Journal of Astrophysics & Aerospace Technology

Ould El Mokhtar, J Astrophys Aerospace Technol 2015, 3:2
DOI: 10.4172/2329-6542.1000126

Research Article Open Access

Existence and Nonexistence for Elliptic Equation with Cylindrical Potentials, Subcritical Exponent and Concave Term

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Abstract

In this paper, we consider the existence and nonexistence of non-trivial solutions to elliptic equations with cylindrical potentials, concave term and subcritical exponent. First, we shall obtain a local minimizer by using the Ekeland's variational principle. Secondly, we deduce a Pohozaev-type identity and obtain a nonexistence result.

Keywords: Existence; Nonexistence; Elliptic equation; Nontrivial solutions

Introduction

In this paper we study the existence, multiplicity and nonexistence of nontrivial solutions of the following problem

$$\left(\mathcal{P}_{\beta,\lambda,\mu}\right)\begin{cases}-\Delta u-\mu|y|^{-\beta}u= \ |y|^{-a\gamma}|u|^{\gamma-2}u+\lambda g(x)|u|^{q-2}u \text{ in } \mathbb{R}^N, y\neq 0\\ u>0,\end{cases}$$

where $\in$ k and N be integers such that $N\geq 3$ and k belongs to $\{1,\dots,N\}\setminus\{0\}$, $2^*=2N/(N-2)$ is the critical Sobolev exponent, $\mu>0$, $\gamma\leq 2^*$, $0\leq a<1$, $1<q<2$, g is a bounded function on $\mathbb{R}^N$, λ and β are parameters which we will specify later.

We denote point x in $\mathbb{R}^N$ by the pair $(y,z)\in\mathbb{R}^k\times\mathbb{R}^{N-k}$, $\mathcal{D}_0^{1,2}=\mathcal{D}_0^{1,2}\left(\left(\mathbb{R}^k\setminus\{0\}\right)\times\mathbb{R}^{N-k}\right)$ and $\mathcal{H}_\mu=\mathcal{H}_\mu\left(\left(\mathbb{R}^k\setminus\{0\}\right)\times\mathbb{R}^{N-k}\right)$, the closure of $C_0^\infty\left(\left(\mathbb{R}^k\setminus\{0\}\right)\times\mathbb{R}^{N-k}\right)$ with respect to the norms

$$\|u\|=\left(\int_{\mathbb{R}^N}|\nabla u|^2\right)^{1/2} \text{ and } \|u\|_{2,\mu}=\left(\int_{\mathbb{R}^N}\left(|\nabla u|^2-\mu|y|^{-2}|u|^2\right)dx\right)^{1/2}$$

We define the weighted Sobolev space $\mathcal{D}:=\mathcal{H}_\mu\cap L^\gamma\left(\mathbb{R}^N,|y|^{-b}dx\right)\cap L^2\left(\mathbb{R}^N,|y|^{-2}dx\right)$ with $b=a\gamma$, which is a Banach space with respect to the norm defined by $\mathcal{N}\|u\|:=\|u\|_{2,\mu}+(\int_{\mathbb{R}^N}|y|^{-b}|u|^\gamma dx)^{1/\gamma}$.

My motivation of this study is the fact that such equations arise in the search for solitary waves of nonlinear evolution equations of the Schrödinger or Klein-Gordon type [1-3]. Roughly speaking, a solitary wave is a nonsingular solution which travels as a localized packet in such a way that the physical quantities corresponding to the invariances of the equation are finite and conserved in time. Accordingly, a solitary wave preserves intrinsic properties of particles such as the energy, the angular momentum and the charge, whose finiteness is strictly related to the finiteness of the L^2- norm. Owing to their particle-like behavior, solitary waves can be regarded as a model for extended particles and they arise in many problems of mathematical physics, such as classical and quantum field theory, nonlinear optics, fluid mechanics and plasma physics [4].

Several existence and nonexistence result are available in the case $k=N$, we quote for example [5,6] and the reference therein. When $\mu=0$ $g(x)\equiv 1$, problem $\left(\mathcal{P}_{\beta,\lambda,\mu}\right)$ has been studied in the famous paper by Brézis and Nirenberg [7] and B. Xuan [8] which consider the existence and nonexistence of nontrivial solutions to quasilinear Brézis-Nirenberg-type problems with singular weights.

Concerning existence result in the case $k<N$ we cite [9,10], and the reference therein. As noticed in [11], for $a=0$ and $\lambda=0$, M. Badiale et al. has considered the problem $\left(\mathcal{P}_{\beta,0,\mu}\right)$. She established the nonexistence of nonzero classical solutions when $k\leq N$ and the pair (β,γ) belongs to the region. i.e: $(\beta,\gamma)\in\mathcal{A}=\mathcal{A}_1\cup\mathcal{A}_2\cup\mathcal{A}_3$ where

$$\mathcal{A}_1:=\left\{(\beta,\gamma)\in\mathbb{R}^2:\beta\in(0,2),\gamma\notin\left(2_\beta,2^*\right),\gamma\geq 2\right\}\setminus\left\{\left(2,2^*\right)\right\},$$

$$\mathcal{A}_2:=\left\{(\beta,\gamma)\in\mathbb{R}^2:\beta\in(2,N),\gamma\notin\left(2^*,2_\beta\right),\gamma\geq 2\right\},$$

$$\mathcal{A}_3:=\left\{(\beta,\gamma)\in\mathbb{R}^2:\beta\in[N,+\infty),\gamma\in\left[2,2^*\right]\right\}.$$

Since our approach is variational, we define the functional $I_{2,\lambda,\mu}$ on $\mathcal{D}$ by

$$I_{2,\lambda,\mu}(u):=(1/2)\|u\|_{2,\mu}^2-(1/\gamma)\int_{\mathbb{R}^N}|y|^{-a\gamma}|u|^\gamma dx-\lambda(1/q)\int_{\mathbb{R}^N}g(x)|u|^q dx.$$

We say that $u\in\mathcal{D}$ is a weak solution of the problem $\left(\mathcal{P}_{2,\lambda,\mu}\right)$ if it is a nontrivial nonnegative function and satisfies

$$\left\langle I'_{2,\lambda,\mu}(u),v\right\rangle:=\int_{\mathbb{R}^N}\left(\nabla u\nabla v-\mu|y|^{-2}uv-|y|^{-b}|u|^{\gamma-2}uv-\lambda g(x)|u|^{q-2}uv\right)$$

$=0$, for $v\in\mathcal{D}$.

Concerning the perturbation g we assume

$(G)\, g\in L^\infty\left(\mathbb{R}^N\right)$ and $g(x)>0$ for all $x\in\mathbb{R}^N$.

In our work, we prove the existence of at least one critical points of $I_{2,\lambda,\mu}$ by the Ekeland's variational in [12]. By the Pohozaev type identities in [12], we show the nonexistence of positive solution for our problem.

We shall state our main result

Theorem 1 *Assume* $2<k\leq N$, $\mu<\bar{\mu}_k=\left((k-2)/2\right)^2$, *β=2, 0<a<1<q<2 and (G) hold.*

If $\gamma\in\left(2,2^*\right)$, then there exist Λ_0 and Λ^* such that the problem $\left(\quad_{2,\lambda,\mu}\right)$ has at least one nontrivial solution for any $\lambda\in\left(\Lambda^*,\Lambda_0\right)$.

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Received July 18, 2015; **Accepted** October 19, 2015; **Published** October 30, 2015

Citation: Ould El Mokhtar MEM (2015) Existence and Nonexistence for Elliptic Equation with Cylindrical Potentials, Subcritical Exponent and Concave Term. J Astrophys Aerospace Technol 3: 126. doi:10.4172/2329-6542.1000126

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Theorem 2 *Let $2 < k \le N$, 0<a<1 and (G) hold.*

If $\beta \in (2,3)$, $\gamma \in (2_{\beta-2a}, 2^*)$ with $2_{\beta-2a} = 2N/(N-2(\beta-2a))$, λ < 0 and *1<q<2*, then $(\mathcal{P}_{\beta,\lambda,\mu})$ has no positive solutions.

This paper is organized as follows. In Section 2, we give some preliminaries. Section 3 is devoted to the proof of Theorem 1. Finally in the last section, we give a nonexistence result by the proof of Theorem 2.

Preliminaries

We list here a few integrals inequalities. The first inequality that we need is the weighted Hardy inequality [13]

$$\bar{\mu}_k \int_{\mathbb{R}^N} |y|^{-2} v^2 dx \le \int_{\mathbb{R}^N} |\nabla v|^2 dx, \forall v \in \mathcal{H}_\mu.$$

The starting point for studying $(\mathcal{P}_{2,\lambda,\mu})$ is the Hardy-Sobolev-Maz'ya inequality that is peculiar to the cylindrical case $k < N$ and that was proved by Maz'ya in [14]. It state that there exists positive constant C_γ such that

$$C_\gamma \left(\int_{\mathbb{R}^N} |v|^\gamma dx \right)^{2/\gamma} \le \int_{\mathbb{R}^N} \left(|\nabla v|^2 - \mu |y|^{-2} v^2 \right) dx,$$

for $\mu = 0$ equation of $(\mathcal{P}_{2,\lambda,\mu})$ is related to a family of inequalities given by Caffarelli, Kohn and Nirenberg [15], for any $v \in C_c^\infty((\mathbb{R}^k \setminus \{0\}) \times \mathbb{R}^{N-k})$. The embedding $\mathcal{H}_\mu \rightarrowtail L^\gamma \left(\mathbb{R}^N, |y|^{-b} dx \right)$ is compact where *b=aγ* and $L^\gamma \left(\mathbb{R}^N, |y|^{-b} dx \right)$ is the weighted L^γ space with respect to the norm

$$|u|_{\gamma,b}^2 = \left(\int_{\mathbb{R}^N} |y|^{-b} |v|^\gamma dx \right)^{2/\gamma}$$

Definition 1 *Assume* $2 \le k < N, 0 < \mu \le \bar{\mu}_k$ *and* $2 < \gamma < 2^*$. *Then the infimum* $S_{\mu,\gamma}$ *defined by*

$$S_{\mu,\gamma} = S_{\mu,\gamma}(k,\gamma) := \inf_{v \in \mathcal{D} \setminus \{0\}} \frac{\int_{\mathbb{R}^N} \left(|\nabla v|^2 - \mu |y|^{-2} v^2 \right) dx}{\left(\int_{\mathbb{R}^N} |y|^{-b} |v|^\gamma dx \right)^{2/\gamma}},$$

is achieved on $\mathcal{H}_\mu$.

Lemma 1 *Let* $(u_n) \subset \mathcal{D}$ *be a Palais-Smale sequence (*$(PS)_\delta$ *for short) of* $I_{2,\lambda,\mu}$ *such that*

$$I_{2,\lambda,\mu}(u_n) \to \delta \text{ and } I'_{2,\lambda,\mu}(u_n) \to 0 \text{ in } \mathcal{D}' \text{ (dual of } \mathcal{D}) \text{ as } n \to \infty, \quad (1)$$

for some $\delta \in \mathbb{R}$. Then if $\lambda < \Lambda_0 = \dfrac{q(\gamma-2)}{2(\gamma-q)}$, $u_n \rightharpoonup u$ in $\mathcal{D}$ and $I'_{2,\lambda,\mu}(u) = 0$.

Proof. From (??), we have

$$(1/2)\|u_n\|_{2,\mu}^2 - (1/\gamma)\int_{\mathbb{R}^N} |y|^{-b} |u_n|^\gamma dx - (\lambda/q)\int_{\mathbb{R}^N} g(x)|u_n|^q dx = \delta + o_n(1)$$

and

$$\|u_n\|_{2,\mu}^2 - \int_{\mathbb{R}^N} |y|^{-b} |u_n|^\gamma dx - \int_{\mathbb{R}^N} g(x)|u_n|^q dx = o_n(1), \text{for } n \text{ large},$$

Where $o_n(1)$ denotes $o_n(1) \to 0$ as $n \to$. Then,

$$\delta + o_n(1) = I_{2,\lambda,\mu}(u_n) - (1/\gamma)\left\langle I'_{2,\lambda,\mu}(u_n), u_n \right\rangle$$

$$= ((\gamma-2)/2\gamma)\|u_n\|_{2,\mu}^2 - \lambda((\gamma-q)/q\gamma)\int_{\mathbb{R}^N} g(x)|u_n|^q dx$$

$$\ge \left[((\gamma-2)/2\gamma) - \lambda((\gamma-q)/q\gamma)|g|_\infty \right] \|u_n\|_{2,\mu}^2.$$

If $\lambda < \Lambda_0 = \dfrac{q(\gamma-2)}{2(\gamma-q)}$ then, (u_n) is bounded in $\mathcal{D}$. Going if necessary to a subsequence, we can assume that there exists $u \in \mathcal{D}$ such that

$u_n \rightharpoonup u$ weakly in $\mathcal{D}$

$u_n \to u$ strongly in $L^\gamma \left(\mathbb{R}^N, |y|^{-b} dx \right)$

$u_n \to u$ strongly in $L^q \left(\mathbb{R}^N \right)$

$u_n \to u$ a.e in $\mathbb{R}^N$.

Consequently, we get for all $v \in C_0^\infty\left((\mathbb{R}^k \setminus \{0\}) \times \mathbb{R}^{N-k} \right)$,

$$\int_{\mathbb{R}^N} \left(\nabla u \nabla v + \mu |y|^{-2} uv - |y|^{-b} |u|^{\gamma-2} uv + \lambda g(x) |u|^{q-2} uv \right) = 0,$$

which means that

$$I'_{2,\lambda,\mu}(u) = 0.$$

Existence Result

Firstly, we require following Lemmas

Lemma 2 *Let* $(u_n) \subset \mathcal{D}$ *be a* $(PS)_\delta$ *sequence of* $I_{2,\lambda,\mu}$ *for some* $\delta \in \mathbb{R}$. *Then,*

$u_n \rightharpoonup u$ in $\mathcal{D}$

and either

$$u_n \to u \text{ or } \delta \ge I_{2,\lambda,\mu}(u) + ((\gamma-2)/2\gamma)(S_{\mu,\gamma})^{\gamma/(\gamma-2)}.$$

Proof. We know that (u_n) is bounded in $\mathcal{D}$. Up to a subsequence if necessary, we have that

$u_n \rightharpoonup u$ in $\mathcal{D}$

$u_n \to u$ a.e in $\mathbb{R}^N$.

Denote $v_n = u_n - u$, then $v_n \rightharpoonup 0$. As in Brézis and Lieb [16], we have

$$\int_{\mathbb{R}^N} |y|^{-b} |u_n|^\gamma = \int_{\mathbb{R}^N} |y|^{-b} |v_n|^\gamma + \int_{\mathbb{R}^N} |y|^{-b} |u|^\gamma$$

and

$$\|u_n\|_{2,\mu}^2 = \|v_n\|_{2,\mu}^2 + \|u\|_{2,\mu}^2$$

From Lebesgue theorem and by using the assumption *(G)*, we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} g(x) |y|^{-b} |u_n|^\gamma dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} g(x) |y|^{-b} |u|^\gamma dx.$$

Then, we deduce that

$$I_{2,\lambda,\mu}(u_n) = I_{2,\lambda,\mu}(u) + (1/2)\|v_n\|_{2,\mu}^2 - (1/\gamma)\int_{\mathbb{R}^N} |y|^{-b} |v_n|^\gamma + o_n(1)$$

and

$$\left\langle I'_{2,\lambda,\mu}(u_n), u_n \right\rangle = \|v_n\|_{2,\mu}^2 - \int_{\mathbb{R}^N} |y|^{-b} |v_n|^\gamma + o_n(1).$$

From the fact that $v_n \rightharpoonup 0$ in $\mathcal{D}$, we can assume that

$$\lim_{n\to\infty} \|v_n\|_{2,\mu}^2 = \lim_{n\to\infty} \int_{\mathbb{R}^N} |y|^{-b} |v_n|^\gamma = \alpha \ge 0.$$

Assume $\alpha > 0$, we have by definition of $S_{\mu,\gamma}$

$$\alpha \ge S_{\mu,\gamma} l^{(2/\gamma)},$$

and so

$$\alpha \ge (S_{\mu,\gamma})^{\gamma/(\gamma-2)}.$$

Then, we get

$$\delta \ge I_{2,\lambda,\mu}(u) + ((\gamma-2)/2\gamma)(S_{\mu,\gamma})^{\gamma/(\gamma-2)}.$$

Therefore, if not we obtain $\alpha = 0$. i.e $u_n \to u$ in $\mathcal{D}$.

Lemma 3 *Suppose* $2 < k \le N$, $\mu < \bar{\mu}_k$ *and (G) hold. There exist* $\Lambda^* > 0$ *such that if* $\lambda > \Lambda^*$, *then there exist ρ and ν positive constants such that,* *i*) there exist $\omega \in \mathbb{R}^N$ such that $I_{2,\lambda,\mu}(\omega) < 0$,

ii) we have

$I_{2,\lambda,\mu}(u) \geq \nu > 0$ for $\|u\|_{2,\mu} = \rho_0$.

Proof. *i*) Let $t_0 > 0$, t_0 small and $\phi \in C_0^\infty\left(\left(\mathbb{R}^k \setminus \{0\}\right)\times \mathbb{R}^{N-k}\right)$ such that $\phi \not\equiv 0$. Choosing $\Lambda^* = |t_0\phi|^{1-q} < \Lambda_0 = \dfrac{q(\gamma-2)}{2(\gamma-q)}$ then, if $\lambda \in (\Lambda^*, \Lambda_0)$

$$I_{2,\lambda,\mu}(t_0\phi) := (t_0^2/2)\|\phi\|_{2,\mu}^2 - (t_0^\gamma/\gamma)\int_{\mathbb{R}^N}|y|^{-b}|\phi|^\gamma 1 - (t_0^q/q)\int_{\mathbb{R}^N}|\phi|^q \lambda g(x)$$

$$< (t_0^2/2)\|\phi\|_{2,\mu}^2 - (t_0^\gamma/\gamma)\int_{\mathbb{R}^N}|y|^{-b}|\phi|^\gamma 1 - \lambda(t_0/q)\int_{\mathbb{R}^N}|\phi| g(x)$$

$$< 0$$

Thus, if $\omega = t_0\phi$, we obtain that $I_{2,\lambda,\mu}(\omega) < 0$.

ii) By the Holder inequality and the definition of $S_{\mu,\gamma}$ and since $\gamma > 2$, we get for all $u \in \mathcal{D}\setminus\{0\}$

$$I_{2,\lambda,\mu}(u) := (1/2)\|u\|_{2,\mu}^2 - (1/\gamma)\int_{\mathbb{R}^N}|y|^{-b}|u|^\gamma dx - (\lambda/q)\int_{\mathbb{R}^N}|u|^q g(x)dx$$

$$\geq (1/2)\|u\|_{2,\mu}^2 - (1/\gamma)S_{\mu,\gamma}\|u\|_{2,\mu}^\gamma - (\lambda/q)|g|_\infty \|u_n\|_{2,\mu}^q$$

If $\lambda > 0$, then there exist $\nu > 0$ and $\rho_0 > 0$ small enough such that

$I_{2,\lambda,\mu}(u) \geq \nu > 0$ for $\|u\|_{2,\mu} = \rho_0$.

We also assume that t_0 is so small enough such that $\|t_0\phi\|_{2,\mu} < \rho_0$. Thus, we have

$$c_1 = \inf\left\{I_{2,\lambda,\mu}(u) : u \in B_{\rho_0}\right\} < 0, \text{where} B_{\rho_0} = \{u \in \mathcal{D}, \mathcal{N}\| u \| \leq \rho_0\}.$$

Using the Ekeland's variational principle, for the complete metric space $\overline{B}_{\rho_0}$ with respect to the norm of $\mathcal{D}$, we can prove that there exists a $(PC)_{c_1}$ sequence $(u_n) \subset \overline{B}_{\rho_0}$ such that $u_n \rightarrow u_1$ for some u_1 with $\mathcal{N}\| u_1 \| \leq \rho_0$.

Now, we claim that $u_n \rightarrow u_1$. If not, by Lemma 2, we have

$$c_1 \geq I_{2,\lambda,\mu}(u_1) + ((\gamma-2)/2\gamma)(S_{\mu,\gamma})^{\gamma/(\gamma-2)}$$

$$\geq c_1 + ((\gamma-2)/2\gamma)(S_{\mu,\gamma})^{\gamma/(\gamma-2)}$$

$$> c_1,$$

which is a contradiction.

Then we obtain a critical point u_1 of $I_{2,\lambda,\mu}$ for all $\lambda \in (\Lambda^*, \Lambda_0)$.

Proof of Theorem 1

Proof. From Lemmas 2 and 3, we can deduce that there exists at least a nontrivial solution u_1 for our problem $(\mathcal{P}_{2,\lambda,\mu})$ with positive energy [17-19].

Nonexistence Result

By a Pohozaev type identity we show the nonexistence of positive solution of $(\mathcal{P}_{2,\lambda,\mu})$ when $\beta \in (2,3)$, $\gamma \in (2_{\beta-2a}, 2^*)$ with $2_{\beta-2a} = 2N/(N-2(\beta-2a))$, $\lambda < 0$, $1<q<2$ and *(G)* hold with $0<a<1$.

First, we need the following Lemma

Lemma 4 *Let* $u \in \mathcal{D}$ *be a positive solution of* $(\mathcal{P}_{2,\lambda,\mu})$ *and. Then the following identity holds*

$$\left(\frac{N-2}{2} - \frac{N-\beta}{2}\right)\int_{\mathbb{R}^N}|\nabla u|^2 dx - \left(\frac{N-a\gamma}{\gamma} - \frac{N-\beta}{2}\right)\int_{\mathbb{R}^N}|y|^{-a\gamma}|u|^\gamma dx$$

$$= \lambda\left(\frac{N}{q} - \frac{N-\beta}{2}\right)\int_{\mathbb{R}^N} g(x)|u|^q dx.$$

Proof. [we shall state the similar proof of proposition 30 and Lemma 31 in [11]].

1) Multiplying the equation of $(\mathcal{P}_{\beta,\lambda,\mu})$ by the inner product $(x\nabla u)$ and integrating on $\mathbb{R}^N$, we obtain

$$\left(\frac{N-2}{2}\right)\int_{\mathbb{R}^N}|\nabla u|^2 dx + \left(\frac{N-\beta}{2}\right)\int_{\mathbb{R}^N}\mu|y|^{-\beta}|u|^2 dx \qquad (2)$$

$$= \left(\frac{N-a\gamma}{\gamma}\right)\int_{\mathbb{R}^N}|y|^{-a\gamma}|u|^\gamma + \lambda\left(\frac{N}{q}\right)\int_{\mathbb{R}^N} g(x)|u|^q dx.$$

2) By multiplying the equation of $(\mathcal{P}_{2,\lambda,\mu})$ by u, using the identity

$$u\Delta u \quad div(u\nabla u) - |\nabla u|$$

in $\left(\mathbb{R}^k \setminus \{0\}\right)\times \mathbb{R}^{N-k}$ and applying the divergence theorem on $\mathbb{R}^N$, we obtain

$$\int_{\mathbb{R}^N}|\nabla u|^2 dx + \int_{\mathbb{R}^N}\mu|y|^{-\beta}|u|^2 dx = \int_{\mathbb{R}^N}|y|^{-a\gamma}|u|^\gamma dx + \lambda\int_{\mathbb{R}^N} g(x)|u|^q dx. \qquad (3)$$

From (3), we have

$$\left(\frac{N-\beta}{2}\right)\int_{\mathbb{R}^N}|\nabla u|^2 dx + \left(\frac{N-\beta}{2}\right)\int_{\mathbb{R}^N}\mu|y|^{-\beta}|u|^2 dx \qquad (4)$$

$$= \left(\frac{N-\beta}{2}\right)\int_{\mathbb{R}^N}|y|^{-a\gamma}|u|^\gamma + \lambda\left(\frac{N-\beta}{2}\right)\int_{\mathbb{R}^N} g(x)|u|^q dx.$$

Combining (??) and (??), we obtain

$$\left(\frac{N-2}{2} - \frac{N-\beta}{2}\right)\int_{\mathbb{R}^N}|\nabla u|^2 dx - \left(\frac{N-a\gamma}{\gamma} - \frac{N-\beta}{2}\right)\int_{\mathbb{R}^N}|y|^{-a\gamma}|u|^\gamma dx$$

$$= \lambda\left(\frac{N}{q} - \frac{N-\beta}{2}\right)\int_{\mathbb{R}^N} g(x)|u|^q dx.$$

Proof of Theorem 2. We proceed by contradictions.

From Lemma 4, since *(G)* hold and $1<q<2$ therefore, if $\beta \in (2,3)$, $\gamma \in (2_{\beta-2a}, 2^*)$ with $2_{\beta-2a} = 2N/(N-2(\beta-2a))$ we obtain that $\lambda > 0$ what contradicts the fact that $\lambda > 0$.

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