

ERROR ESTIMATION FOR ILL-POSED PROBLEMS WITH PRIORI INFORMATION

Anatoly Yagola

Lomonosov Moscow State University, Moscow, Russia
Email: yagola@inverse.phys.msu.ru

Received March 2009, Revised May 2009, Accepted July 2009

Abstract

The theory of solving linear and nonlinear ill-posed problems is advanced greatly today (see, e.g., [1, 2]). A general scheme for constructing regularizing algorithms using Tikhonov's variational approach is considered in [2]. It is very well known that ill-posed problems have unpleasant properties even in the cases when stable methods (regularizing algorithms) of their solution exist. E.g., it is impossible to estimate an error of an approximate solution of an ill-posed problem without very strong assumptions concerning the unknown solution. The following assumptions are under consideration: 1) the unknown solution is an element of the given compact set; 2) the unknown solution is sourcewise represented with a compact operator. For these cases the theory of error estimation or a posteriori error estimation was developed and applied for solving operator equations including integral equations and some inverse problems for differential equations. Numerical methods for solving ill-posed problems and their error estimation are based on convex programming. The results above were used for the solution of practical problems in astrophysics, acoustics, physical chemistry, electron microscopy, nuclear physics, etc.

Keywords: ill-posed problems, regularizing algorithms, error estimation, constrained minimization

1. Introduction

Let us consider an operator equation:

$$Az = u, \quad (1)$$

where A is a linear operator acting from a Hilbert space Z into a Hilbert space U . It is required to find a solution of the operator equation z corresponding to a given inhomogeneity (or right-hand side) u .

This equation is a typical mathematical model for many physical so called inverse problems if it is supposed that unknown physical characteristics z cannot be measured directly. As results of experiments, it is possible to obtain only data u connected with z with help of an operator A .

French mathematician J. Hadamard formulated the following conditions of well-posedness of mathematical problems. Let

us consider these conditions for the operator equation above. The problem of solving the operator equation is called to be well-posed (according to Hadamard) if the following three conditions are fulfilled:

- 1) the solution exists $\forall u \in U$;
- 2) the solution is unique;
- 3) if $u_n \rightarrow u$, $Az_n = u_n$, $Az = u$, then $z_n \rightarrow z$.

The condition 2) can be realized then and only then the operator A is one-to-one (injective). The conditions 1) and 2) imply that an inverse operator A^{-1} exists, and its domain $D(A^{-1})$ (or the range of the operator $A R(A)$) coincides with U . It is equivalent to that the operator A is bijective. The condition 3) means that the inverse operator A^{-1} is continuous, i.e., to "small" perturbations of the right-hand side u "small" changes of the solution z correspond. Moreover, J. Hadamard believed that well-posed problems only can be considered while solving practical problems. However, there are well known a lot of examples of ill-posed problems that should be numerically solved when practical problems are investigated. It should be noted that stability or instability of solutions depends on definition of the space of solutions Z . Usually, a choice of the space of solutions (including a choice of the norm) is determined by requirements of an applied problem. A mathematical problem can be ill-posed or well-posed depending on a choice of a norm in a functional space.

Numerous inverse (including ill-posed) problems can be found in different branches of physics. E.g., an astrophysicist has no possibility to influent actively on processes in remote stars and galaxies. He is induced to make conclusions about physical characteristics of very remote objects using their indirect manifestations measured on the Earth surface or near the Earth on space stations. Excellent examples of ill-posed problems are in medicine. Firstly, let us point out computerized tomography. A lot of applications of ill-posed problems are in geophysics. Indeed, it is easier and cheaper to judge about what is going under the Earth surface solving

inverse problems than drilling deep boreholes. Other examples are in radio astronomy, spectroscopy, nuclear physics, plasma diagnostics, etc., etc.

2. Regularizing algorithms

In 1963 A.N. Tikhonov (see, e.g., [1, 2] formulated a famous definition of the regularizing algorithm (RA) that is a basic conception in the modern theory of ill-posed problems.

Definition. Regularizing algorithm (regularizing operator) $R(\delta, u_\delta) \equiv R_\delta(u_\delta)$ is called an operator possessing two properties:

- 1) $R_\delta(u_\delta)$ is defined for any $\delta > 0$, $u_\delta \in U$, and is mapping $(0, +\infty) \times U$ into Z ;
- 2) For any $z \in Z$ and for any $u_\delta \in U$ such that $Az = u$, $\|u - u_\delta\| \leq \delta$, $\delta > 0$, $z_\delta = R_\delta(u_\delta) \xrightarrow{\delta \rightarrow 0} z$.

A problem of solving an operator equation is called to be regularizable if there exists at least one regularizing algorithm. Directly from the definition it follows that if there exists one regularizing algorithm then number of them is infinite.

At the present time, all mathematical problems can be divided into following classes:

- 1) well-posed problems;
- 2) ill-posed regularizable problems;
- 3) ill-posed nonregularizable problems.

All well-posed problems are regularizable since it can be taken $R_\delta(u_\delta) = A^{-1}$. Let us note that knowledge of $\delta > 0$ is not obligatory in this case.

Not all ill-posed problems are regularizable, and it depends on a choice of spaces Z, U at that. Russian mathematician L.D. Menikhes constructed an example of an integral operator with a continuous closed kernel acting from $C[0,1]$ into $L_2[0,1]$ such that an inverse problem (that is, solving a Fredholm integral equation of the 1st kind) is nonregularizable. It depends on properties of the space $C[0,1]$. Below it would be shown that if Z is the Hilbert space, and an operator A is bounded and injective, then the problem of solving of the operator equation is regularizable. This result is valid for some Banach spaces, not for all (for reflexive Banach spaces only). In particular, the space $C[0,1]$ does not belong to such spaces.

An equivalent definition of the regularizing algorithm is following. Let be given an operator (mapping) $R_\delta(u_\delta)$ defined for any $\delta > 0$, $u_\delta \in U$, and reflecting $(0, +\infty) \times U$ into Z . An accuracy of solving an operator equation in a point $z \in Z$ using an operator $R_\delta(u_\delta)$ under condition that the right-hand side defined with an error $\delta > 0$ is defined as

$$\Delta(R_\delta, \delta, z) = \sup_{u_\delta \in U: \|u_\delta - u\| \leq \delta, Az = u} \|R_\delta u_\delta - z\|. \text{ An operator}$$

$R_\delta(u_\delta)$ is called a regularizing algorithm (operator) if for

any $z \in Z$ $\Delta(R_\delta, \delta, z) \xrightarrow{\delta \rightarrow 0} 0$. This definition is equivalent to the definition above. Similar definitions can be formulated if the operator is specified with an error.

It is very important to get an answer to the following question: is it possible to solve an ill-posed problem (i.e., to construct a regularizing algorithm) without knowledge of an error level δ . Evidently, if a problem is well posed then a stable method of its solution can be constructed without knowledge of an error δ . E.g., if an operator equation is under consideration then it can be taken $z_\delta = A^{-1}u_\delta \rightarrow z = A^{-1}u$ as $\delta \rightarrow 0$. It is impossible if a problem is ill posed. The next very important property of ill-posed problems is impossibility of error estimation for a solution even if an error of a right-hand side of an operator equation is known. These basic results were obtained by A.B. Bakushinsky

From the definition of the regularizing algorithm it follows immediately if one exists then infinite number of them exists. While solving ill-posed problems it is impossible to choose a regularizing algorithm that finds an approximate solution with the minimal error. It is impossible also to compare different regularizing algorithms according to errors of approximate solutions. Only including a priori information in a statement of the problem can give such a possibility, but in this case a reformulated problem is well-posed in fact. We will consider examples below.

3. ILL-Posed problems on compact sets

Let us consider an operator equation (1) in the case when A is a linear injective operator acting between normed spaces Z and U . Let \bar{z} is an exact solution of an operator equation, $A\bar{z} = \bar{u}$, \bar{u} is an exact right-hand side, and it is given an approximate right-hand side such that $\|\bar{u} - u_\delta\| \leq \delta$, $\delta > 0$.

A set $Z_\delta = \{z_\delta : \|Az_\delta - u_\delta\| \leq \delta\}$ is a set of approximate solutions of the operator equation. For linear ill-posed problems

$$\text{diam } Z_\delta = \sup \{\|z_1 - z_2\| : z_1, z_2 \in Z_\delta\} = \infty$$

for any $\delta > 0$ since the inverse operator A^{-1} is not bounded.

The question is that: is it possible to use a priori information in order to restrict a set of approximate solutions or (it is better) to reformulate a problem to be well-posed. A.N. Tikhonov proposed a following idea: if it is known the set of solutions is a compact then a problem of solving an operator equation is well-posed under condition that an approximate right-hand side belongs to the image of the compact. A.N. Tikhonov proved this assertion using as basis the following theorem.

Theorem. Let an injective continuous operator A be mapping: $D \in Z \rightarrow AD \in U$, where Z, U are normed

spaces, D is a compact. Then the inverse operator A^{-1} is continuous on AD .

The theorem is true for nonlinear operators also. So, a problem of solving an operator equation is well-posed under condition that an approximate right-hand side belongs to AD . This idea made possible to M.M. Lavrentiev to introduce a conception of a well-posed according to A.N. Tikhonov mathematical problem (it is supposed that a set of well-posedness exists), and to V.K. Ivanov to define a quasisolution of an ill-posed problem.

The theorem above is not valid if $u_\delta \notin R(A)$. So, it should be generalized.

Definition. An element $z_\delta \in D$ such that $z_\delta = \arg \min_{z \in D} \|Az - u_\delta\|$ is called a quasisolution of an operator equation on a compact D ($z_\delta = \arg \min_{z \in D} \|Az - u_\delta\|$ means that $\|Az_\delta - u_\delta\| = \min\{\|Az - u_\delta\| : z \in D\}$).

A quasisolution exists but maybe is nonunique. Though, any quasisolution tends to an exact solution: $z_\delta \rightarrow \bar{z}$ as $\delta \rightarrow 0$. In this case, knowledge of an error δ is not obligatory. If δ is known then:

- 1) any element $z_\delta \in D$ satisfying an inequality : $\|Az_\delta - u_\delta\| \leq \delta$, can be chosen as an approximate solution with the same property of convergence to an exact solution (δ -quasisolution);
- 2) it is possible to find an error of an approximate solution solving an extreme problem:
find $\max_{z \in D} \|z - z_\delta\|$ maximizing on all $z \in D$ satisfying an inequality: $\|Az - u_\delta\| \leq \delta$ (it is obviously that an exact solution satisfying the inequality).

Thus, the problem of quasisolving an operator equation does not differ strongly from a well-posed problem. A condition of uniqueness only maybe does not satisfy.

If an operator A is specified with an error then the definition of a quasisolution can be easily modified.

If Z and U are Hilbert spaces then many numerical methods of finding quasisolutions of linear operator equations are based on convexity and differentiability of the discrepancy functional $\|Az - u_\delta\|^2$. If D is a convex compact set then finding a quasisolution is a problem of convex programming. The inequalities written above and defining approximate solutions can be used as stopping rules for minimizing the discrepancy procedures. The problem of calculating errors of an approximate solution is a nonstandard problem of convex

programming because it is necessary to maximize (not to minimize) a convex functional.

Some sets of correctness are very well known in applied sciences. First of all, if an exact solution belongs to a family of functions depending on finite number of bounded parameters then the problem of finding parameters can be well-posed. The same problem without such a priori information can be ill-posed.

If an unknown function $z(s)$, $s \in [a, b]$, is monotonic and bounded then it is sufficient to define a compact set in the space $L_2[a, b]$. After finite-dimensional approximation the problem of finding a quasisolution is a quadratic programming problem. For numerical solving, known methods such as a method of projections of conjugate gradients or a method of conditional gradient can be applied. Similar approach can be used also when the solution is monotonic and bounded, or monotonic and convex, or has given number of maxima and minima. In these cases an error of an approximate solution can be calculated.

4. ILL-posed problems with source wise represented solutions

Let an operator A be linear injective continuous and mapping $Z \rightarrow U$; Z, U normed spaces. Let the following a priori information be valid: it is known that an exact solution \bar{z} over an equation $\bar{u} = A\bar{z}$ is represented in the form $B\bar{v} = \bar{z}$, $\bar{v} \in V$; $B : V \rightarrow Z$; B is an injective completely continuous operator; V is a Hilbert space. Let suppose that an approximate right-hand side u_δ such that $\|\bar{u} - u_\delta\| \leq \delta$, and its error $\delta > 0$ is known. Such a priori information is typical for many physical problems.

V.K. Ivanov and I.N. Dombrovskaya proposed an idea of a method of extending compacts. Let describe a version of this method below.

Let preset an iteration number $n=1$, and define a closed ball in the space V : $\bar{S}_n(0) = \{v : \|v\| \leq n\}$. Its image $Z_n = B\bar{S}_n(0)$ is a compact since B is a completely continuous operator and V is a Hilbert space. After that let us find $\min_{z \in B(\bar{S}_n(0))} \|Az - u_\delta\|$, where u_δ is given approximate right-

hand side $\|\bar{u} - u_\delta\| \leq \delta$, $\delta > 0$. Existence of the minimum is guaranteed by compactness of Z_n and continuity of A . If

$\min_{z \in B(\bar{S}_n(0))} \|Az - u_\delta\| \leq \delta$, then the iteration process should be

stopped, and the number $n(\delta) = n$ defined. An approximate solution of the operator equation can be chosen as any element

$z_{n(\delta)} : z_{n(\delta)} \in B(\bar{S}_{n(\delta)}(0))$ satisfying

$\|Az_{n(\delta)} - u_\delta\| \leq \delta$. If $\min_{z \in B(\bar{z}_n(0))} \|Az - u_\delta\| > \delta$ then the

compact should be extended. For this purpose n changes to $n+1$, and the process repeats.

Theorem [3]. The process described above converges: $n(\delta) < +\infty$. There exists $\delta_0 > 0$ (generally speaking, depending on \bar{z}) such that $n(\delta) = n(\delta_0) \quad \forall \delta \in (0, \delta_0]$.

Approximate solutions $z_{n(\delta)}$ strongly converge to the exact solution \bar{z} as $\delta \rightarrow 0$.

It is clear why the method is referred to as “an extending compacts method”. It appears that using this method so called an a posteriori error estimate can be defined. It means that there exists a function $\chi(u_\delta, \delta)$ such that $\chi(u_\delta, \delta) \rightarrow 0$ as $\delta \rightarrow 0$, and $\chi(u_\delta, \delta) \geq \|z_{n(\delta)} - \bar{z}\|$ at least for sufficiently small $\delta > 0$. As an a posteriori error estimate $\chi(u_\delta, \delta) = \max\{\|z_{n(\delta)} - z\| : z \in Z_{n(\delta)}, \|Az - u_\delta\| \leq \delta\}$ can be taken.

An a posteriori error estimate is not an error estimate in a general sense, error estimates cannot be constructed for ill-posed problems. However, for sufficiently small $\delta > 0$ (notably $\forall \delta \in (0, \delta_0]$) an a posteriori error estimate is an error estimate for a solution of an ill-posed problem if an a priori information about source wise represent ability is available.

This approach was generalized to cases when both operators A and B are specified with errors, also to nonlinear ill-posed problems under condition of sourcewise representation of an exact solution.

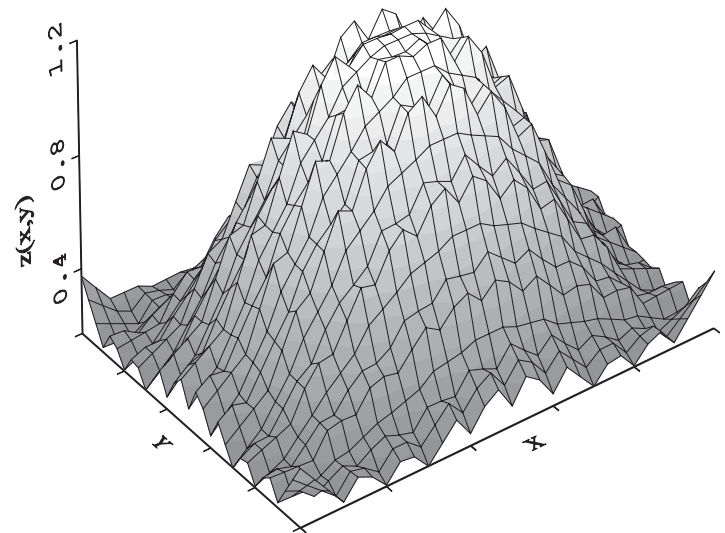
Numerical methods for solving linear ill-posed problems under condition of sourcewise representation were constructed, including methods for an a posteriori error estimation. To use a sequence of natural numbers as radii of balls in the space V is not obligatory. Any unbounded monotonically increasing sequence of positive numbers can be taken.

5. Applications

Methods for solving ill-posed problems on functional compact sets of special structure (monotonic functions, convex functions, monotonic convex functions, piecewise convex-concave functions) so as under condition of sourcewise representability were effectively applied to solution of ill-posed problems in astrophysics, acoustics, physical chemistry, electron microscopy, nuclear physics, etc.

On the picture 1 it can be found an example of solving an inverse problem for 2D heat conductivity equation under condition that the initial temperature is 2D concave function. In this case it is possible to calculate not only an approximate solution but also functions that define errors of the solution from above and from below. The first function is on the picture.

Many other practical applications and numerical examples can be found in [4-10].



Picture 1

6. Conclusion

We have described in brief fundamentals of the theory of ill-posed problems and error estimation if a priori information is available.

Acknowledgements

Author thanks the RFBR (grants 08-01-00160 and 07-01-92103-NFSC) for partial financial support.

References

1. Tikhonov A.N., Goncharky A.V., Stepanov V.V., Yagola, A.G. Numerical Methods for the Solution of Ill-Posed Problems. Kluwer Academic Publ., Dordrecht, 1995.
2. Tikhonov A.N., Leonov A.S., Yagola, A.G. Nonlinear Ill-Posed Problems. Chapman and Hall, London, 1995.
3. Yagola A.G., Dorofeev K.Yu. Sourcewise representation and a posteriori error estimates for ill-posed problems. - In “A. G. Ramm, P. N. Shivakumar, A. V. Strauss (Eds.), Fields Institute Communications: Operator Theory and Its Applications”, Providence, RI: American Mathematical Society, 2000, v. 25, pp. 543-550.
4. Titarenko V.N., Yagola A.G. Cauchy problems for Laplace equation on compact sets. – Inverse Problems in Engineering, 2002, v. 10, N 3, p. 235-254.
5. Dorofeev K.Yu., Yagola A.G. The method of extending compacts and a posteriori error estimates for nonlinear ill-posed problems. – Journal of Inverse and Ill-Posed Problems, 2004, v. 12, No 6, pp. 627-636.

6. Nikolaeva N.N., Titarenko V.N., Yagola A.G. An error estimation of a solution of Abel equation. – Numerical Functional Analysis and Optimization, 2004, v. 25, N 3-4, pp. 259-269.
7. Yagola A., Titarenko V. Using a priori information about a solution of an ill-posed problem for constructing regularizing algorithms and their applications. - Inverse Problems in Science and Engineering, 2007, v. 15, No 1, pp. 3 – 17.
8. Nikolaeva N.N., Ruchkin S.V., Rychagov M.N., Yagola A.G. Symmetric velocity profiles reconstruction in channels with a circular cross-section by ultrasonic flow measurements. – Inverse Problems in Science and Engineering, 2006, v. 14, No 7, pp. 711 – 723.
9. Titarenko V., Yagola A. Error estimation for ill-posed problems on piecewise convex functions and sourcewise represented sets. - Journal of Inverse and Ill-posed Problems, 2008, Vol. 16, No. 6, pp. 625–638.
10. Titarenko V., Yagola A. Solution of ill-posed problems on sets of functions convex along all lines parallel to coordinate axes. - Journal of Inverse and Ill-posed Problems, 2008, Vol. 16, No. 8, pp. 805–824.