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Estimations of s-Numbers of an Infinite Matrix Operator on H_β^1 Spaces

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Abstract

In this paper we study conditions that an infinite matrix $T=\left(\tau_{ij}\right)_{i=1\ j=1}^{\infty\ \ \infty}$ has to satisfy in order to define a linear bounded operator from any weighted H_β^1 with a weight sequence $\beta=\{\beta_i\}$ of positive numbers into itself, and give upper and lower estimations of the approximation numbers of such an infinite matrix.

Keywords: Approximation numbers; Infinite matrix; Weighted sequence

Introduction

In [1-8] an exact estimation of approximation numbers of a multiplication operator $T=\left(\tau_i\right)_{i=1}^{\infty}$ on the space ℓ^p $(1 \le p < \infty)$ was given by the formula $\alpha_r(T)=\sup_{card\,F=r+1}\ \inf_{i\in F\subset N} |\tau_i|$, where F is any subset with (r+1) elements of the set of natural numbers N, and the same result was found true for any multiplication operator $T=(_i)_{i\in I}$ defined on the space $\ell_¥(I)$ of bounded families of real numbers.

In [3] upper and lower estimations of the approximation numbers of an infinite matrix $T=\left(\tau_{nk}\right)_{n=1,k=1}^{\infty\ \ \infty}$ on the space ℓ^1 were given by

$$\sup_{\lambda_{nk}}\ \sup_{card\,F=r+1}\ \inf_{n}\frac{1}{\sum_{k\in F}|\lambda_{nk}|}\le\alpha_r(T)\le\sup_{card\,F=r+1}\ \inf_{k\in F}\sum_{n=1}^{\infty}|\tau_{nk}|,$$

where, $\{\lambda=(\lambda_{nk}),\ k\in F\}$ is a biorthogonal family of functionals to the family $\{\tau_k=(\lambda_{kj}),\ k\in F\}\subseteq\ell^1$. In [5] the same result was found true for an infinite matrix $T=\left(\tau_{nk}\right)_{n=1\ k=1}^{\infty\ \ \infty}$ on the space $\ell_¥$.

The main objective of this paper is finding upper and lower estimations of the approximation numbers of an infinite matrix operator $T=\left(\tau_{nk}\right)_{n=1\ k=1}^{\infty\ \ \infty}$ from the weighted H_β^1 space into itself [1-6].

Notations and Basic Definitions

1 - By card (F) we denote the number of elements belonging to any subset F.

2 - For any two Banach spaces X and Y we denote by L(X, Y) the Banach space of all continuous linear operators from X into Y equipped with the usual operator norm.

3 - For a weight sequence $\beta=\{\beta_i\}$ of positive numbers, we denote by H_β^1, H_β^∞ the weighted spaces,

$$H_\beta^1=\left\{f(z):f(z)=\sum_{i=1}^{\infty}a_i\frac{z^i}{\beta_i},\ \sum_{i=1}^{\infty}|a_i|<\infty\right\}$$

$$H_\beta^\infty=\left\{f(z)\ :f(z)=\sum_{i=1}^{\infty}a_i\frac{z^i}{\beta_i},\sup_i|a_i|<\infty\right\}$$

Equipped with the following norms:

$\|f\|=\left(\sum_{i=1}^{\infty}|a_i|\right)$ and $\|f\|=\sup_i|a_i|$ respectively.

Preliminary Lemmas and Propositions

In this section we mention some terminologies and state several auxiliary lemmas and propositions which we use.

Proposition

A map which assigns to every operator T∈L(X,Y) a unique sequence $\{s_n(T)\}$, $n=0,1,2,\ldots\ldots..$ is called an s-function if the following conditions are satisfied: (8,9)

1 - $s_{n+m}(U+V)\le s_n(U)+s_m(V)$ for U,V∈L(X,Y).

2 - $|s_r(U)-s_r(V)|\le\|U-V\|$ for U,V∈L(X,Y).

3 - $s_r(\lambda U)=|\lambda|\,s_r(U)$, for all real numbers λ and U∈L(X,Y).

4 - $s_n(U)\ge s_n(U^*)$.If U is compact operator then $s_n(U)\ge s_n(U^*)$.

5 - $\|T\|=s_0(T)\ge s_1(T)\ge s_2(T)\ge\ldots..\ge 0$, for all T∈L(X,Y).

6 - $s_n(UTV)\le\|U\|\,s_n(T)\,|V\|$ for V∈L (X_0,X), $T\in L(X,Y)$, $U\in L(Y,Y_0)$.

7 - $s_n(T)=0$ if rank (T)≤n for $T\in L(X,Y)$.

8 - $s_r(I_n)=\begin{cases}1 & \text{for } r\prec n\\ 0 & \text{for } r\ge n\end{cases}$.

Where I_n is the identity operator on the Euclidean space l_2^n. We call $s_n(T)$ the n[th] s-number of the operator T.

As examples of s-numbers, we mention approximation numbers $\alpha_r(T)$, Gelfand numbers $c_r(T)$, Kolmogorov numbers $d_r(T)$ and Tichomirov numbers $d_n^*(T)$ defined by:

i) Approximation numbers of operators,

For any arbitrary normed spaces X and Y, $A_r(X,Y)$ for r=0,1,2,…. denotes the collection of all finite mappings A∈L(X,Y) whose range is at most r-dimensional. For any arbitrary operator T∈L(X,Y) the r-th approximation numbers of T is defined by

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$$\alpha_r(T) = \inf\{\ \|T - A\| : A \in A_r(X,Y)\ \}\ \ (r = 0,1,2,3,...).$$

ii) $c_r(T)=\alpha_r(J_Y T)$, where J_y is a metric injection (a metric injection is a one to one operator with closed range and with norm equal one)from the space Y into a higher space $l^\infty(I)$ for suitable index set I [7-9].

iii) $d_n(T) = \inf_{\dim Y \le n} \sup_{\|x\|\le 1} \inf_{y\in Y} \|Tx - y\|$. for T∈L(X,Y).

iv) $d_r^*(T) = d_r(J_Y T)$ for T∈L(X,Y).

Lemma 1

Let $T = (\tau_{nk})_{n=1}^{\infty}{}_{k=1}^{\infty}$ be an infinite matrix. If $\sup_n \sum_{k=1}^{\infty}|\tau_{nk}| \prec \infty$ then the expression $T(f(z)) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\tau_{nk}a_k)\frac{z^n}{\beta_n},\ \forall f(z)\in H_\beta^\infty$ defines a linear continuous operator T: $H_\beta^\infty \to H_\beta^\infty$ with $\|T\| = \sup_n \sum_{k=1}^{\infty}|\tau_{nk}|$.

Proof

For any element f(z)∈ H_β^∞ we get,

$$\|T(f(z))\| = \left\|\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\tau_{nk}a_k)\frac{z^n}{\beta_n}\right\|_{H_\beta^\infty} = \sup_n\left|\sum_{k=1}^{\infty}\tau_{nk}a_k\right|$$

$$\le \sup_k|a_k|.\sup_n\left|\sum_{k=1}^{\infty}\tau_{nk}\right|$$

Therefore, $\|T\| \le \sup_n \sum_{k=1}^{\infty}|\tau_{nk}|$.

On the other hand, noting that $f_n=(sign\ \tau_{n1}, sign\ \tau_{n2},\ldots)$ is an element belonging to H_β^∞ we get,

$$\|T\| \ge \frac{\|T(f_n)\|_{H_\beta^\infty}}{\|f_n\|_{H_\beta^\infty}}.\ Hence\ \|T\| \ge \sup_n \sum_{k=1}^{\infty}|\tau_{nk}|.$$

Lemma 2

Let $T = (\tau_{nk})_{n=1,k=1}^{\infty\ \ \infty}$ be aninfinite matrix. If $\sup_k \sum_{n=1}^{\infty}|\tau_{nk}| \prec \infty$ then the expression $T(f(z)) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\tau_{nk}a_k)\frac{z^n}{\beta_n},\ \forall f(z)\in H_\beta^1$ defines a linear continuous operator T: $H_\beta^1 \to H_\beta^1$ with $\|T\| = \sup_k \sum_{n=1}^{\infty}|\tau_{nk}|$.

Proof

$$\|T(f(z))\| = \left\|\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\tau_{nk}a_k)\frac{z^n}{\beta_n}\right\| \le \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}|\tau_{nk}a_k|$$

$$= \sum_{k=1}^{\infty}|a_k|.\sum_{n=1}^{\infty}|\tau_{nk}| \le \sum_{k=1}^{\infty}|a_k|.\sup_k \sum_{n=1}^{\infty}|\tau_{nk}|$$

Therefore, $\|T\| \le \sup_k \sum_{n=1}^{\infty}|\tau_{nk}|$.

On the other hand writing $\beta^{-1} = \left(\beta_n^{-1}\right)$ and noting that the point

f_n=(sign τ_{n1}, sign τ_{n2}, ...)∈ $H^\infty_{\beta^{-1}}$ with $\|f_n\|_{H^\infty_{\beta^{-1}}} = 1$ we get,

$$\left\|T : H_\beta^1 \to H_\beta^1\right\| = \left\|T^* : H^\infty_{\beta^{-1}} \to H^\infty_{\beta^{-1}}\right\| \ge \left\|T^* f_n\right\|_{H^\infty_{\beta^{-1}}} \ge \sup_k \sum_{n=1}^{\infty}|\tau_{nk}|$$

Lemma 3

For $f = (a_i) \in H_\beta^\infty$ we get

$$\sup_{card\ F=n+1}\ \inf_{i\in F}|a_i| = \inf_{card\ F=n}\ \sup_{i\notin F}|a_i|.$$

Main Theorems

Theorem 1

Let $T = (\tau_{nk})_{n=1,k=1}^{\infty\ \ \infty}$ be an infinite matrix with linearly independent rows and columns satisfying the conditions of Lemmas (1), (2), (3) and considering T as an operator such that:

$T : H_\beta^1 \longrightarrow H_\beta^1$ then we get the following inequality

$$\sup_{cardF=r+1}\ \inf_n\left(\frac{1}{\sum_{k=1}^{\infty}|\gamma_{nk}|}\right) \le \alpha_r(T) \le \sup_{cardF=r+1}\ \inf_{k\in F}\sum_{n=1}^{\infty}|\tau_{nk}|.$$

Where F is any subset of natural numbers with (r+1) elements and $(\gamma_{ij})_{i=1}^{\infty}$ is the biortogonal function to $(\tau_{nk})_{k=1}^{\infty}$.

Proof

For any set E of indices with card E=r we define an operator $A_E : H_\beta^1 \longrightarrow H_\beta^1$ with rank (A_E) ≤ r by :

$$A_E = (a_{nk}),\ a_{nk} = \begin{cases} 0 & , k \notin E \\ \tau_{nk} & , k \in E \end{cases}$$

From the definition of approximation numbers and using Lemmas (1),(2),(3) we get

$$\alpha_r(T) \le \inf_{card\ E=r}\|T - A_E\| \le \|T - A_E\| = \sup_{k\notin E}\sum_{n=1}^{\infty}|\tau_{nk}|$$

$$\le \sup_{cardE=r+1}\ \inf_{k\in E}\sum_{n=1}^{\infty}|\tau_{nk}|$$

To prove the other side, let F={n_1, n_2,..., n_{r+1}} ⊂ N be a set of indices with (r+1)-elements and define a projection $P_F \in L\left(H^\infty_{1/\beta}, H^\infty_{1/\beta}\right)$ as follows :

$$P_F(f_i(z)) = y_i,\ y_i = \begin{cases} f_i(z), & i \in F \\ 0, & i \notin F \end{cases}$$

Clearly the range of P_F is the subspace

Y_F={ ($f_i(z)$) : $f_i(z)$=0 ∀ i ∉ F }.

The set $\{\tau_n = (\tau_{nk})_{k=1}^{\infty} \in H_\beta^1,\ n \in F\}$ is a subset of linearly independent elements in H_β^1 then using Hahn-Banach Theorem there exist linear continuous functionals [10] $g_k = (\gamma_{nk})_{n=1}^{\infty} \in H^\infty_{1/\beta}$, k∈F satisfying, $g_k(\tau_n) = \sum_{i=1}^{\infty}\tau_{ni}\gamma_{ik} = \delta_{nk}$.

Let us construct the operator

$$S_F = (\delta_{ij})_{i=1,\ j=1}^{\infty\ \ \infty} \in L\left(Y_F, H^\infty_{1/\beta}\right) \text{ such that } \delta_{ij} = \begin{cases} 0 & , j \notin F \\ \gamma_{ij} & , j \in F \end{cases}$$

Therefore we get the following diagram

$$Y_F \xrightarrow{S_F} H^\infty_{1/\beta} \xrightarrow{T^*} H^\infty_{1/\beta} \xrightarrow{P_F} Y_F$$

Since $P_F T^* S_F = I_{Y_F}$ is the identity on the (r+1)-dimensional subspace Y_F then 1=$\alpha_r(I_{Y_F})$=$\alpha_r(P_F T^* S_F)$≤$\alpha_r(T^*)$||S_F||.

Hence, $\alpha_r(T) \ge \alpha_r(T^*) \ge \frac{1}{\|S_F\|} \ge \left(\sup_n \sum_{k=1}^{\infty}|\gamma_{nk}|\right)^{-1} = \sup_{cardF=r+1}\ \inf_n\left(\frac{1}{\sum_{k=1}^{\infty}|\gamma_{nk}|}\right)$

Theorem 2

Let $T = (\tau_{nk})_{n=1,k=1}^{\infty\ \ \infty}$ be an infinite matrix with linearly independent rows and columns satisfying the conditions of Lemmas (1),(2),(3) and considering T as an operator such that

$T : H_\beta^1 \longrightarrow H_\beta^1$ whose dual $T^* : H_{\beta^{-1}}^\infty \rightarrow H_{\beta^{-1}}^\infty$ we get

$$\sup_{cardF=r+1} \inf_k \left(\frac{1}{\sum_{n=1}^{\infty} |\gamma_{nk}|} \right) \leq \alpha_r(T^*) \leq \sup_{cardF=r+1} \inf_{n\in F} \sum_{k=1}^{\infty} |\tau_{nk}|$$

Where F is any subset of natural numbers with (r+1) elements and $(\gamma_{ij})_{i=1}^{\infty}$ *is* the biortogonal functional to $(\tau_{nk})_{k=1}^{\infty}$.

Proof:

The proof of the theorem is similar to theorem 1.

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