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Euler-Poincare Formalism of Peakon Equations with Cubic Nonlinearity

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Abstract

We present an Euler-Poincar´e (EP) formulation of a new class of peakon equations with cubic nonlinearity, viz., Fokas-Qiao and V. Novikov equations, in two almost equivalent ways. The first method is connected to flows on the spaces of Hill's and first order differential operator and the second method depends heavily on the flows on space of tensor densities. We give a comparative analysis of these two methods. We show that the Hamiltonian structures obtained by Qiao and Hone and Wang can be reproduced by EP formulation. We outline the construction for the 2+1-dimensional generalization of the peakon equations with cubic nonlinearity using the action of the loop extension of *Vect*(S^1) on the space of tensor densities.

Keywords: Spaces of tensor densities; Fokas-Qiao equation; Novikov equation; Bi-Hamiltonian; 2+1-dimensional peakon equation

Mathematics Subject Classifications (2000): 53A07, 53B50

Introduction

The one-parameter family of shallow water equations

$$u_t - u_{xxt} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx}, \quad (1)$$

where b is a real parameter, has recently drawn some attention. This equation is called the b-field equation by Degasperis, Holm and Hone [1,2], who introduced this equation. They showed the existence of multi-peakon solutions for any value of b, although only the special cases b=2, 3 are integrable, having bi-Hamiltonian formulations. In the literature the partial differential equation (1) is also known as the Degasperis-Holm-Hone (or DHH) equation. The b=2 case is the well-known Camassa-Holm (CH) equation [3] and b=3 is the integrable system discovered by Degasperis and Procesi [4]. The most interesting feature of the CH equation is to admit peaked soliton (peakon) solutions. A peakon is a weak solution in some Sobolev space with corner at its crest. The stability and interaction of peakons were discussed in several references [5-7]. Using the Helmholz field $m=u-u_{xx}$, the DHH equation (1) allows reformulation in a more compact form

$$m_t + um_x + bu_x m = 0, \quad (2)$$

where the three terms correspond respectively to evolution, convection and stretching of the one-dimensional flow. Recently Lundmark and Szmigielski [8,9] used an inverse scattering approach to determine a completely explicit formula for the general n-peakon solution of the DP equation.

An Euler-Poincare formalism has been studied for the Degasperis and Procesi (DP) equation [10]. It has been shown that DP equation is a superposition of two flows, on the space of Hill's operators and the first order differential operators. Thus the Poisson operators of the Degasperis-Procesi flow are the pencil of two operators. Moreover, the Hamiltonian structure obtained from the EP framework exactly coincides with the Hamiltonian structues of the DP equation obtained by Degasperis, Holm and Hone. More recently we have given a short derivation [11,12] of the DP equation using algebra of tensor densities on S^1.

In an interesting paper Fokas et al. [13] proposed an algorithmic construction of (2+1) dimensional integrable systems which yield peakon/dromion type solutions.

$$q_{xt} - \nu q_{xxxt} + aq_{xy} + bq_{xxxy} + c\left(q_{xx}q_y + 2q_x q_{xy}\right) - c\nu\left(q_{xxxx}q_y + 2q_{xxx}q_{xy}\right). \quad (3)$$

This equation can also be identified [11,12] with the potential form of the Camassa-Holm or peakon analogue of the Calogero-Bogoyavlenskii-Schiff (CBS) equation (for a=0), one of the most well-known (2+1)-dimensional KortewegdeVries (KdV) type system. Equation (3) reduces CBS equation for ν=0. It is known [12] that (3) is an Euler-Poincare flow on the co-adjoint orbit of loop Virasoro algebra with respect to H^1-Sobolev norm.

In addition to the CH equation and DP equation, other integrable models with peakon solutions have been studied in recent years. The topic of this communication is to formulate the Euler-Poincare theory of the peakon type equations with cubic nonlinearity. One must note that the nonlinear term in the Camassa-Holm type systems and their two component generalizations are quadratic. So it is natural to ask whether there exist integrable systems admitting peakon solutions with cubic nonlinear terms. Among these models, there are two integrable peakon equations with cubic nonlinearity. Recently Qiao found a peakon equation with cubic nonlinearity, which can also be written as:

$$m_t + (u^2 - u_x^2)m_x + 2u_x m^2 = 0. \quad (4)$$

This inspired Novikov to find another peakon equation with cubic nonlinearity while studying symmetry classification of nonlocal PDEs with quadratic or cubic nonlinearity.

This equation can also be expressed as

$$u_t - u_{xxt} + 4u^2 u_x = 3uu_x u_{xx} + u^2 u_{xxx}. \quad (5)$$

Hone and Wang [14] gave the Lax pair, bi-Hamiltonian structure, and peakon solutions of this equation.

Note that the equation (4) was proposed independently by Fokas

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[15], Fuchssteiner [16], Olver and Rosenau [17]. Qiao [18] derived Lax pair, M/W-shaped soliton and peaked/cusped solitons. Recently, the peakon stability of equation (5) in the case of b=0 was worked out by Gui et al. [19]. In this communication we will show that the Hamiltonian structures obtained by Qiao [18,20] and independently Hone and Wang [14] for these two peakon equations with cubic nonlinearity can be obtained from the Euler-Poincare framework.

Result and plan

In section 2 we present the Euler-Poincare formulation of the cubic peakon equations. We compute the Hamiltonian structures of Camassa-Holm-Whitham- Burger equation and b-field equation using two different methods. The first method is based on Euler-Poincar´e flows on the space of first and second order differential operators and second is associated to the flows on tensor densities. Using these two equivalent methods we derive the Hamiltonian structure of the b-field equation. We use this result to compute the Hamiltonian structures of the cubic peakon equations in Section 3. We also give a formulation of 2+1-dimensional cubic peakon equation using the action of the loop extension of $Vect(S^1)$ on the space of tensor densities. We actually demonstrate that the algebra of tensor densities is more effective than the first method for the construction of higher-dimensional cubic peakon equation.

Euler-Poincare Construction of *b*-field Equations and Hamiltonian Structures

Our goal is to derive Hamiltonian structures of the cubic peakon equations, and this has to go through the construction of b-field structure. So in this section we give an Euler-Poincare (EP) derivation of the b-field equation in two different methods: (A) one of them is related to the flows on the space of first and second order differential operators with respect to H^1-Sobolev norm [10] and (B) other one is connected to the flows on tensor densities [11,12]. One of the main reasons that we are shifting to Lie derivative approach is that there is no equivalent description of EP flows on the space first order differential operators in terms of co-adjoint orbit. We collate all the definitions and background materials in the next section.

Lie-derivative method: a different way of interpreting *Vect(S¹)* action

Denote $F_\mu(S^1)$ the space of tensor-densities of degree μ on S^1

$$F_\mu = \{a(x)dx^\mu \mid a(x) \in C^\infty(S^1)\},$$

where μ is the degree, x is a local coordinate on S^1. As a vector space, $F_\mu(S^1)$ is isomorphic to $C^\infty(S^1)$.

Let $\Omega = T^*S^1$ be the cotangent bundle of S^1. Geometrically we say $F_\lambda \in \Gamma(\Omega^{\otimes\lambda})$, where $\Omega^{\otimes\lambda} = (T^*S^1)^{\otimes\lambda}$. This space plays an important role in equivariant quantization. This space is endowed with a structure of $Diff(S^1)$ and $Vect(S^1)$-module. Here $F_0(M) = C^\infty(M)$, the space $F_1(M)$ and $F_{-1}(M)$ coincide with the spaces of differential forms and vector fields respectively. The section of Ω^λ is locally given by $s = g(x)dx^\lambda$, where $g(x) = g(x+2\pi)$. The action of a vector field $\hat{f} = f\frac{d}{dx}$ on s is given by the Lie derivative of degree λ

$$L_{\hat{f}} s = (fg' + \lambda f' g)dx^\lambda, \tag{6}$$

which describes action of a vector field $f(x)\frac{d}{dx}$ on the space of tensor densities F_λ.

By Lazutkin and Penkratova [21], the dual space of the Virasoro algebra can be identified with the space of Hill's operator or the space of projective connections

$$\Delta = \frac{d^2}{dx^2} + u(x), \tag{7}$$

where u is a periodic potential: $u(x+2\pi) = u(x) \in C^\infty(R)$. The Hill's operator maps

$$\Delta : F_{-\frac{1}{2}} \to F_{\frac{3}{2}}. \tag{8}$$

The action of $Vect(S^1)$ on the space of Hill's operator Δ is defined by the commutation with the Lie derivative

$$[L_{f(x)\frac{d}{dx}}, \Delta] := L^{\frac{3}{2}}_{f(x)\frac{d}{dx}} \circ \Delta - \Delta \circ L^{-\frac{1}{2}}_{f(x)\frac{d}{dx}}. \tag{9}$$

Thus, right hand side denotes the co-adjoint action of $Vect(S^1)$ on its dual Δ with respect to L^2 norm on the space of algebra.

Lemma 2.1: The Lie derivative action on Δ yields

$$[L_{f(x)\frac{d}{dx}}, \Delta]_{L^2} = (\frac{1}{2}\partial^3 + 2u\partial + u')f, \tag{10}$$

and this gives the (second) Hamiltonian structure of the KdV equation

$$O_{KdV} = (\frac{1}{2}\partial^3 + 2u\partial + u')$$

Proof: By direct computation.

It is clear from the definition of the Lie derivative on the space of Hill's operator that this coincides with the co-adjoint action of a vector $\eta = f(x)\frac{d}{dx}$ on its dual udx^2:

$$ad^*_\eta(udx^2) = [L_{f(x)\frac{d}{dx}}, \Delta]_{L^2},$$

and for this reason one obtains the same Hamiltonian operator from two different computations.

Lemma 2.2: The Hamiltonian vector field on $udx^2 \in g^*$ corresponding to a Hamiltonian function H, computed with respect to the Lie-Poisson structure is given by

$$\frac{du}{dt} = -ad^*_{dH}u = -O_{KdV}(u) \tag{11}$$

Proposition 2.3: The KdV equation is the Euler-Poincare equation for $H(u) = \frac{1}{2}\int_{S^1} u^2\,dx$ and it is given by

$$u_t + u_{xxx} + 6uu_x = 0. \tag{12}$$

Camassa-Holm in lie derivative method

Before we are going to embark the Camassa-Holm equation let us briefly recapitulate the usual co-adjoint method with respect to H^1 norm.

On the Virasoro algebra we consider the H^1 inner product, which is defined as $(f(x)\frac{d}{dx}, \nu) \in Vir$ and a point $(udx^{\otimes 2}, \lambda)$ or (udx^2, λ) on the dual space is given by

$$< (f(x)\frac{d}{dx}, \nu), (udx^{\otimes 2}, \lambda) >_{H^1} = \lambda\nu + \int_{S^1} f(x)u(x)dx + \int_{S^1} f_x u_x. \tag{13}$$

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Let us compute the co-adjoint action $ad^*_{\hat{f}}\hat{u}|_{H^1}$ of the vector field $(f(x)\frac{d}{dx},\nu)\in Vir$ its dual $(udx^{\otimes 2},\lambda)$ with respect to H^1 norm.

Lemma 2.4:

$$ad^*_{\hat{f}}\hat{u}|_{H^1}=(1-\partial^2)^{-1}[f(x)(1-\partial^2)u_x+2f'(1-\partial^2)u+\lambda f'''], \quad (14)$$

where $\hat{u}=(u\,dx^{\otimes 2},\lambda)$ and $\hat{f}=(f(x)\frac{d}{dx},\nu)$.

Proof: By direct computation.

Corollary 2.5: Using the Helmholtz function $m=u-u_{xx}$ Equation (14) can be rewritten as

$$ad^*_{\hat{f}}\hat{u}|_{H^1}=(1-\partial^2)^{-1}[fm_x+2f_xm+\lambda f_{xxx}], \quad (15)$$

and corresponding Hamiltonian operator is given by

$$O^{H^1}=(1-\partial^2)^{-1}(\lambda\partial^3+\partial m+m\partial). \quad (16)$$

Let us state co-adjoint action of *Vect(S¹)* on its dual in terms of Lie derivative language. It is clear that Lie derivative of *Vect(S¹)* on the space of Hill's operator should reflect the co-adjoint action with respect to H^1 norm, hence the Lie derivative equation must be expressed in terms m, i.e., Helmholtz operator acting on *u*.

Definition 2.6: *The Vect(S¹) action on the space of Hill's operator Δ with respect to H^1 –metric is defined as*

$$[L_{f(x)\frac{d}{dx}},\Delta]_{H^1}:=L^{\frac{3}{2}}_{f(x)\frac{d}{dx}}\circ\tilde{\Delta}-\tilde{\Delta}\circ L^{-\frac{1}{2}}_{f(x)\frac{d}{dx}}, \quad (17)$$

where

$$\tilde{\Delta}=\lambda\frac{d^2}{dx^2}+m(x) \qquad m=(u-u_{xx}).$$

Therefore Lie derivative $L_{f(x)\frac{d}{dx}}$ action yields the following scalar operator, i.e. the operator of multiplication by a function.

Proposition 2.7:

$$[L_{f(x)\frac{d}{dx}},\Delta]_{H^1}=(\frac{1}{2}\lambda\partial^3+2m\partial+m')f. \quad (18)$$

The L.H.S. of equation denotes the co-adjoint action evaluated with respect to H^1 norm. Thus we obtain the R.H.S. of Equation (18).

Lemma 2.8: The co-adjoint action of vector field $f(x)\frac{d}{dx}$ on its dual with respect to the right invariant H¹ metric can be realized as

$$ad^*_{\hat{f}}\hat{u}|_{H^1}=(1-\partial^2)^{-1}[L_{f(x)\frac{d}{dx}},\Delta]_{H^1}$$

This yields the Hamiltonian structure of the Camassa-Holm equation

$$O^{H^1}=(1-\partial^2)^{-1}(\frac{1}{2}\lambda\partial^3+2m\partial+m'). \quad (19)$$

At this stage we assume λ=0, since we do not require the cocycle term to compute the Camassa-Holm equation. It is clear that the term $\frac{1}{2}\lambda\partial^3$ manufactures from the cocycle term.

Therefore, the Euler-Poincare equation

$$u_t=-O^{H^1}_{CH}\frac{\delta H}{\delta u} \text{ where } O^{H^1}_{CH}=2m\partial+m'$$

$$=-(1-p^2)^{-1}(2m\partial+m')\text{——]} \text{ with } H=\frac{1}{2}\int_{S^1}u^2\,dx$$

$$\Rightarrow m_t+2mu'+m'u=0$$

yields the Camassa-Holm equation.

Euler-Poincare formalism of Whitham-Burgers equations

Let us consider a first order differential operator $\Delta_1=\frac{d}{dx}+u(x)$ acting on the space of tensor densities of degree $\frac{-1}{2}$, i.e., $F_{-\frac{1}{2}}\in\Gamma(\Omega^{-\frac{1}{2}})$.

This Δ_1 maps

$$\Delta_1=\frac{d}{dx}+u(x):F_{-\frac{1}{2}}\to F_{\frac{1}{2}}$$

Definition 2.9: The Vect(S¹)-action on Δ_1 is defined by the commutator with the Lie derivative

$$[L_{f(x)\frac{d}{dx}},\Delta_1]:=L^{\frac{1}{2}}_{f(x)\frac{d}{dx}}\circ\Delta_1-\Delta_1\circ L^{-\frac{1}{2}}_{f(x)\frac{d}{dx}}$$

The result of this action is a scalar operator, i.e. the operator of multiplication by a function, given by

$$[L_{f(x)\frac{d}{dx}},\Delta_1]=\frac{1}{2}f''(x)+uf'(x)+u'f(x). \quad (22)$$

This action yields the operator of the Burgers equation

$$O_B=\frac{1}{2}\frac{d^2}{dx^2}+u\frac{d}{dx}+u'(x). \quad (23)$$

Remark: The operator (23) is not a Poisson operator, since it does not satisfy the skew symmetric condition. When a vector field *Vect(S¹)* acts on the space of Hill's operator, it generates a Poisson flow, that is, operator involves in this flow is Poisson operator. But when *Vect(S¹)* acts on the space of first order differential operators, it does not generate a Poisson flow. Thus we obtain an almost Poisson operator.

Now we study the H^1 analogue of our previous construction Let us normalize the first order differential operator as $\Delta_1=2\frac{d}{dx}+u(x)$.

Lemma 2.10:

$$[L_{f(x)\frac{d}{dx}},2\frac{d}{dx}+u(x)]_{H^1}=f''(x)+mf'(x)+m'f(x), \qquad m=u-u_{xx} \quad (24)$$

Again, we can interpret this equation as an action of the vector field $L_{f(x)\frac{d}{dx}}$ on the space of modified first order scalar differential operator $2\frac{d}{dx}+m$. The factor "2" is just the normalization constant.

The L.H.S. denotes co-adjoint action with respect to H^1 norm. Once again we convert this to L^2 action, given as

$$[L_{f(x)\frac{d}{dx}},2\frac{d}{dx}+u(x)]_{L^2}=(1-\partial^2)^{-1}[L_{f(x)\frac{d}{dx}},2\frac{d}{dx}+u(x)]_{H^1}.$$

Therefore, the Hamiltonian operator of the Whitham-Burgers equation becomes

$$O^{H^1}_{WB}=-(1-\partial^2)^{-1}(\partial^2+\partial m). \quad (25)$$

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The Euler-Poincare flow on the space of first order operators with respect to H^1 norm yields the Whitham Burgers equation

$$m_t+u_{xx}+(mu)_x=0. \quad (26)$$

Euler-Poincare formalism and b-field equation and the Camassa-Holm-Whitham-Burgers equation

In this section we will state the Euler-Poincar´e construction for the the Degasperis-Procesi equation

$$m_t+um_x+3m_{ux}=0, \quad \text{with} \quad m=u-u_{xx}, \quad (27)$$

b-field equation and the Camassa-Holm-Whitham-Burgers equation. Latter one is the peakon analogue of the KdV-Burgers equation.

We need to combine the $Vect(S^1)$ action on both second and first order differential operators, Δ_2 and Δ_1 respectively, with respect to H^1 norm.

Definition 2.1: The Vect(S^1) action the pencil of operators $\Delta^{\lambda,\mu} := \lambda\Delta_2 + \mu\Delta_1$ is given by

$$[L_{f(x)\frac{d}{dx}}, \Delta^{\lambda,\mu}]_{H^1} = \lambda[L_{f(x)\frac{d}{dx}}, \Delta_2]_{H^1} + \mu[L_{f(x)\frac{d}{dx}}, \Delta_1]_{H^1}, \quad (28)$$

where

$$\Delta_2 = k_1\frac{d^2}{dx^2} + m(x), \quad \Delta_1 = 2k_2\frac{d}{dx} + m(x).$$

The pencil of Hamiltonian structures corresponding to $Vect(S^1)$ action on $\Delta^{\lambda,\mu}$ is given by

$$O^{H^1}_{\lambda,\mu} = -(1-\nu\partial^2)^{-1}\lambda(\frac{1}{2}k_1\partial^3 + \partial m + m\partial) + \mu(k_2\partial^2 + \partial m), \quad m = u - \nu u_{xx}. \quad (29)$$

If we assume $k_1=k_2=0$ and $\lambda=2$ and $\mu=-1$, we obtain the operator of Degasperis-Procesi equation

$$O_{DP} = -(1-\nu\partial^2)^{-1}(\partial m + 2m\partial). \quad (30)$$

The operator of the b-field equation can be obtained from $k_1=k_2=0$ and $\lambda=b$-1 and $\mu=-(b-2)$, given by

$$O_B = -(1-\nu\partial^2)^{-1}(m_x + bm\partial). \quad (31)$$

It is clear from this expression and the construction that for $b=3$ case we obtain the Degasperis-Procesi operator and for $b=2$ we recover the famous Camassa-Holm equation.

The Euler-Poincare equation

$$m_t = -O_B\frac{\delta H}{\delta u}, \quad O_B = (m_x + bm\partial), \quad H = \frac{1}{2}\int_{S^1} u^2\,dx \quad (32)$$

yields the b -field equation

$$m_t+m_xu+bmu_x=0=0, \quad m=u-\nu u_{xx}.$$

This is the derivation of the b-field equation using tensor algebra, and it was given in [11,12]. Consider the dual space of F_b with a frozen m structure. In other words, we fix some point $m_0 \in F_b$ and define the generalized Hamiltonian structure. This immediately yields the first or frozen Hamiltonian structure

$$O_{Frozen}=bm_0\partial. \quad (33)$$

This can be easily normalized as $O_{Frozen}=\partial$.

The Camassa-Holm-Whitham-Burgers equation: The KdV-Burgers equation is given by

$$u_t+\kappa uu_x+\beta u_{xxx}+\eta u_{xx}=0, \quad (34)$$

where κ,β *and* η are constants. It appears naturally in unmagnetized dusty plasma and yields shock waves.

Proposition 2.12: The Euler-Poincare flow $u_t = -O^{H^1}_{\lambda,\mu}\frac{\delta H}{\delta u}$ on space of Hill's and first-order differential operators with respect to H^1-metric yields the Camassa-Holm-Whitham- Burgers equation

$$m_t + (\lambda+\mu)m_xu + (2\lambda+\mu)mu_x + \kappa u_{xxx} + \beta u_{xx} = 0, \qquad m = u - \nu u_{xx},$$

where $\kappa = k_1\lambda/2$ and $\beta = k_2\mu$. This reduces to the KdV-Burgers equation for $\nu=0$.

Computation of Hamiltonian structure via deformed bracket

Let us introduce a new algebraic structure, called b-algebra. The commutator (or Lie bracket) is defined in a following way:

Definition 2.13: The b-bracket between $v(x)\frac{d}{dx}$ and $w(x)\frac{d}{dx}$ is defined as

$$[v, w]_b=vw_x-(b-1)v_xw \quad (35)$$

This b-bracket can also be expressed as

$$[v,w]_b = \frac{b}{2}[v,w] - \frac{b-2}{2}[v,w]^{sym}, \quad (36)$$

where $[v, w]=vw_x-v_xw$ *and* $[v, w]^{sym}=vw_x+v_xw$.

Remark: The b -bracket can be interpreted as an action of $Vect(S^1)$ on $F_{-(b-1)}(S^1)$, a tensor densities on S^1 of degree -(b-1). For b=2 this is just a vector field action corresponding to a Lie algebra. Moreover because of $[v, w]^{sym}$ term b -bracket is not a skew-symmetric bracket, it is a deformation of the bracket of vector fields.

There is a pairing

$$\langle,\rangle : F_\mu \otimes F_{1-\mu} \to \mathbb{R}$$

given by

$$\langle a(x)(dx)^\mu, b(x)(dx)^{1-\mu}\rangle = \int_{S^1} a(x)b(x)dx \quad (37)$$

which is $Diff(S^1)$ -invariant.

$$\langle a(x)(dx)^\mu, b(x)(dx)^{1-\mu}\rangle = \int_{S^1} a(x)b(x)dx.$$

We denote b-algebra by $F_{-(b-1)}$ and its dual by F_b. Thus we can define a pairing according to (9)

$$\langle a(x)(dx)^{-(b-1)}, b(x)(dx)^b\rangle = \int_{S^1} a(x)b(x)dx.$$

Let us compute the co-adjoint action with respect to the b-field equation.

Lemma 2.14:

$$(ad^{H^1})^*_{f(dx)^{-(b-1)}}(udx^b) = (1-\nu\partial^2)^{-1}[f(1-\nu\partial^2)u_x + bf_x(1-k\partial^2)u. \quad (38)$$

Proof: We suppress all the density terms, thus from the definition we obtain

$$< ad^*_f(u), g >_{H^1} = -< u, [f,g]_b >_{H^1}$$

$$\equiv -< udx^b, (fg' - (b-1)f'g)(dx)^{1-b} >_{H^1},$$

hence the pairing is well-defined. Let us compute

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$$\text{R.H.S.} = \int_{S^1} (ufg' - (b-1)uf'g)dx + v\int_{S^1} u'(fg' - (b-1)f'g)'dx$$

$$= \int_{S^1} [f(1-v\partial^2)u' + bf'(1-v\partial^2)u$$

$$L.H.S. = \int_{S^1} (ad^{H^1})^*_f u)gdx + v\int_{S^1} (ad^{H^1})^*_f u'g'dx$$

$$= \int_{S^1} [f(1-v\partial^2)ad^{H^1})^*_f u]gdx\,.$$

Thus by equating the R.H.S. and L.H.S. we obtain the above formula.

Proposition 2.15: The Euler-Poincare flow generated by the action of generalized vector field $\gamma \in F_{-(b-1)}$ on the dual space of tensor densities $u \in F_b$ yields the b-field equation

$m_{t+}m_x u + bmu_x = 0,$

where O_B is the second generalized Hamiltonian structure.

Hamiltonian Structures of Integrable Peakon Equations with Cubic Nonlinearity

The subject of this section is to study the Hamiltonian structure of the partial differential equation

$$m_t + \left(m\left(u^2 - u_x^2\right)\right)_x = 0\,, \quad (39)$$

which was recently obtained by Qiao [18,20]. One must note that this equation was first appeared in the paper of Thanasis Fokas [15]. Qiao described more details of its properties in the dispersion less case. So we propose to call this equation as the Fokas-Qiao equation. This equation has a cubic (rather than quadratic) nonlinear terms and was found to admit tri-Hamiltonian structure by Olver and Rosenau [17] and Qiao gave its Lax pair and cusp soliton solutions. This inspired Novikov [22] to seek other integrable equations of this kind, given by

$$m_{t+}u^2 m_x + 3uu_x m = 0. \quad (40)$$

Hone and Wang [14] gave a matrix Lax pair for Novikov's equation, and showed how it is related by a reciprocal transformation to a negative flow in the Sawada-Kotera hierarchy.

Hamiltonian structures of the Fokas-Qiao equation

Qiao studied Hamiltonian structures for b=1 case of DHH equation with cubic nonlinearity. Recently Qiao and his coauthors showed that this cubic nonlinear equation possesses the bi-Hamiltonian structure, namely, it can be written as

$$m_t = J\frac{\delta H_0}{\delta m} = K\frac{\delta H_1}{\delta m}\,, \quad (41)$$

where

$$J = \partial m\partial^{\ 1} m\partial, \qquad K = \partial^3\partial,\,, \quad (42)$$

with

$$H_0 = 2\int_{S^1} umdx, \qquad H_1 = \frac{1}{4}\int_{S^1} (u^4 + 2u^2u_x^2 - \frac{1}{3}u_x^4 0)dx\,. \quad (43)$$

The first Hamiltonian structure can be simplified to $J = -(m_x\partial^{-1}m\partial + m^2\partial)\,.$

Proposition 3.1: The Fokas-Qiao equation in the Hamiltonian form

1. $m_t = J\dfrac{\delta H_0}{\delta m}$, *with* $H_0 = 2\int umdx$ *and* $J = -\partial m\partial^{-1}m\partial$ is equivalent to

$$m_t = O\frac{\delta \mathbb{H}}{\delta u} \text{ for } H = \int_{S^1} (u^2 - u_x^2)dx\,, \quad (44)$$

where $O = (m_x + m\partial) = \partial m\,.$

2. $m_t = K\dfrac{\delta H_1}{\delta m}$ with $H_1 = \dfrac{1}{4}\int_{S^1} (u^4 + 2u^2u_x^2 - \dfrac{1}{3}u_x^4 0)dx$ and $K = \partial^3\partial$ is equivalent to

$$m_t = -\partial\frac{\delta \mathbb{H}}{\delta u}\,. \quad (45)$$

Proof: By simple computation we obtain

$$m_t = -\left(m_x\partial^{-1}m\partial + m^2\partial\right)\frac{\delta H_0}{\delta m} = -2(m_x\partial^{-1}mu_x + m^2u_x)\,.$$

Then using $mu_x = \frac{1}{2}(u^2 - u_x^2)_x$ we obtain our first result.

The second part can be proved in a similar way. The variational derivatives with respect to m and u are connected by

$$\frac{\delta H}{\delta u} = (1-\partial^2)\frac{\delta H}{\delta m}\,.$$

Thus by straight forward calculation we obtain

$$m_t = -\partial\frac{\delta H}{\delta u} = -(1-\partial^2)\frac{\delta H}{\delta m} = K\frac{\delta H}{\delta m}\,.$$

Hamiltonian structure of V. Novikov equation

Novikov [22] studied Hamiltonian structures for b=3/2 case of DHH equation with cubic nonlinearity. The V. Novikov equation can be expressed as

$$m_t = B_1\frac{\delta H_i}{\delta m}\,, \quad (46)$$

where the second Hamiltonian structure is given by

$$B_1 = -2(3mD_x + 2m_x)(4D_x - D_x^3)^{-1}(3mD_x + m_x)\,. \quad (47)$$

Proposition 3.2: The V. Novikov equation is equivalent to

$$m_t = O\frac{\delta H}{\delta u} \text{ for } H = \frac{1}{12}\int_{S^1} u^3 dx$$

where $O = -2(m_x + 3m\partial)$ for b=3/2 case.

Proof: Our goal is to show

$$(4D_x - D_x^3)^{-1}(3mD_x + m_x)\frac{\delta H_1}{\delta m} = \frac{\delta H}{\delta u}\,, \text{ where } H_1 = \frac{1}{2}\int_{S^1} mudx\,.$$

We insert $\dfrac{\delta H_1}{\delta m} = \dfrac{1}{2}u$ to left hand side of above equation. By simple computation one can check that

$$(3mu_x + m_x u) = 4uu_x - 3u_x u_{xx} - u_{xxx}u = \frac{1}{2}(4\partial - \partial^3)(u^2)\,,$$

thus we obtain

$$\frac{1}{2}(4\partial - \partial^3)^{-1}(3mu_x + m_x u) = \frac{1}{4}u^2$$

Finally we obtain the V. Novikov equation via

$$m_t = -2(3m\partial + m_x)\frac{\delta H}{\delta u}.$$

Therefore the Hamiltonian structure found by Hone and Wang [14] coincides with our Hamiltonian structure. The first or frozen Hamiltonian structure can be easily found as

$$m_t = -\partial\frac{\delta H_0}{\delta u}, \text{ where } \frac{\delta H_0}{\delta u} = 2u^2 - u_x^2 - uu_{xx}. \tag{48}$$

This equation can also be expressed as $m_t = -(4-\partial^2)(uu_x)$.

Proposed new peakon equation with cubic nonlinearity: Let us propose a new peakon equation with cubic nonlinearity which is a generalization of both the Fokas-Qiao equation and Novikov equation. It is given by

$$m_t = -(m_x + bm\partial)\frac{\delta H}{\delta u}, \quad \text{where } \frac{\delta H}{\delta u} = (\mu u^2 - \nu u_x^2), \tag{49}$$

or

$$m_t + m_x(\mu u^2 - \nu u_x^2) + bm(\mu u^2 - \nu u_x^2)_x = 0, \text{ where } m=u-u_{xx}. \tag{50}$$

This equation reduces to the Fokas-Qiao equation for $b=\mu=\nu=1$ and *V*. Novikov equation [22] for $b=3/2$, $\mu=2$ and $\nu=0$.

2+1 cubic peakon equations

It is known that the Virasoro algebra can be extended to two space variables [23]. A natural way to do this is to consider the loops on it. One defines the loop group on $Diff(S^1)$ as follows

$$L\left(Diff(S^1)\right) = \left\{\phi : S^1 \to Diff(S^1) | \quad \phi \quad \text{is differentiable}\right\},$$

the group law being given by

$$(\phi \circ \psi)(y) = \phi(y) \circ \psi(y), \qquad y \in S^1.$$

We also know that the corresponding Lie algebra $L(Vect(S^1))$ consisting of vector fields on S^1 depending on one more independent variable $y \in S^1$. The loop variable is thus denoted by y and the variable on the "target" copy of S^1 by x. The elements of $L(Vect(S^1))$ are of the form: $f(x,y)\frac{\partial}{\partial x}$ where $f \in C^\infty(S^1 \times S^1)$ and the Lie bracket reads as follows [?]

$$\left[f(x,y)\frac{\partial}{\partial x}, g(x,y)\frac{\partial}{\partial x}\right] = \left(f(x,y)g_x(x,y) - f_x(x,y)g(x,y)\right)\frac{\partial}{\partial x}.$$

We extend this scheme to the space of tensor densities [12]. Consider $\tilde{G}_1 = LG_1$ be the associated loop group corresponding to G_1 whose Lie algebra is given by

$$\widetilde{g}_1 = L(F_{-(b-1)}).$$

Consider an action of $L(Vect(S^1))$ on $L(F_{-(b-1)})$

$$L_{f\frac{\partial}{\partial x}}(g(dx)^{-(b-1)}) == (fg_x - (b-1)f_x g)(dx)^{-(b-1)}, \tag{51}$$

this yields a new bracket.

Let us introduce H^1 norm on the algebra $\widetilde{g}_1$.

Definition 3.3: The H^1-Sobolev norm on the loop tensor density algebra is defined as

$$< f(x,y)(dx)^{-(b-1)}, u(x,y)(dx)^b >_{H^1} = \int_{S^1} fudx + \nu\int_{S^1} d_x f d_x u dx, \tag{52}$$

Proposition 3.4: The co-adjoint action with respect to H^1 metric of the Lie algebra $\hat{g}$ is given by

$$\widehat{ad}^*_{\left(f\frac{d}{dx}\right)}(u(x,y)(dx)^b) = +\left(fm_x + bf_x m\right)dx^b,$$

where $m = (1-\nu\partial^2)u$.

Corollary 3.5: The Hamiltonian operator corresponding to the co-adjoint action of the cotangent loop Virasoro algebra with respect to H^1 metric is given by

$$\hat{O} = -(1-\nu\partial_x^2)^{-1}\left(\partial_x m + (b-1)m\partial_x\right) \tag{53}$$

Proposition 3.6: The Euler-Poincare flow on the $\hat{g}_1$ orbit yields the 2+1 – dimensional b-field equation

$$m_t + m_x\partial_x^{-1}(\mu u^2 - \nu u_x^2)_y + bm(\mu u^2 - \nu u_x^2)_y = 0 \tag{54}$$

where the Hamiltonian is given by $\frac{\delta H}{\delta u} = \partial_x^{-1}(\mu u^2 - \nu u_x^2)_y$, which reduces to 1+1-dimensional b–field equation for y=x.

It is clear that the equation becomes the Fokas-Qiao equation for y=x, b=μ=ν=1 and V. Novikov equation for y=x, b=3/2, μ=2 and ν=0.

Conclusion

We have studied the Euler-Poincare formalism of the cubic peakon equations using two different but equivalent methods. The first Euler-Poincare framework is based on the flows defined on the spaces of Hill's and first order differential operators and the second one is studied using the algebra of tensor densities. We have explicitly derived the Hamiltonian structures of the cubic peakon equations as given by Qiao, Hone and Wang using the Euler- Poincare formalism. We also derived 2+1-dimensional cubic peakon equations using the action of the loop extension of $Vect(S^1)$ on the space of tensor densities. It would be interesting to derive other novel features of these equations using EP theory.

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