alpha}+D_2\ddot{\beta}$$

where,

$$A_2=\frac{\sqrt{3}}{2c^2}(1-2\mu),$$

$$B_2=-\frac{1}{2c^2}\left(4-\mu+\mu^2\right),$$

$$C_2=\frac{1}{4c^2}\left(17-2\mu+2\mu^2\right),$$

$$D_2=-\frac{\sqrt{3}}{4c^2}(1-2\mu).$$

$$\frac{d}{dt}\left(\frac{\partial W}{\partial\dot{\eta}}\right)_{\xi=a+\alpha,\,\eta=b+\beta}=A_3\dot{\alpha}+B_3\dot{\beta}+C_3\ddot{\alpha}+D_3\ddot{\beta}$$

where,

$$A_3=\frac{1}{2c^2}\left(6-5\mu+5\mu^2\right),$$

$$B_3=-\frac{\sqrt{3}}{2c^2}(1-2\mu),$$

$$C_3=-\frac{\sqrt{3}}{4c^2}(1-2\mu),$$

$$D_3=\frac{3\left(5-2\mu+2\mu^2\right)}{4c^2}.$$

The characteristic equation of the variational equations of motion corresponding to (1) can be expressed as

$$\lambda^4+b\lambda^2+d=0 \tag{7}$$

where,

$$b=\left(1-\frac{9}{c^2}\right)+3\sigma_1+\frac{3}{2}(2\mu-3)\sigma_2+3\sigma_1'+\left(3\mu+\frac{3}{2}\right)\sigma_2'$$

$$d=\frac{27\mu(1-\mu)}{4}+\frac{9\mu\left(-65+77\mu-24\mu^2+12\mu^3\right)}{8c^2}+\frac{9\left(-89\mu^2+99\mu-10\right)}{16}\sigma_1+\frac{9\left(37\mu^2-47\mu+10\right)}{16}\sigma_2$$
$$-\frac{9\mu(89\mu-79)}{16}\sigma_1'+\frac{9\mu(37\mu-27)}{16}\sigma_2'+\frac{3}{2}\mu(\delta_1+\delta_2)-\frac{3}{2}\mu^2(\delta_1+\delta_2)$$

For $\frac{1}{c^2}\to 0$ and when the primaries are non-luminous and spherical $(i.e.\ \delta_1=\delta_2=\sigma_1=\sigma_2=\sigma_1'=\sigma_2'=0)$, Eq. (7) reduces to its well-known classical restricted problem form (See e.g. Szebehely [20]).

$$\lambda^4+\lambda^2+\frac{27}{4}\mu(1-\mu)=0.$$

The discriminant of (7) is

$$\Delta=\frac{-54}{c^2}\mu^4+\frac{108}{c^2}\mu^3+\left(27+6\delta_1+6\delta_2+\frac{801}{4}\sigma_1-\frac{333}{4}\sigma_2+\frac{801}{4}\sigma_1'-\frac{333}{4}\sigma_2'-\frac{693}{2c^2}\right)\mu^2$$
$$+\left(-27-6\delta_1-6\delta_2-\frac{891}{4}\sigma_1+\frac{447}{4}\sigma_2-\frac{771}{4}\sigma_1'+\frac{219}{4}\sigma_2'+\frac{585}{2c^2}\right)\mu+1+\frac{57}{2}\sigma_1-\frac{63}{2}\sigma_2+6\sigma_1'-3\sigma_2'-\frac{18}{c^2} \tag{8}$$

Its roots are

$$\lambda^2=\frac{-b\pm\sqrt{\Delta}}{2} \tag{9}$$

where,

$$b=\left(1-\frac{9}{c^2}\right)+3\sigma_1+\frac{3}{2}(2\mu-3)\sigma_2+3\sigma_1'-\left(3\mu+\frac{3}{2}\right)\sigma_2'$$

From (8), we have

$$\frac{d\Delta}{d\mu}=\frac{-216}{c^2}\mu^3+\frac{324}{c^2}\mu^2+2\left(27+6\delta_1+6\delta_2+\frac{801}{4}\sigma_1-\frac{333}{4}\sigma_2+\frac{801}{4}\sigma_1'-\frac{333}{4}\sigma_2'-\frac{693}{2c^2}\right)\mu$$
$$\left(-27-6\delta_1-6\delta_2-\frac{891}{4}\sigma_1+\frac{447}{4}\sigma_2-\frac{771}{4}\sigma_1'+\frac{219}{4}\sigma_2'+\frac{585}{2c^2}\right)<0\;\forall\mu\in\left(0,\frac{1}{2}\right). \tag{10}$$

From (10), it can be easily seen that Δ is monotone decreasing in $\left(0,\frac{1}{2}\right)$.

But

$$(\Delta)_{\mu=0}=1+\frac{57}{2}\sigma_1-\frac{63}{2}\sigma_2+6\sigma_1'-3\sigma_2'-\frac{18}{c^2}>0 \tag{11}$$

$$(\Delta)_{\mu=\frac{1}{2}}=-\frac{23}{4}-\frac{525}{16}\sigma_1+\frac{57}{16}\sigma_2-\frac{525}{16}\sigma_1'+\frac{57}{16}\sigma_2'-\frac{3}{2}(\delta_1+\delta_2)+\frac{207}{4c^2}<0$$

Since $(\Delta)_{\mu=0}$ and $(\Delta)_{\mu=\frac{1}{2}}$ are of opposite signs, and Δ is monotone decreasing and continuous, there is one value of μ, e.g. μ_c in the interval $\left(0,\frac{1}{2}\right)$ for which Δ vanishes.

Solving the equation $\Delta=0$ using (8), we obtain the critical value of the mass parameter as

$$\mu_c=\frac{1}{2}-\frac{1}{18}\sqrt{69}-\frac{17\sqrt{69}}{486c^2}+\frac{1}{2}\left(\frac{5}{6}+\frac{59}{9\sqrt{69}}\right)\sigma_1-\frac{1}{2}\left(\frac{19}{18}+\frac{85}{9\sqrt{69}}\right)\sigma_2-\frac{1}{2}\left(\frac{5}{6}-\frac{59}{9\sqrt{69}}\right)\sigma_1'+\frac{1}{2}\left(\frac{19}{18}-\frac{85}{9\sqrt{69}}\right)\sigma_2'$$
$$-\frac{2}{27\sqrt{69}}(\delta_1+\delta_2) \tag{12}$$

$$\mu_c=\mu_0-\frac{17\sqrt{69}}{486c^2}+\frac{1}{2}\left(\frac{5}{6}+\frac{59}{9\sqrt{69}}\right)\sigma_1-\frac{1}{2}\left(\frac{19}{18}+\frac{85}{9\sqrt{69}}\right)\sigma_2-\frac{1}{2}\left(\frac{5}{6}-\frac{59}{9\sqrt{69}}\right)\sigma_1'+\frac{1}{2}\left(\frac{19}{18}-\frac{85}{9\sqrt{69}}\right)\sigma_2'$$
$$-\frac{2}{27\sqrt{69}}(\delta_1+\delta_2)$$

where $\mu_0=0.03852...$ is the Routh's value.

We consider the following three regions of the values of μ separately.

When $0\le\mu<\mu_c$, $\Delta>0$, the values of λ^2 given by (9) are negative and therefore all the four characteristic roots are distinct pure imaginary numbers. Hence, the triangular points are stable.

When $\mu_c<\mu\le\frac{1}{2},\Delta<0$, the real parts of the characteristic roots are positive. Therefore, the triangular points are unstable.

When $\mu=\mu_c,\Delta=0$, the values of λ^2 given by (9) are the same. This induces instability of the triangular points.

Hence, the stability region is

$$0<\mu<\mu_0+p \tag{13}$$

with

$$p=-\frac{17\sqrt{69}}{486c^2}-\frac{2}{27\sqrt{69}}(\delta_1+\delta_2)+\frac{1}{2}\left(\frac{5}{6}+\frac{59}{9\sqrt{69}}\right)\sigma_1-\frac{1}{2}\left(\frac{19}{18}+\frac{85}{9\sqrt{69}}\right)\sigma_2-\frac{1}{2}\left(\frac{5}{6}-\frac{59}{9\sqrt{69}}\right)\sigma_1'$$
$$+\frac{1}{2}\left(\frac{19}{18}-\frac{85}{9\sqrt{69}}\right)\sigma_2'$$

Discussion

In this section, we discuss the triangular libration points in the relativistic restricted three-body problem, under the assumption that the primaries are luminous and triaxial. The positions of the triangular points in equation (6) are obtained. It can be seen that they are affected by the relativistic, radiation and triaxiality factors. It is important to note that these triangular libration points cease to be classical one i.e., they no longer form equilateral triangles with the primaries. Rather they form scalene triangles with the primaries. Equation (12) gives the critical value of the mass parameter μ_c of the system which depends upon relativistic factor, triaxiality parameters $\sigma_i, \sigma_i' \quad (i = 1, 2)$ and radiation factors $\delta_i \quad (i = 1, 2)$. In the absence of relativistic factor, the results obtained in this study are in agreement with those of Sharma et al. [10] Singh [8] when there is no perturbations in the Coriolis and centrifugal forces $(i.e.\ \varepsilon = \varepsilon' = 0)$. When the primaries are non triaxial, the results of the present study tally with those of Singh and Bello [18].

It is noticeable from (13) that that radiation and triaxiality all have destabilizing effects, and therefore the size of the range of stability decreases with increase of the values of these parameters. Evidently, it can also seen that the relativistic factor reduces the size of stability region. When the primaries are non-luminous and non-triaxial, the stability results obtained in this study are in accordance with those of Douskos and Perdios [15] and disagree with Bhatnagar and Hallan [14]. In the absence of relativistic factor, the results obtained in this study are in agreement with those of Sharma [14] and those of Singh [8] when the perturbations are absent. When the primaries are oblate spheroids $(i.e.\ \sigma_1 = \sigma_2, \sigma_1' = \sigma_2')$, the results of equation (6) in this study differ from those of katour et al. [16] when the radiation terms are absent in the realistic part of the potential W.

Conclusion

By considering the primaries as triaxial rigid bodies and sources of radiation in the relativistic CR3BP, we have determined the positions of the triangular points and their linear stability. It is found that their positions and stability region are affected by relativistic, triaxiality and radiation factors. It is further observed that the relativistic, triaxiality and radiation factors have destabilizing tendencies resulting in a decrease in the size of the region of stability. We have noticed that the expressions for A, D, A_2, C_2 in Bhatnagar and Hallan [14] differ from the present study when the radiation pressure factors are absent and the primaries are spherical $(i.e.\ \delta_i = \sigma_i = \sigma_i' = 0, i = 1, 2)$. Consequently, the characteristic equation is also different. This led them Bhatnagar and Hallan [14] to infer that the triangular points are unstable, contrary to Douskos and Perdios and our results. Our results are also in disagreement with those of Katour et al. [16]. One major distinction is that the expression of the mean motion which they used in their study differ from our own. It seems that there is an error in their expression.

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