Journal of
Applied & Computational Mathematics
ISSN: 2168-9679

Hichar et al., J Appl Computat Math 2014, 3:7
DOI: 10.4172/2168-9679.1000186

Research Article | Open Access

Estimates for Solutions of Semilinear Elliptic Equation in Two Dimensions

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Abstract

In this paper we study a family of nonlinear Elliptic problems in two dimensions, we give some estimates for the solutions of this problem, and we decompose it on two problems, the first is the Poisson's equation and the second is the Liouville equation.

Keywords: Liouville equation; Nonlinear problem; Elliptic equation

Introduction

In this paper we study the problem

$$\begin{cases} -\Delta u = \lambda e^{u} + f(\mathrm{x}) & \text{in}\, \Omega \subset \mathbb{R}^2 \\ u = g & on\, \partial\Omega \end{cases} \tag{1}$$

Where Ω is a bounded domain, λ is a positive parameter, $g \in L^{\infty}(\partial\Omega)$ and $f \in L^{q}(\Omega)$ for some q>1.

Equation of Liouville-type is used in this study, it has the form:

$$\begin{cases} -\Delta u = v(\mathrm{x})\,\mathrm{e}^{\mathrm{u}} + f(\mathrm{x}) & \text{in}\, \Omega \subset \mathbb{R}^2 \\ u = 0 & on\ \partial\Omega \end{cases}$$

$v(\mathrm{x}) \in \mathrm{L}^{\mathrm{q}}(\Omega)$ For some q>1.

This related equation has been received much attention in the recent years. On the one hand, this is due to the wide range of application of this equation: it used in astrophysics [1] and combution theory [2], it is also related to the prescribed Gaussian curvature problem in Riemannian geometry [3], to the mean _led limit of vortices in Euler ows [4], to onsager's formulation in statistical mechanics [5], to the Keller-Siegel system of chemotaxis [6], to the Chern-Simon-Higgs gauge theory [7, 8], and it has many other physical applications. On the other hand, Liouville equation is mathematically appealing since it has an interesting solution structure [9-16].

Preliminaries

Assume $\Omega \subset \mathbb{R}^2$ is a bounded domain and let u be a solution of

$$\begin{cases} -\Delta u = f(\mathrm{x}) & \text{in}\, \Omega \subset \mathbb{R}^2 \\ \quad u = g & on\, \partial\Omega \end{cases} \tag{2}$$

With $f \in L^{1}(\Omega).\mathrm{set} \| \mathrm{f} \| 1 = \int\limits_{\Omega} \| f(\mathrm{x}) | \,\mathrm{dx}$

Theorem 2.1 For every $\delta \in (0, 4\pi)$ we have

$$\int\limits_{\Omega} \exp[\frac{(4\pi-\delta)\,|\,\mathrm{u}(\mathrm{x})\,|}{\| f \|_1}] dx \le \frac{4\pi^2}{\delta}(\operatorname{diam}\Omega)^2 \tag{3}$$

Proof: Let $R = \frac{1}{2} diam\,\Omega$ so that $\Omega \subset B_R$ for some ball of radius R.

Extend f to be zero outside Ω and set, for $x \in \mathbb{R}^2$

$$\bar{u}(\mathrm{x})) = \frac{1}{2\pi}\int_{B_R} \log(\frac{2R}{|\,\mathrm{x}-\mathrm{y}\,|})\,|\,\mathrm{f}(\mathrm{y})\,|\,\mathrm{dy}$$

so that

$$-\Delta\bar{u}(\mathrm{x}) = |\,\mathrm{f}\,|, \text{on}\, \mathbb{R}^2$$

note that $\bar{u}(\mathrm{x}) \ge 0$ for $\mathrm{x} \in B_R$ since $\frac{2R}{|\,\mathrm{x}-\mathrm{y}\,|} \ge 1 \forall x, y \in B_R$ it follows from the maximum principle that $|\,\mathrm{u}\,| \le \bar{\mathrm{u}}\, on\, \Omega$ and thus

$$\int_{\Omega} \exp[\frac{(4\pi-\delta)\,|\,\mathrm{u}(\mathrm{x})\,|}{\| \mathrm{f} \|_1}] dx \le \int_{B_R} \exp[\frac{(4\pi-\delta)\bar{\mathrm{u}}\,(\mathrm{x})}{\| \mathrm{f} \|_1}]\mathrm{dx}$$

using Jensen in quality

$$F(\int w(\mathrm{y})\varphi(\mathrm{y})\,\mathrm{dy}) \le \int w(\mathrm{y}) F(\varphi(\mathrm{y}))\,\mathrm{dy}$$

With $F(\mathrm{t}) = \exp \mathrm{t}, \mathrm{w}(\mathrm{y}) = \frac{|\,\mathrm{f}(\mathrm{y})\,|}{\| \mathrm{f} \|_1} and\ \varphi(\mathrm{y}) = \frac{(4\pi-\delta)}{2\pi}\log\frac{2R}{|\,\mathrm{x}-\mathrm{y}\,|}$

We obtain

$$\int_{B_R} \exp[\frac{(4\pi-\delta)\bar{\mathrm{u}}\,(\mathrm{x})}{\| f \|_1}] dx \le \int_{B_R} dx \int_{B_R} (\frac{2R}{|\,\mathrm{x}-\mathrm{y}\,|})^{2-\frac{\delta}{2\pi}} \frac{|\,\mathrm{f}(\mathrm{y})\,|}{\| \mathrm{f} \|_1} dy$$

$$\int_{B_R} \exp[\frac{(4\pi-\delta)\bar{\mathrm{u}}\,(\mathrm{x})}{\| f \|_1}] dx \le \int_{B_R} \frac{|\,\mathrm{f}(\mathrm{y})\,|}{\| \mathrm{f} \|_1}[\int_{B_R} (\frac{2R}{|\,\mathrm{x}-\mathrm{y}\,|})^{2-\frac{\delta}{2\pi}} dx] dy$$

but, for $y \in B_R$ we have

$$\int_{B_R} (\frac{2R}{|\,\mathrm{x}-\mathrm{y}\,|})^{2-\frac{\delta}{2\pi}} dx \le \int_{B_R} (\frac{2R}{|\,\mathrm{x}\,|})^{2-\frac{\delta}{2\pi}} dx = \frac{4\pi^2}{\delta}(\operatorname{diam}\Omega)^2$$

and the estimate (3) follows.

Corollary 2.2 Let u be a solution of (2) with $f \in L^{1}(\Omega)$ Then for every constant k>0

$$e^{k|\mathrm{u}|} \in L^{1}(\Omega)$$

Proof: Let $0 < \varepsilon < \frac{1}{k}$ We may split f as f=f1+f2 with

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Received August 06, 2014; **Accepted** September 25, 2014; **Published** September 30, 2014

Citation: Hichar S, Guerfi A, Meftah MT (2014) Estimates for Solutions of Semilinear Elliptic Equation in Two Dimensions. J Appl Computat Math 3: 186. doi:10.4172/2168-9679.1000186

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$\| f_1 \|_1 < \varepsilon \; and \; f_2 \in L^{\infty}(\Omega)$

Write u = u1 + u2 where ui are the solutions of

$$\begin{cases} -\Delta u_i = f_i \; in \, \Omega \\ u_i = 0 \quad on \, \partial\Omega \end{cases}$$

choosing, for example , $\delta = (4\pi - 1)$ in Theorem (2.1) we find

$$\int_{\Omega} \exp[\frac{|\mathrm{u}_1(\mathrm{x})|}{\| \mathrm{f}_1 \|_1}] < \infty$$

And thus

$$\int_{\Omega} \exp[\mathrm{k} \, | \, \mathrm{u}_1 \, |] < \infty$$

The conclusion follows since

$$|\,\mathrm{u}\,| \le |\,\mathrm{u}_1\,| + |\,\mathrm{u}_2\,|$$

And $u_2 \in L\infty(\Omega)$

Statement of the Results

Let u satisfy the nonlinear equation (Liouville Equation)

$$\begin{cases} -\Delta u = v(\mathrm{x})\,\mathrm{e}^{\mathrm{u}} \quad in \, \Omega \\ u = 0 \quad on \, \partial\Omega \end{cases} \tag{4}$$

Where Ω is a bounded domain in $\mathbb{R}^2$ and v(x) a given function on Ω

Corollary 3.1 suppose u is a solution of (4) with $v \in L^p(\Omega)$ and $e^u \in L^{p'}(\Omega)$ for some $1 < p \le \infty$ Then $u \in L^{\infty}(\Omega)$

Proof: By corrollary (2:2), we know that $e^{ku} \in L^1(\Omega), \forall \, \mathrm{k} > 0$,

i.e., $e^u \in L^r(\Omega)$

$\forall r < \infty$ It follows that $ve^u \in L^{p-\delta}(\Omega) \forall \delta > 0$ if $p < \infty \, and \, ve^u \in L^r(\Omega) \, \forall \, \mathrm{r} < \infty \, \mathrm{if} \, p = \infty$. Standard elliptic estimates imply that $u \in L^{\infty}(\Omega)$

Resolution of the equation $-\Delta u = \lambda e^u + f(\mathrm{x})$

The Corollary (3:1) still holds for a solution u of

$$\begin{cases} -\Delta u = \lambda e^u + f(\mathrm{x}) \;\; \mathrm{in} \, \Omega \\ u = g \quad on \, \partial\Omega \end{cases} \tag{5}$$

With $\Omega \subset \mathbb{R}^2$ is a bounded domain, $g \in L\infty(\partial\Omega)$ and $f \in L^q(\Omega)$ for some q>1.

Let w be a solution of

$$\begin{cases} -\Delta w = f(\mathrm{x}) \;\; \mathrm{in} \, \Omega \\ w = g \quad on \, \partial\Omega \end{cases} \tag{6}$$

So that $w \in L^{\infty}(\Omega)$

The function k=u-w satisfies:

$$\begin{cases} -\Delta k = (\lambda \, \mathrm{e}^{\mathrm{w}})\, \mathrm{e}^{\mathrm{k}} \quad in \, \Omega \\ k = 0 \quad on \;\; \partial\Omega \end{cases} \tag{7}$$

The solution k is of the problem of liouville (7). Thus the solution of the problem (5) is u=k+w with w the solution of the problem (6) and k the solution of the problem (7).

Remark 3.2

Corollary (3:1) is not valid for p=1 for that we have this example.

Example 3.3 Let 0 < a < 1. The function $u = -a\log(\log\frac{e}{r})$, with $r = |\,\mathrm{x}\,|$ satisfies

$$\begin{cases} -\Delta u = ve^u \quad in \, \Omega = B_1 \\ u = 0 \quad on \;\partial\Omega \end{cases} \tag{8}$$

With

$$v = -\frac{a}{r2(\log\frac{e}{r})^{2-\mathrm{a}}}$$

Note that $v \in L^1(\Omega), \mathrm{e}^{\mathrm{u}} \in L\infty(\Omega)$ and nevertheless $u \notin L^{\infty}(\Omega)$ since $u(\mathrm{x}) \to -\infty \, \mathrm{as} \, \mathrm{x} \to 0$. The same function u with a<0 provides an example where u satisfies (8) with $v \in L^1(\Omega), ve^u \in L^1(\Omega)$ and nevertheless $u^+ \notin L^{\infty}(\Omega)$ since $u(\mathrm{x}) \to +\infty \, \mathrm{as} \, \mathrm{x} \to 0$

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