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Examples of Freudenthal-Kantor triple systems

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Abstract

Symmetry group of Lie algebras and superalgebras constructed from (ϵ,δ) Freudenthal-Kantor triple systems has been studied. Also, the definition and examples of hermitian triple systems is introduced in this note.

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Introduction

It seems that the concept of a triple system (or called a ternary algebra) in nonassociative algebras started from the metasymplectic geometry due to Freudenthal. After a generalization of the concept has been studied by Tits, Koecher, Kantor, Yamaguti, Allison and authors ([1,2] for many earlier references on the subject). Also it is well known the object of investigation of Jordan and Lie algebras with application to symmetric spaces or domains [3] and to physics [4,5].

Nonassociative algebras are rich in of mathematics, not only for pure algebra differential geometry, but also for representation theory and algebraic geometry. Specialy, the Lie algebras and Jordan algebras plays an important role in many mathematical and physical objects. As a construction of Lie algebras as well as Jordan algebras, we are interested in characterizing the Lie algebras from view point of triple systems [6-9]). These imply that we are considering to structure of the subspace L_1 of the five graded Lie (super)algebra $L (\epsilon,\delta) = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ satisfying $[L_i, L_j] \subseteq L_{i+j}$, associated with an (ϵ,δ) Freudenthal-Kantor triple system which contains a class of Jordan triple systems related 3 graded Lie algebra $L_{-1} \oplus L_0 \oplus L_1$. For these consideretions without untilizing properties of root systems or Cartan matrices, we would like to refer to the articles of the present author and earlier references quoted therein [1,2,10,11].

In particular, for an characterizing of Lie algebras, an applying to geometry and physics, we will introduce couple topics about a symmetry of Lie algebras (section 1) and a definition of hermitian triple systems (section 2) in this note, which is a survey and an announcement of new results.

More precisely speaking, first, the symmetry group of Lie algebras and superalgebras constructed from (ϵ,δ) Freudenthal-Kantor triple systems has been studied. Especially, for a special (ϵ, ϵ) Freudenthal-Kantor triple, it is *SL(2)* group. Secondly, we will give a definition of hermitian * generalized Jordan triple systems and the examples of their tripotents defined by elements c of triple systems satisfying $CCC=C$.

Symmetry of Lie algebras associated with triple sytems

A triple system V is a vector space over a field F of characteristic $\neq$ 2 or 3 with a trilinear map $V \otimes V \otimes V \to V$. We denote the trilinear product (or ternary product) by juxtaposition *(xyz)* $\in$ *V(or<xyz>,[xyz],etc).*

A well studied triple system is the (ϵ,δ) Freudenthal-Kantor triple systems (abbreviated hereafter as to (ϵ,δ) FKTS) with ϵ and δ being either +1 or -1 [1,2,9,12].

Since (ϵ,δ) Freudenthal-Kantor triple systems offer a simple method of constructing Lie algebras (for the case of $(\delta=+1)$) and Lie superalgebras (for the case of $\delta = -1$) with 5-graded structure, it may be of some interest to study its symmetry group in this note. In order to facilitate the discussion, let us briefly sketch its definition.

We introduce two linear mappings L and $K:V \oplus V \to End\ V$ by

$$L (x,y) = xyz,\ K (x,y)z= - \delta yzx \quad (1.1)$$

for $\delta = +1\ or\ -1$ If they satisfy

$$[L(u,v), L(x,y)] = L(L(u,v)x,y)+ \epsilon L(x,L(v,u)y),\ (1.2)$$

$$K(K(u,v)x,y) = L(y,x)K(u,v)- \epsilon K(u,v)L(x,y) \quad (1.3)$$

for any $u,v,x,y \in V$ and $\epsilon = +1$ we call the triple system to be (ϵ,δ) FKTS.

One consequence of Eqs.(1.2) and (1.3) is the validity of the following important identity (see [Y-O.] Eqs.(2.9) and (2.10))

$$K(u,v)K(x,y) = \epsilon\ \delta L(K(u,v)y,x- \epsilon L(K(u,v)x,y) \quad (1.4)$$

$$= L(v,K(x,y)u) – \delta L(u,K(x,y)v) \quad (1.5)$$

We note that (-1,1)FKTS is said to be a generalized Jordan triple system of second order [12] in section 2, becaue that is a generalization of concept of Jordan triple systems.

We can then construct a Lie algebra for $\delta =+1$ and a Lie superalgebra for $\delta =-1$ as follows:

Let W be a space of 2×1 matrices over V

$$W = \begin{pmatrix} V \\ V \end{pmatrix}$$

and define a tri-linear product:

$W \oplus W \oplus W \to W$ by

$$\left[\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \right] =$$

$$\begin{pmatrix} L(x_1, y_2) - \delta L(x_2, y_1) & \delta K(x_1, x_2) \\ -\varepsilon K(y_1, y_2) & \varepsilon L(y_2, x_1) - \varepsilon \delta L(y_1, x_2) \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} . (1.6)$$

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Then, it defines a Lie triple system for $\delta = +1$ and an anti-Lie triple system for $\delta = -1$ We then note

$$\hat{L}_{\bar{0}} = \text{span}\left\{\begin{pmatrix} L(x,y) & \delta K(z,w) \\ -\varepsilon K(u,v) & \varepsilon L(y,x) \end{pmatrix} \Big| x,y,z,w,u,v \in V \right\} (1.7)$$

is a Lie subalgebra of $Mat_2(End(V))^-$ where B^- for an associative algebra B implies a Lie algebra with bracket; $[x,y] = xy-yx$ We note also then

$$\hat{D} = \begin{pmatrix} L(x,y), & \delta K(z,w) \\ -\varepsilon K(u,v), & \varepsilon L(y,x) \end{pmatrix} \in L(W,W) (1.8)$$

is a derivation of the triple system. Setting

$$L_{\bar{T}} = \text{span}\left\{ X = \begin{pmatrix} x \\ y \end{pmatrix} \Big| x,y \in V \right\} (=W), (1.9)$$

then L defined by

$$L = L_{\bar{0}} \oplus L_{\bar{T}} (1.10)$$

gives a Lie algebra for $\delta = +1$ and a Lie superalgebra for $\delta = -1$ where

$$L_{\bar{0}} = \{D \mid D \text{ is a derivation of } L\}, (1.11)$$

i.e., D satisfies

$$= D[X_1, X_2, X_3]$$

$$[DX_1, X_2, X_3] + [X_1, DX_2, X_3] + [X_1, X_2, DX_3] (1.12a)$$

and hence induces also

$$[D,[X,Y]] = [DX,Y] + [X,DY], (1.2b)$$

if we define the bracket by

$$[D_1 \oplus X_1, D_1 \oplus X_2] = ([D_1, D_2] + L(X_1, X_2)) \oplus (D_1 X_2 - D_2 X_1). (1.13)$$

where

$$[D_1, D_2] = D_1, D_2 - D_2, D_1$$

and

$$L(X_1, X_2) = [X_1, X_2] = \left[\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right] =$$

$$\begin{pmatrix} L(x_1, y_2) - \delta L(x_2, y_1) & \delta K(x_1, x_2) \\ -\varepsilon K(y_1, y_2) & \varepsilon L(y_2, x_1) - \varepsilon \delta L(y_1, x_2) \end{pmatrix}. (1.14)$$

Note that the endomorphism $L(X,Y)$ is then an inner derivation of the triple system.

Since $L_{\bar{0}} \supset \hat{L}_{\bar{0}}$, we will mainly discuss a subsystem $\hat{L}$ of L, given by rather than the larger L. Then, $\hat{L}$ is 5-graded

$$\hat{L} = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2 (1.16)$$

where

$$L_{-2} = \text{span}\left\{ \begin{pmatrix} 0 & 0 \\ -\varepsilon K(x,y) & 0 \end{pmatrix} \Big| x,y \in V \right\} (1.17a)$$

$$L_{-1} = \text{span}\left\{ \begin{pmatrix} 0 \\ x \end{pmatrix} \Big| x \in V \right\} (1.17b)$$

$$L_0 = \text{span}\left\{ \begin{pmatrix} L(x,y) & 0 \\ 0 & \varepsilon L(y,x) \end{pmatrix} \Big| x,y \in V \right\} (1.17c)$$

$$L_1 = \text{span}\left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \Big| x \in V \right\} (1.17d)$$

$$L_2 = \text{span}\left\{ \begin{pmatrix} 0 & \delta K(x,y) \\ 0 & 0 \end{pmatrix} \Big| x,y \in V \right\}. (1.17e)$$

Here, we utilized the following Proposition for some of its proof.

Theorem 1: ([K-O.], [K-M-O.]) *Let $(V, (xyz))$ be a (ϵ,δ)-Freudenthal-Kantor triple system with an endomorphism P such that $P^2 = -\epsilon\delta Id$ and $P(xyz) = (PxPyPz)$ Then, $(V,[xyz])$ is a Lie triple system (for $\delta=1$) and anti-Lie triple system (for $\delta=-1$) with respect to the product*

$$[xyz] = (xPyz) - \delta(yPxz) + \delta(xPzy) - (yPzx).$$

In passing, we note that the standard $\hat{L} = \Sigma_{i=-2}^2 \oplus L_i$ is a result of Theorem 1 immediately with

$$P = \begin{pmatrix} 0, & \delta \\ -\varepsilon, & 0 \end{pmatrix} \text{ and } x \to X \text{ etc.}$$

Next, we introduce $\theta, \sigma(\lambda) \in End(\hat{L})$ for any $\lambda \in F (\lambda \neq 0)$, being non-zero constant by

$$\theta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\varepsilon y \\ \delta x \end{pmatrix} (1.18a)$$

$$\sigma(\lambda) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda x \\ \frac{1}{\lambda} y \end{pmatrix} (1.18b)$$

in $W = L_{\bar{T}} = L_{-1} \oplus L_1$. We may easily verify that they are automorphism of $[W,W,W]$, i.e, we have for example

$$\theta([X,Y,Z]) = [\theta X, \theta Y, \theta Z]$$

for $X,Y,Z \in W$ We then extend their actions to the whole of $\hat{L}$ in a natural way to show that they will define automorphism of L. They moreover satisfy

(i) σ

$$\sigma(1) = Id, \theta^4 = Id (1.19a)$$

where Id is the identity mapping

(ii)

$$\theta^2 = -\varepsilon\delta Id \text{ for } L_{\bar{T}} \text{ but } \theta^2 = Id \text{ for } L_{\bar{0}} (1.19b)$$

(iii)

$$\sigma(\mu)\sigma(v) = \sigma(\mu v) \text{ for } \mu, v \in F, \mu v \neq 0$$

(iv)

$$\sigma(\lambda) \theta \sigma(\lambda) = \theta \text{ for any } \lambda \in F, \lambda \neq 0. (1.19d)$$

We call the group generated by $\sigma(\lambda)$ and θ satisfying these conditions simply as D (ϵ,δ) due to a lack of better terminology. If the field F contains $\omega \in F$ satisfying $\omega^3 = 1$ but $\omega \neq 1$, then a finite sub-group of D (ϵ,δ) generated by θ and $\sigma(\omega)$ defines DiC_3 group for $\epsilon = \delta$ but S_3 for $\epsilon = -\delta$, as they have noted already [13].

Conversly any 5-graded Lie algebra (or Lie superalgebra) with such automorphism θ and $\sigma(\lambda)$ satisfying Eqs.(1.19) lead essentially to a $(\epsilon \delta)$ FKTS in L_1 with a triple product defined by $\{x,y,z\} = [[x, \theta y,]z]$ for $x,y, z \in L_1$ [6].

We note that the corresponding local symmetry of D (ϵ,δ) yields a derivation of L, given by

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (1.20)$$

which satisfies

$$h[X,Y,Z] = [hX,Y,Z]+[X,hY,Z]+[X,Y,hZ] (1.21a)$$

as well as *[h,[X,Y]] = [hX,Y] + [X,hY] (1.21b)* for $X,Y,Z \in W$.

We can find a larger automorphism group of $\hat{L}$, if we impose some additional conditions. First suppose that $K(x,y)$ is now expresed as

$$K(x,y) = \epsilon \, \delta \, L(y,x) - \epsilon \, L(x,y) \,(1.22)$$

for any $x,y \in V$. We call then the triple system to be a special (ϵ, δ) FKTS [13]. Moreover for the case of special (ϵ, ϵ) FKTS (i.e. $\epsilon = \delta$), the automorphism group of L turns out to be a larger $SL(2,F)(=Sp(2,F))$ group which contains $D(\epsilon,\epsilon)$ as its subgroup. In this case, the triple system $[W,W,W]$ becomes invariant under

$$\begin{bmatrix} x \\ y \end{bmatrix} \to U \begin{bmatrix} x \\ y \end{bmatrix} (1.23a)$$

for any 2×2 SL(2,F) matrix U, i.e.

$$U = \begin{pmatrix} \alpha & \beta \\ \mu & v \end{pmatrix}, \mathrm{Det}\, U = \alpha v - \beta\mu = 1. (1.23b)$$

Also, the associated Lie algebras or superalgebras are BC_1-graded algebra of type C_1.

Finally, we consider a ternary system (V, xy, xyz) *(V,xy,xyz)* where xy and xyz are binary and ternary products, respectively, in the vector space V. Suppose that they satisfy

(1) the triple system *(V,xyz)* is a (-1,1) FKTS.

(2) The binary algebra *(V,xy)* is unital and involution $\overline{(\overline{xy} = \overline{yx})}$ with the involutive map $_{xyz = (z\bar{y})x - (z\bar{x})y + (x\bar{y})z.(1.24)}$

(3) The triple product xyz is expressed in terms of the bi-linear products by

$$xyz = (z\bar{y})x - (z\bar{x})y + (x\bar{y})z. (1.24)$$

We may call the ternary system *(V,xy,xyz)* to be Allison-ternary algebra or simply A-ternary algebra, since $A = (V,xy)$ is then the structurable algebra [14-16]. This case is of great interest, first because structurable algebras exhbit a triality relation [16,17], and second because we can construct another type of Lie algebras independently of the standard construction of (-1,1) FKTS, which is S_4-invariant and of BC_1 graded Lie algebra of type B_1. The relationship between the Lie algebra constructed in the new way and that given as in Eq.(1.17) is by no means transparent. Note that the group D(-1,1) contains S_3 but not S_4 symmetry. We may show that if the field F contains the square root $_{\sqrt{-1} \in F,}$ of-1, then Eqs.(1.17) can be prolonged to yield the Lie algebra for the structurable algebra.

The Lie algebras constructed as in Eqs.(1.13) and (1.14) is also a BC_1-graded Lie algebra of type B_1 without assuming $_{\sqrt{-1} \in F,}$ ([6]). Also, if F is an algebraically closed field of characteristic zero, then any simple Lie algebra is known to be S_4-invariant and can be constructed by some structurable algebra, so that any such Lie algebra is also a BC_1-graded Lie algebra of type B_1, (as well as of type C_1). Of course, the underlying *sl(2)* symmetry is different for both B_1 and C_1 cases. Roughly speaking, it seems that these concept are a version of Lie algebras theory corresponding with a Galois group of algebraic numbers theory.

The contents of this section is a cowork with Prof. Okubo, and the details will be discussed in other papers.

Hermitian triple systems

For a geometrical object based on triple systems in this section, first we note that the symmetric bounded domains are a one to one, correspondence to hermite Jordan triple systems, such that a certain trace form is positive definite hermitian. Hence as a generalization of these triple systems, we are interesting to investigate for structure theory of hermitian generalized Jordan triple systems, in particular, the case of hermitian generalized Jordan triple systems of second order (i.e. hermitian (-1,1) -Freudenthal-Kantor triple system).

We shall concerned with algebras and triple systems which are infinite over a complex number field, unless otherwise specified in this section.

Definition: A triple system V is said to be a *-generalized Jordan triple system of second order if relations (0)--(iv) satisfy;

0) V is a Banach space,

i) $[L(a,b),L(c,d)] = L(<abc>,d) - L(c,<bad>)$,

ii) $K(<abc>,d)+K(c,<abd>)+K(a,K(c,d)b)= 0$,

where $L<(a,b)c = <abc>$ and $K(a,b)c = <acb> - <bca>$,

iii) $<xyz>$ is C-linear operator on x,z and C-anti-linear operator on y,

iv) $<abc>$ continuous with respect to a norm $\| \,\|$ that is, there exists $K>0$ such that

$$\|<xxx>\| \le K \|x\|^3 \text{ for all } x \in V.$$

Furthermore a *-generalized Jordan triple system of second order is said to be hermitian if it satisfies the following condition,

v) all operator $L(x,y)$ is a positive hermitian operator with a hermitian metrix

$$(x,y) = \text{trace } L(x,y), (2.1)$$

that is, $(L(x,y)z,w) = (z,L^*(x,y)w)$, and $(x, y) = \overline{(y, x)}$. In particular, if a triple system V satisfies the condition (o),(i),(iii),(iv) and (v), then it is said to be a hermitain *-generalized Jordan triple system.

Furthermore, as a generalization of the generalized Jordan triple system of second order, it is said to be a . (ϵ, δ) -Freudenthal-Kantor triple system if the following relations satisfy;

i) ′ $[L(a,b),L(c,d)] = L(<abc>,d) \,\epsilon L(c,<bad>)$

ii) ′ $K(<abc>,d)+K(c,<abd>) + \delta K(a,K(c,d)b) =0$,

where $K(a,b)c = <abc> - \delta<bca>$, $\epsilon = \pm 1$, $\delta = \pm 1$.

We note that these identities *(i)′* and *(ii)′* are equivalent to the identities (1.2) and (1.3) in section 1 [2,7].

Example 1: Let V be a J^*-triple (for the definition, to Loos [3]. Then V is a hermitian *-generalized Jordan triple system of second order, because they satisfy the condition (i) and a special case of (ii), that is,

$$K(x,y)z = <xyz> - <yzx> \equiv 0$$

(identicallly zero).

Example 2: Let $T^*_{n,n}$ be the space of diagonal matrix of n×n with element of the complex number. Then $T^*_{n,n}$ is a *-generalized Jordan triple system of second order, with respect to the product and the norm

$$\| \,\|$$

$$<xyz> = x\bar{y}^T z + z\bar{y}^T x - \bar{y}x^T z$$

for all $x,y,z \in T^*_{n,n}$

$$\| x \| = max(| \lambda_i |, | \mu_i |)$$

where $x = \Sigma(\lambda_i e_i + \mu_i \sqrt{-1} e_i)$, $\lambda_i, \mu_i \in R$ (real number) and e_i are matrix unit element of $T^*_{n,n}$ and x^T is the transpose of x.

Next let V be a *-generalized Jordan triple system. Then we may define the notation of a tripotent and a bitripotent as follows.

Definition: It is said to be a tripotent of V if

$<ccc> = c, c \in V (2.2)$.

Definition: It is said to be a strongly bitripotent of V if a pair (c_1, c_2) of

$<c_1 c_1 c_2> = -1/2 c_2$, $< c_1 c_1 c_2> = -1/2 c_1$,

and other porducts are zero.

Definition: It is said to be a bitripotent of V if a pair (c_1, c_2) of the tripotens satisfy the relations

$<c_1 c_1 c_2> = \alpha c_2$, $<c_2 c_2 c_1> = \alpha c_1$,

$< c_1 c_2 c_1> = \beta c_2$, $< c_2 c_2 c_2> = \beta c_1$,

$<c_2 c_1 c_1> = \gamma c_2$, $<c_1 c_2 c_2> = \gamma c_1$,

and other products are zero, where

$\alpha^2 + \beta^2 + \gamma^2 \neq 0, \alpha, \beta, \gamma \in R$ (real number).

For *-generalized Jordan triple system V, we can define a norm $\| \|$ as follows;

$$\| x \| = max |\lambda_i|, if x = \Sigma \lambda_i e_i \in V,$$

where e_i are tripotents or bitripotents. We note that

$$\| x \| \geq 0 \text{ and } \| x + y \| \leq \| x \| + \| y \|.$$

Example 3: (a generalization of Example 2.2) Let $D^*_{n,k}$ be the set of all $n \times k$ matrices with operation

$< xyz > := x\bar{y}^T z + z\bar{y}^T x - \bar{y}x^T z$, (2.3)

where x^T and $\bar{x}$ mean transpose and conjugation of x respectively.

Then D^*_{nk} is a hermitian *-generalized Jordan triple system of second order. In fact, the conditions (i),(ii) and (iii) are evident, by straightfoward calculations.

For (v) considering, we have $(x,y) := tr (x,y)$ linear on x and anti linear on y by the condition (iii). Thus we may get $tr L(x, y) = tr \overline{L(y,x)}$ as follows.

From the relations;

$(L(x, y) + L(y,x))z = x\bar{y}^T z + z\bar{y}^T x - \bar{y}x^T z + y\bar{x}^T z + z\bar{x}^T y - \bar{x}y^T z$

$= (x\bar{y}^T + y\bar{x}^T)z - (\bar{x}y^T + \bar{y}x^T)z + z(\bar{y}^T x + \bar{x}^T y)$,

$(L(x, y) - L(y,x))z = x\bar{y}^T z + z\bar{y}^T x - \bar{y}x^T z - y\bar{x}^T z - z\bar{x}^T y + \bar{x}y^T z$

$= (x\bar{y}^T - y\bar{x}^T)z + (\bar{x}y^T - \bar{y}x^T)z + z(\bar{y}^T x - \bar{x}^T y)$,

$= (x\bar{y}^T - y\bar{x}^T)z + (\bar{x}y^T - \bar{y}x^T)z + z(\bar{y}^T x - \bar{x}^T y)$,

it follows that $L(x,y)+L(y,x)$, and $L(y,x)-L(y,x)$ have are a real trace form and an imagenary trace form respectively. Thus we have $(x, y) = \overline{(y,x)}$.

To prove positive definite, we consider

$L(x,x)z = x\bar{x}^T z + z\bar{x}^T x - \bar{x}x^T z = (x\bar{x}^T - \bar{x}x^T)z + z\bar{x}^T x$.

Since $tr \, x\bar{x}^T = tr \, \bar{x}x^T$, we get $trace L(x,x) = k \, trace \, \bar{x}^T x$, this

means that the trace from (x,y) is positive definite. It is enough to show the condition (iv). By means of result of the property of matrix, we can write

$x = \Sigma \mu_i e_i$, $\| x \| = max |\mu_i|$,

e_i are tripotents or bitripotens if e_i is the unit element of matrix..

Moreprecisely speaking, we have

$\| x \| = max(| \lambda_{ij} |, | \mu_{ij} |)$ for $x = \Sigma(\lambda_{ij} E_{ij} + \mu_{ij}\sqrt{-1} E_{ij})$,

where the notations are denoted by $\lambda ij, \mu ij \in R$ (real number field) and also E_{ij} means that (i,j) element is 1 and othere element is zero. E_{ij} and $\sqrt{-1}E_{ij}$ are tipotents, i.e., $<E_{ij}E_{ij}E_{ij}> = E_{ij}$, and $< \sqrt{-1}E_{ij}\sqrt{-1}E_{ij}\sqrt{-1}E_{ij} > = \sqrt{-1}E_{ij}$. Furthermore, we note $(E_{ij}, \sqrt{-1}E_{ij})$ are bitripotents.

Then we have

$\| < xxx > \| = \| 2x\bar{x}^T x - \bar{x}x^T x \| \leq 2 \| x\bar{x}^T x \| + \| \bar{x}x^T x \| \leq 3 \| x \|^3$.

These show that $D^*_{n,k}$ is a hermilian *- generalized Jordan triple system of second order, that is, a hermitian (-1,1)-FKTS.

Examples 4: Let $S^*_{2n,k}$ be the set of all $2n \times k$ matrices with operation

$< XYZ > = X(\bar{Y})^T Z + Z(\bar{Y})^T X - \phi(\bar{Y})\phi(X)^T Z (2.4)$

where $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, x_i is an $n \times k$- matrix, and $( ) = \begin{pmatrix} \end{pmatrix}$.

Then $S^*_{2n,k}$ is a hermitian *-generalized Jordan triple system of second order.

In fact, the elements $\frac{1}{\sqrt{2}}\begin{pmatrix} E_{ij}^{(1)} \\ E_{ij}^{(2)} \end{pmatrix} \in S^*_{2n,k}$ are tripotents.

For any element X of $S_{2n,k}$, we may represent as follows;

$X = \Sigma(\lambda_{ij}^{(1)} E_{ij}^{(1)} + \mu_{ij}^{(1)}\sqrt{-1} E_{ij}^{(1)}) + \Sigma(\lambda_{lm}^{(2)} E_{lm}^{(2)} + \mu_{lm}^{(2)}\sqrt{-1} E_{lm}^{(2)})$

and the norm is defined by

$\| X \| = max(| \lambda_{ij}^{(1)} |, | \lambda_{lm}^{(2)} |, | \mu_{ij}^{(1)} |, | \mu_{lm}^{(2)} |)$

where $E_{ij}^{(1)}$ is the matrix unit of $\begin{pmatrix} S_{n,k} \\ 0 \end{pmatrix}$ and $E_{lm}^{(2)}$ $E_{lm}^{(2)}$ is the matrix unit of $\begin{pmatrix} 0 \\ S_{n,k} \end{pmatrix}$.

By straightforward calculations as well as Example 2.3, we can show that $S^*_{2n,k}$ is a hermitian *-generalized Jordan triple system of second order.

Example 4: Let $A^*_{kn} \oplus A^*_{nl} A^*_{kn}$ be the set fo all pairs $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, where x_1 is a $k \times n$ matrix and x_2 is a

$n \times l$ matrix with operation given by foumula

$$< XYZ > = \begin{pmatrix} x_1\bar{y}_1^T z_1 + z_1\bar{y}_1^T x_1 - z_1 x_2\bar{y}_2^T \\ x_2\bar{y}_2^T z_2 + z_2\bar{y}_2^T x_2 - \bar{y}_1^T x_1 z_2 \end{pmatrix}.(2.5)$$

Then $A^*_{kn} \oplus A^*_{nl}$ is a hermitian *-generalized Jordan triple system of second order.

In fact, for $X = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$. we have

$(X, X) = trace L(X, X) = n \, trace \, u_1\bar{u}_1^T + l \, trace \, u_2\bar{u}_2^T$,

hence it follows that the trace form $(,)$ is positive definite.

For any element of $A^*_{kn} \oplus A^*_{nl}$, we may represent as follows;

$X = \Sigma(\lambda_{ij}^{(1)} E_{ij}^{(1)} + \mu_{ij}^{(1)}\sqrt{-1} E_{ij}^{(1)}) + \Sigma(\lambda_{st}^{(2)} E_{st}^{(2)} + \mu_{st}^{(2)}\sqrt{-1} E_{st}^{(2)})$

and $(E_{ij}^{(1)}, E_{jt}^{(2)}) \epsilon A^{*}_{kn} \oplus A^{*}_{nl}$ is a tripotent.

Furthermore, the norm is defined by

$$\| X \| = \sqrt{2} max(| \lambda_{ij}^{(1)} |, | \lambda_{st}^{(2)} |, | \mu_{ij}^{(1)} |, | \mu_{st}^{(2)} |).$$

Example 6: Let $C^{*}_{nn} \oplus A^{*}_{nl}$ be the set of all pairs $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, where x_1 is a skew-symmetric $n \times n$ matrix and x_2 is a column,i.e.,is a $n \times 1$ matrix, with operation given by formula,

$$< XYZ > := \begin{pmatrix} x_1 \overline{y}_1^T z_1 + z_1 \overline{y}_1^T x_1 - \overline{y}_2 x_2^T z_1 - z_1 x_2 \overline{y}_2^T \\ x_2 \overline{y}_2^T z_2 + z_2 \overline{y}_2^T x_2 - \overline{y}_1^T x_1 z_2 \end{pmatrix} (2.6)$$

Then $C^{*}_{nn} \oplus A^{*}_{nl}$ is a hermitian *-generalized Jordan triple system of second order.

In fact, we put $c_{ij} :=$ the matrix of (i,j) -element is $1,(j,i)$ -element is -1 and other element is zero $(1 \le i \le n)$, further more $ek := (0$- - - $,1,$ - - - $0)T,(k$ th element is 1). Then $\sqrt{2} \begin{pmatrix} c_{ij} \\ 0 \end{pmatrix}$ are tripotents and also $\sqrt{2} \begin{pmatrix} 0 \\ e_k \end{pmatrix}$ are tripotents but these are not bitripotents. For any element of $C^{*}_{nn} \oplus A^{*}_{nl}$, we may represent as follow;

$$X = \Sigma(\lambda_{ij}^{(1)} c_{ij} + \mu_{ij}^{(1)} \sqrt{-1} c_{ij}) + \Sigma(\lambda_{k}^{(2)} e_k + \mu_{k}^{(2)} \sqrt{-1} e_k)$$

and the norm is defined by

$$\| X \| = \sqrt{2} max(| \lambda_{ij}^{(1)} |, | \mu_{ij}^{(1)} |, | \lambda_{k}^{(2)} |, | \mu_{k}^{(2)} |).$$

In the end of this section, we note a Peirce decomposition as follows ([12] for the case of $\delta = 1$).

Theorem 1: *For $\delta = \pm 1$ and hermitian $(-1, \delta)$ Freudenthal-Kantor triple system V with a tripotent C s.t. $<ccc> = c$ we have*

$$V = V_0 \oplus V_{1/2} \oplus V_1$$

where denoted by $V_0 = \{x \mid N_\delta = 0, T_\delta = 0 \}$, $V_{1/2} = \{x \mid N_\delta \neq 0, T_\delta = 0 \}$, $V_1 = \{x \mid N_\delta = 0, T_\delta \neq 0 \}$ and also defined by $R(x,y)z = <zxy>$, $N_\delta = L(c,c)$ $- \delta R(c,c)$ and $T_{\delta =} (1+\delta)L(c,c)-R(c,c)+Id.$,

This section is a cowork with Dr M.Sato with an application to physics and the details will be considered in future paper [18].

Concluding Remark

In this section, we shall briefy describe a correspondence with the triple systems and the Lie algebras or superalgebras of simple classical type associated with their triple systems [1,2,7,10,11].

Let V be the matrix set of $Mat(m,n;F)$ and the triple product is defined by

$$<xyz> = xy^T z + \delta (zy^T x - \mu z x^T y) x,y,z \epsilon V.(*)$$

Then there exists 4 cases of $(-1, \delta)$-Freudentahl-Kantor triple systems with $\mu = 0$ or 1. and $\delta = \pm 1$.

The standard embeding Lie algebras or superalgebras $\hat{L} = L(W,W) \oplus W$, where $W = V \oplus V$ (cf. section 1) associated with the triple product $(*)$ are appeared by 4 types as follows;

(i) 5 graded Lie algebra $(\delta = 1, \mu = 1)$

$B_{m+1} = so(2m+2l+1) (n= 2l+1), D_{m+1} = so(2m+2l) (n=2l).$

(ii) 3 graded Lie algebra $(\delta = 1, \mu = 0)$

$A_{n+m-1} = sl(n+m).$

(iii) 5 graded Lie superalgebra $(\delta = -1, \mu = 1)$

$. . .B(l,m) = osp(2l+1|2m) (n = 2l+1), D(l,m) = osp(2l|2m) (n=2l).$

In particular, if $m=1$, then the triple product

$<xyz> = <x,y> z - <z,y> x + <z,x> y$

is a $(-1,-1)$-Freudentahl-Kantor triple system satisfying $K(x,z) = <xyz> + <zyx> = 2<x,z>y$, and so $dim\{(x,z\}span = 1$ (called a balanced triple system), where $<x,y> = x_1 y_1 + . . . + x_n y_n$.

(iv) 3 graded Lie superalgebra $(\delta = -1, \mu = 0)$

$A(m-1,n-1) = sl(m|n) (m \neq n), A(m-1|m-1) = psl (m|m) (m= n).$

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