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Research Article

Dynamical Yang-Baxter Maps Associated with Homogeneous Pre-Systems*

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Dedicated to Professor Susumu Okubo on the occasion of his eightieth birthday

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Abstract We construct dynamical Yang-Baxter maps, which are set-theoretical solutions to a version of the quantum dynamical Yang-Baxter equation, by means of homogeneous pre-systems, that is, ternary systems encoded in the reductive homogeneous space satisfying suitable conditions. Moreover, a characterization of these dynamical Yang-Baxter maps is presented.

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1 Introduction

The quantum dynamical Yang-Baxter equation (QDYBE for short) [9,10], a generalization of the quantum Yang-Baxter equation (QYBE for short) [2,3,40,41], has been studied extensively in recent years (see [7] and the references therein). Dynamical Yang-Baxter maps [31,32,34] are set-theoretical solutions to a version of the QDYBE.

Let H and X be nonempty sets with a map $(\cdot) : H \times X \ni (\lambda, x) \mapsto \lambda \cdot x \in H$. A map $R(\lambda) : X \times X \to X \times X$ ($\lambda \in H$) is a dynamical Yang-Baxter map associated with H, X and $(\cdot)$, if and only if, for every $\lambda \in H$, $R(\lambda)$ satisfies the following equation on $X \times X \times X$:

$$R_{23}(\lambda)R_{13}\left(\lambda \cdot X^{(2)}\right)R_{12}(\lambda) = R_{12}\left(\lambda \cdot X^{(3)}\right)R_{13}(\lambda)R_{23}\left(\lambda \cdot X^{(1)}\right). \tag{1.1}$$

Here $R_{12}(\lambda)$, $R_{12}(\lambda \cdot X^{(3)})$, $R_{23}(\lambda \cdot X^{(1)})$, and others are the maps from $X \times X \times X$ to itself defined as follows: for $u, v, w \in X$,

$$\begin{aligned} R_{12}(\lambda)(u,v,w) &= \big(R(\lambda)(u,v), w\big), \\ R_{12}\left(\lambda \cdot X^{(3)}\right)(u,v,w) &= R_{12}(\lambda \cdot w)(u,v,w), \\ R_{23}\left(\lambda \cdot X^{(1)}\right)(u,v,w) &= \big(u, R(\lambda \cdot u)(v,w)\big). \end{aligned}$$

Set-theoretical solutions to the QYBE [6], also known as Yang-Baxter maps [39], are dynamical Yang-Baxter maps constant for the parameter λ of any set H; indeed, the dynamical Yang-Baxter map is a generalization of the set-theoretical solution to the QYBE.

This dynamical Yang-Baxter map yields a bialgebroid [4]. Every dynamical Yang-Baxter map with some conditions gives birth to an (H, X)-bialgebroid [35], a generalization of the quantum group [5,11], through the Faddeev-Reshetikhin-Takhtajan construction [8].

It is worth pointing out that a ternary system (Definition 1(3)) can produce the dynamical Yang-Baxter map [32]. Each triple (L, M, π) consisting of a left quasigroup $L = (L, \cdot)$ (Definition 1(1)), a ternary system M satisfying (2.2) and (2.3), and a (set-theoretic) bijection $\pi : L \to M$ gives a dynamical Yang-Baxter map $R(\lambda)$ associated with L, L and $(\cdot)$ (see Section 2 for more details).

Homogeneous systems [18,19,20,21,22,23] are algebraic features of the reductive homogeneous space [24,28] satisfying suitable conditions. Let A be a group with its subgroup K. We assume that a subset G of the group A satisfies the following:

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(1) the group A is uniquely factorized as $A = GK$,
(2) $G^{-1} = G$,
(3) $kGk^{-1} = G$, for all $k \in K$.

Let $p : A \to G$ denote the canonical projection with respect to the factorization $A = GK$, and $(\cdot)$ the binary operation on G defined by $x \cdot y = p(xy)$ $(x, y \in G)$. Because the map $L_x : G \ni y \mapsto x \cdot y \in G$ is bijective, we define the ternary operation η on G by

$$\eta(x, y, z) = L_x\Big(\big(L_x\big)^{-1}(y) \cdot \big(L_x\big)^{-1}(z)\Big), \quad x, y, z \in G.$$

This ternary system $G = (G, \eta)$ is a homogeneous system [23, Proposition 1] (see also Definition 7), and every homogeneous system is constructed in such a way.

If G is a connected and second countable C^∞-manifold with a C^∞-map $\eta : G \times G \times G \to G$, then the homogeneous system $G = (G, \eta)$ is isomorphic to a reductive homogeneous space A/K for a connected Lie group A with its closed subgroup K [19, Theorem 1]. The homogeneous system is a ternary system, an algebraic structure, encoded in the reductive homogeneous space (for ternary systems in differential geometry and mathematical physics, see [13, 14, 15, 29]).

It is natural to relate this homogeneous system to the dynamical Yang-Baxter map through the ternary system.

The aim of this paper is to produce the dynamical Yang-Baxter maps by means of homogeneous pre-systems, which generalize the homogeneous system. Furthermore, we characterize such dynamical Yang-Baxter maps.

The organization of this paper is as follows.

Section 2 contains a brief summary of the dynamical Yang-Baxter map. We focus on its construction by means of the ternary system. This construction yields a category $\mathcal{A}$ concerning the ternary systems, which is equivalent to a category $\mathcal{D}$ consisting of the dynamical Yang-Baxter maps.

Section 3 presents the notion of a homogeneous pre-system, together with examples.

In Section 4, our main results are stated and proved. Every homogeneous pre-system satisfying (4.1) can produce a dynamical Yang-Baxter map via the ternary system. More precisely, we construct a category $\mathcal{H}$, isomorphic to the category $\mathcal{A}$, by means of the homogeneous pre-systems with (4.1). Because the category $\mathcal{A}$ is equivalent to the category $\mathcal{D}$, each object of $\mathcal{H}$ gives a dynamical Yang-Baxter map; in particular, we demonstrate dynamical Yang-Baxter maps provided by a certain left quasigroup and the examples in Section 3.

The last section, Section 5, deals with a relation between the homogeneous pre-system satisfying (4.1) and the left quasigroup with (5.1), which is due to the work in [32, Section 6]. We introduce a category $\mathcal{B}$ concerning the left quasigroups satisfying (5.1) and an essentially surjective functor $J : \mathcal{B} \to \mathcal{H}$ to construct the dynamical Yang-Baxter maps by means of quasigroups of reflection [17, 27].

Our viewpoint sheds some light on the relation between geometry and the dynamical Yang-Baxter map.

2 Dynamical Yang-Baxter maps

In this section, we briefly summarize without proofs the relevant material in [32] on the construction of the dynamical Yang-Baxter map.

Definition 1. (1) $(L, \cdot)$ is a left quasigroup (resp. right quasigroup [38, Section I.4.3]), if and only if L is a nonempty set, together with a binary operation $(\cdot)$ on L having the property that, for all $u, w \in L$, there uniquely exists $v \in L$ such that $u \cdot v = w$ (resp. $v \cdot u = w$). For the simplicity, one uses the notation uv instead of $u \cdot v$ $(u, v \in L)$.
(2) A quasigroup $(Q, \cdot)$ is a left and right quasigroup (see [30, Definition I.1.1] and [38, Section I.2]).
(3) A ternary system (M, μ) is a pair of a nonempty set M and a ternary operation $\mu : M \times M \times M \to M$.

By this definition, the left quasigroup $L = (L, \cdot)$ has another binary operation $\backslash_L$ called a left division [38, Section I.2.2]. For $u, w \in L$, we denote by $u\backslash_L w$ the unique element $v \in L$ satisfying $uv = w$,

$$u\backslash_L w = v \iff uv = w. \tag{2.1}$$

The binary operation on the quasigroup is not always associative.

Example 2. We define the binary operation $(*)$ on the set $Q = \{1, 2, 3, 4, 5\}$ of five elements by Table 1. Here $1 * 2 = 3$. This $Q = (Q, *)$ is a quasigroup, because each element in Q appears once and only once in each row and in each column of Table 1 [30, Theorem I.1.3]. The binary operation $(*)$ is not associative, since $(1*2)*3 \neq 1*(2*3)$. This quasigroup Q is due to Nobusawa [27, Section 6, type 1]. However, the order of the binary operation $(*)$ in Table 1 is reversed.

$*$	1	2	3	4	5
1	1	3	5	2	4
2	5	2	4	1	3
3	4	1	3	5	2
4	3	5	2	4	1
5	2	4	1	3	5

Table 1: Binary operation $(*)$ on Q.

Each ternary system $M = (M, \mu)$ satisfying

$$\mu\big(a, \mu(a,b,c), \mu\big(\mu(a,b,c), c, d\big)\big) = \mu\big(a, b, \mu(b,c,d)\big), \tag{2.2}$$

$$\mu\big(\mu\big(a, b, \mu(b,c,d)\big), \mu(b,c,d), d\big) = \mu\big(\mu(a,b,c), c, d\big), \tag{2.3}$$

for any $a, b, c, d \in M$, can provide a dynamical Yang-Baxter map [32, Theorem 3.2]. Let $L = (L, \cdot)$ be a left quasigroup isomorphic to M as sets, and $\pi : L \to M$ a (set-theoretic) bijection. For $\lambda, u \in L$, we define the maps $\xi_\lambda^{(L,M,\pi)}(u) : L \to L$ and $\eta_\lambda^{(L,M,\pi)}(u) : L \to L$ as follows: for $v \in L$,

$$\xi_\lambda^{(L,M,\pi)}(u)(v) = \lambda \backslash_L \pi^{-1}\Big(\mu\Big(\pi(\lambda), \pi(\lambda u), \pi\big((\lambda u)v\big)\Big)\Big), \tag{2.4}$$

$$\eta_\lambda^{(L,M,\pi)}(u)(v) = \big(\lambda \xi_\lambda^{(L,M,\pi)}(v)(u)\big) \backslash_L \big((\lambda v)u\big). \tag{2.5}$$

Let $R^{(L,M,\pi)}(\lambda)$ $(\lambda \in L)$ denote the map from $L \times L$ to itself defined by

$$R^{(L,M,\pi)}(\lambda)(u,v) = \big(\eta_\lambda^{(L,M,\pi)}(v)(u), \xi_\lambda^{(L,M,\pi)}(u)(v)\big), \quad u, v \in L. \tag{2.6}$$

Theorem 3. *The map* $R^{(L,M,\pi)}(\lambda)$ (2.6) *is a dynamical Yang-Baxter map* (1.1) *associated with* L, L *and* $(\cdot)$.

We now introduce two categories $\mathcal{A}$ and $\mathcal{D}$ concerning a special class of the dynamical Yang-Baxter maps, which play a central role in this article.

The first task is to explain the category $\mathcal{A}$ (cf. the category $\mathcal{A}_2$ in [32, Section 6]). We follow the notation of [16, Chapter XI]. Let $L = (L, \cdot)$ be a left quasigroup, $M = (M, \mu)$ a ternary system satisfying (2.2) and

$$\mu(a, b, b) = a, \quad \forall a, b \in M, \tag{2.7}$$

$$\mu\big(\mu(a,b,c), c, d\big) = \mu(a, b, d), \quad \forall a, b, c, d \in M, \tag{2.8}$$

and $\pi : L \to M$ a bijection. The object of $\mathcal{A}$ is, by definition, a triple (L, M, π).

The morphism $f : (L, (M, \mu), \pi) \to (L', (M', \mu'), \pi')$ of $\mathcal{A}$ is a homomorphism $f : L \to L'$ of left quasigroups such that $h := \pi' \circ f \circ \pi^{-1} : M \to M'$ is a homomorphism of ternary systems; that is, the map $f : L \to L'$ satisfies

$$\begin{gathered} f\big(a \cdot_L b\big) = f(a) \cdot_{L'} f(b), \quad \forall a, b \in L, \\ h\big(\mu(a,b,c)\big) = \mu'\big(h(a), h(b), h(c)\big), \quad \forall a, b, c \in M. \end{gathered} \tag{2.9}$$

The identity id, the source $s(f)$ and the target $b(f)$ of a morphism $f : (L, M, \pi) \to (L', M', \pi')$, and the composition $g \circ f$ for morphisms $f : (L, M, \pi) \to (L', M', \pi')$ and $g : (L', M', \pi') \to (L'', M'', \pi'')$ are defined as follows: for an object $(L, M, \pi) \in \mathcal{A}$,

$$\begin{gathered} \mathrm{id}_{(L,M,\pi)} = \mathrm{id}_L, \\ s\big(f : (L, M, \pi) \longrightarrow \big(L', M', \pi'\big)\big) = (L, M, \pi), \\ b\big(f : (L, M, \pi) \longrightarrow \big(L', M', \pi'\big)\big) = \big(L', M', \pi'\big), \end{gathered}$$

the composition $g \circ f$ is the usual one of the maps $f : L \to L'$ and $g : L' \to L''$.

Proposition 4. *$\mathcal{A}$ is a category.*

The next task is to describe the category $\mathcal{D}$, which is exactly the category $\mathcal{D}_2$ in [32, Section 6]. The object of this category $\mathcal{D}$ is a pair (L, R) of a left quasigroup $L = (L, \cdot)$ and a dynamical Yang-Baxter map $R(\lambda) : L \times L \to L \times L$ $(\lambda \in L)$ satisfying

$$\xi_\lambda(u)\big((\lambda u)\backslash_L(\lambda u)\big) = \lambda\backslash_L\lambda, \quad \forall \lambda, u \in L,$$
$$\big(\lambda\xi_\lambda(u)(v)\big)\eta_\lambda(v)(u) = (\lambda u)v, \quad \forall \lambda, u, v \in L,$$
$$\big(\lambda\xi_\lambda(u)(v)\big)\xi_{\lambda\xi_\lambda(u)(v)}\big(\eta_\lambda(v)(u)\big)(w) = \lambda\xi_\lambda(u)\big((\lambda u)\backslash_L\big(\big((\lambda u)v\big)w\big)\big), \quad \forall \lambda, u, v, w \in L.$$

Here, $(\eta_\lambda(v)(u), \xi_\lambda(u)(v)) := R(\lambda)(u, v)$ $(\lambda, u, v \in L)$.

The morphism $f : (L, R) \to (L', R')$ of $\mathcal{D}$ is a homomorphism $f : L \to L'$ of left quasigroups satisfying

$$R'\big(f(\lambda)\big) \circ f \times f = f \times f \circ R(\lambda), \quad \forall \lambda \in L.$$

Proposition 5. *$\mathcal{D}$ is a category; the definitions of the identity, the source, the target and the composition are similar to those of the category $\mathcal{A}$.*

We can construct functors $S : \mathcal{A} \to \mathcal{D}$ and $T : \mathcal{D} \to \mathcal{A}$, which establish an equivalence of the categories $\mathcal{A}$ and $\mathcal{D}$ (cf. [32, Proposition 6.15]): for $(L, M, \pi) \in \mathcal{A}$, set $S(L, M, \pi) = (L, R^{(L,M,\pi)})$. Here, $R^{(L,M,\pi)}(\lambda)$ is defined by (2.4), (2.5) and (2.6); for a morphism f of $\mathcal{A}$, write $S(f) = f$; for $(L, R) \in \mathcal{D}$, $T(L, R)$ denotes the triple $(L, (M, \mu), \mathrm{id}_L)$, where $M = L$ as sets and $\mu(a, b, c) = a\xi_a(a\backslash_L b)(b\backslash_L c)$ $(a, b, c \in M(= L))$; for a morphism f of $\mathcal{D}$, set $T(f) = f$.

These functors S and T satisfy $ST = \mathrm{id}_{\mathcal{D}}$, and $\theta(L, M, \pi) := \mathrm{id}_L$ $((L, M, \pi) \in \mathcal{A})$ gives a natural isomorphism $\theta : TS \to \mathrm{id}_{\mathcal{A}}$. Thus, the following theorem holds.

Theorem 6. *$S : \mathcal{A} \to \mathcal{D}$ is an equivalence of categories.*

3 Homogeneous pre-systems

This section is devoted to introducing homogeneous pre-systems.

Definition 7. (1) A ternary system $G = (G, \eta)$ (Definition 1(3)) is a homogeneous pre-system if and only if the ternary operation η satisfies

$$\eta(x, y, x) = y, \quad \forall x, y \in G, \tag{3.1}$$
$$\eta\big(x, y, \eta(u, v, w)\big) = \eta\big(\eta(x, y, u), \eta(x, y, v), \eta(x, y, w)\big), \tag{3.2}$$

for all $x, y, u, v, w \in G$.

(2) A homogeneous system $G = (G, \eta)$ [18] is a homogeneous pre-system satisfying

$$\begin{aligned} \eta(x, x, y) &= y, \quad \forall x, y \in G, \\ \eta\big(x, y, \eta(y, x, z)\big) &= z, \quad \forall x, y, z \in G. \end{aligned} \tag{3.3}$$

We explain two examples in this section: one homogeneous pre-system and one homogeneous system, which imply dynamical Yang-Baxter maps in the next section.

Let G be an abelian group. We define the ternary operation η on G by

$$\eta(x, y, z) = x + y - z, \quad x, y, z \in G. \tag{3.4}$$

A trivial verification shows that $G = (G, \eta)$ is a homogeneous pre-system, which is not always a homogeneous system because of (3.3) (cf. [18, Remark 4]).

Another example is a homogeneous system on an arbitrary group G [18, Example in Section 1]. We define the ternary operation η on the group G by

$$\eta(x, y, z) = yx^{-1}z, \quad x, y, z \in G. \tag{3.5}$$

It is clear that this $G = (G, \eta)$ is a homogeneous system.

Remark 8. The homogeneous system (G, η) (3.5) is equivalent to the notion of a torsor [25, 33, 36], also known as the principal homogeneous space, up to the choice of the unit element. Hence, the principal homogeneous space provides a homogeneous system.

4 A relation between dynamical Yang-Baxter maps and homogeneous pre-systems

In this section, we construct dynamical Yang-Baxter maps (2.6) by means of homogeneous pre-systems $G = (G, \eta)$ satisfying

$$\eta(x, y, z) = \eta\big(w, \eta(x, y, w), z\big), \quad \forall x, y, z, w \in G. \tag{4.1}$$

In fact, we present a category $\mathcal{H}$ concerning the homogeneous pre-systems with (4.1); this $\mathcal{H}$ is isomorphic to the category $\mathcal{A}$ in Section 2, and, on account of Theorem 6, every object of $\mathcal{H}$ consequently gives a dynamical Yang-Baxter map.

Let $L = (L, \cdot)$ be a left quasigroup, $G = (G, \eta)$ a homogeneous pre-system satisfying (4.1) and $\pi : L \to G$ a (set-theoretic) bijection. The object of $\mathcal{H}$ is a triple (L, G, π).

The morphism $f : (L, (G, \eta), \pi) \to (L', (G', \eta'), \pi')$ of $\mathcal{H}$ is a homomorphism $f : L \to L'$ of left quasigroups such that $h := \pi' \circ f \circ \pi^{-1} : G \to G'$ is a homomorphism of ternary systems; that is, the map $f : L \to L'$ satisfies (2.9) and

$$h\big(\eta(x, y, z)\big) = \eta'\big(h(x), h(y), h(z)\big), \quad \forall x, y, z \in G.$$

Proposition 9. *$\mathcal{H}$ is a category; the definitions of the identity, the source, the target and the composition are similar to those of the category $\mathcal{A}$.*

In order to prove that the category $\mathcal{H}$ is isomorphic to the category $\mathcal{A}$, we construct functors $F : \mathcal{A} \to \mathcal{H}$ and $F' : \mathcal{H} \to \mathcal{A}$.

We first introduce the functor $F : \mathcal{A} \to \mathcal{H}$. Let $(L, (M, \mu), \pi) \in \mathcal{A}$. Define the ternary system $G = (G, \eta)$ by $G = M$ as sets and

$$\eta(x, y, z) = \mu(y, x, z), \quad x, y, z \in G(= M). \tag{4.2}$$

Proposition 10. *$(L, G, \pi) \in \mathcal{H}$.*

Proof. We need only show that G is a homogeneous pre-system satisfying (4.1).

An easy computation shows (3.1) and (4.1).

By virtue of (4.2), the left-hand side of (3.2) is $\mu(y, x, \mu(v, u, w))$, and, with the aid of (2.2), (2.7) and (2.8),

$$\begin{aligned}
\mu\big(y, x, \mu(v, u, w)\big) &= \mu\big(\mu(y, x, v), v, \mu(v, u, w)\big) \\
&= \mu\big(\mu(y, x, v), \mu\big(\mu(y, x, v), v, u\big), \mu\big(\mu\big(\mu(y, x, v), v, u\big), u, w\big)\big) \\
&= \mu\big(\mu(y, x, v), \mu(y, x, u), \mu(y, x, w)\big),
\end{aligned}$$

which is the right-hand side of (3.2). This proves the proposition. □

By setting $F(L, (M, \mu), \pi) = (L, G, \pi)$ and $F(f) = f$ for a morphism f of $\mathcal{A}$, the following proposition holds.

Proposition 11. *$F : \mathcal{A} \to \mathcal{H}$ is a functor.*

The next task is to construct a functor $F' : \mathcal{H} \to \mathcal{A}$. Let $(L, (G, \eta), \pi) \in \mathcal{H}$. We define the ternary system $M_G = (M_G, \mu)$ by $M_G = G$ as sets and

$$\mu(a, b, c) = \eta(b, a, c), \quad a, b, c \in M_G(= G). \tag{4.3}$$

Proposition 12. *$(L, M_G, \pi) \in \mathcal{A}$.*

Proof. It suffices to prove that M_G satisfies (2.2), (2.7) and (2.8).

A trivial verification shows (2.7) and (2.8).

Due to (4.1) and (4.3), the left-hand side of (2.2) is

$$\mu\big(a, \mu(a, b, c), \mu\big(\mu(a, b, c), c, d\big)\big) = \eta\big(\eta(b, a, c), a, \eta(b, a, d)\big).$$

From (3.1) and (3.2),

$$\eta\big(\eta(b, a, c), a, \eta(b, a, d)\big) = \eta\big(\eta(b, a, c), \eta(b, a, b), \eta(b, a, d)\big) = \eta\big(b, a, \eta(c, b, d)\big),$$

which is exactly the right-hand side of (2.2). □

By setting $F'(L,(G,\eta),\pi) = (L, M_G, \pi)$ and $F'(f) = f$ for a morphism f of $\mathcal{H}$, the following proposition holds.

Proposition 13. *$F' : \mathcal{H} \to \mathcal{A}$ is a functor.*

Since the functors F and F' satisfy $F'F = \mathrm{id}_{\mathcal{A}}$ and $FF' = \mathrm{id}_{\mathcal{H}}$, the following theorem holds.

Theorem 14. *The categories $\mathcal{A}$ and $\mathcal{H}$ are isomorphic.*

By taking account of Theorem 6, we have the following corollary.

Corollary 15. *Each object of $\mathcal{H}$ provides a dynamical Yang-Baxter map.*

The proof of the following proposition is straightforward.

Proposition 16. *The ternary operations* (3.4) *and* (3.5) *satisfy* (4.1).

As a consequence of Corollary 15 and Proposition 16, the homogeneous pre-system G (3.4) and the homogeneous system G (3.5) imply dynamical Yang-Baxter maps. Let $L = (G, \cdot)$ denote the left quasigroup whose binary operation $(\cdot)$ is defined by

$$u \cdot v = v, \quad u, v \in L, \tag{4.4}$$

and let $\pi : L(= G) \to G$ be the identity map on G. The corresponding dynamical Yang-Baxter maps are as follows: if G is a homogeneous pre-system (3.4), then

$$R^{(L,M_G,\pi)}(\lambda)(u,v) = (v, \lambda + u - v), \quad \lambda, u, v \in L(= G),$$

and if G is a homogeneous system (3.5), then

$$R^{(L,M_G,\pi)}(\lambda)(u,v) = \big(v, \lambda u^{-1} v\big), \quad \lambda, u, v \in L(= G).$$

5 A relation between homogeneous pre-systems and left quasigroups

Because of the work in [32, Section 6] and the fact that the categories $\mathcal{A}$ and $\mathcal{H}$ are isomorphic, every homogeneous pre-system G (Definition 7(1)) in the object $(L, G, \pi) \in \mathcal{H}$ is a left quasigroup (Definition 1(1)) whose binary operation gives the ternary operation of G. This last section demonstrates it by constructing a category $\mathcal{B}$ concerning the left quasigroups with (5.1) and an essentially surjective functor $J : \mathcal{B} \to \mathcal{H}$ (see [32, Proposition 6.17]). The functors $J : \mathcal{B} \to \mathcal{H}$, $S : \mathcal{A} \to \mathcal{D}$ in Section 2, and $F' : \mathcal{H} \to \mathcal{A}$ in Section 4, together with quasigroups of reflection [17,27], provide examples of the dynamical Yang-Baxter map.

The first task is to introduce a category $\mathcal{B}$. Let $L_1, L_2 = (L_2, *)$ be left quasigroups. We assume that the left quasigroup L_2 satisfies

$$(a * c)\backslash_{L_2}\big((a * b) * c\big) = (a' * c)\backslash_{L_2}\big((a' * b) * c\big), \quad \forall a, a', b, c \in L_2. \tag{5.1}$$

Here the symbol $\backslash_{L_2}$ is the left division (2.1) of L_2. Let $\pi : L_1 \to L_2$ be a (set-theoretic) bijection. An object of $\mathcal{B}$ is such a triple (L_1, L_2, π).

A morphism $f : (L_1, L_2, \pi) \to (L_1', L_2', \pi')$ is a homomorphism $f : L_1 \to L_1'$ of left quasigroups such that $\pi' \circ f \circ \pi^{-1} : L_2 \to L_2'$ is also a homomorphism of left quasigroups.

Proposition 17. *$\mathcal{B}$ is a category; the definitions of the identity, the source, the target and the composition are similar to those of the category $\mathcal{A}$.*

The next task is to construct a functor $J : \mathcal{B} \to \mathcal{H}$. Let $(L_1, (L_2, *), \pi) \in \mathcal{B}$. We define the ternary system $G_{L_2} = (G_{L_2}, \eta_{L_2})$ by $G_{L_2} = L_2$ as sets and

$$\eta_{L_2}(x, y, z) = z * \big(x\backslash_{L_2} y\big), \quad x, y, z \in G_{L_2}\big(= L_2\big). \tag{5.2}$$

Proposition 18. $(L_1, G_{L_2}, \pi) \in \mathcal{H}$.

Proof. It suffices to prove that G_{L_2} is a homogeneous pre-system satisfying (4.1).

We give a proof only for (3.2) because the rest of the proof is straightforward. Let $x, y, u, v, w \in G(= L_2)$. From (5.2) we have

$$\begin{aligned} \eta_{L_2}\big(x, y, \eta_{L_2}(u, v, w)\big) &= \big(w * \big(u \backslash_{L_2} v\big)\big) * \big(x \backslash_{L_2} y\big) \\ &= \big(w * \big(x \backslash_{L_2} y\big)\big) * \big(\big(w * \big(x \backslash_{L_2} y\big)\big) \backslash_{L_2} \big(\big(w * \big(u \backslash_{L_2} v\big)\big) * \big(x \backslash_{L_2} y\big)\big)\big). \end{aligned} \tag{5.3}$$

With the aid of (5.1), the right-hand side of (5.3) is

$$\begin{aligned} &\big(w * \big(x \backslash_{L_2} y\big)\big) * \big(\big(w * \big(x \backslash_{L_2} y\big)\big) \backslash_{L_2} \big(\big(w * \big(u \backslash_{L_2} v\big)\big) * \big(x \backslash_{L_2} y\big)\big)\big) \\ &\quad = \big(w * \big(x \backslash_{L_2} y\big)\big) * \big(\big(u * \big(x \backslash_{L_2} y\big)\big) \backslash_{L_2} \big(\big(u * \big(u \backslash_{L_2} v\big)\big) * \big(x \backslash_{L_2} y\big)\big)\big) \\ &\quad = \big(w * \big(x \backslash_{L_2} y\big)\big) * \big(\big(u * \big(x \backslash_{L_2} y\big)\big) \backslash_{L_2} \big(v * \big(x \backslash_{L_2} y\big)\big)\big), \end{aligned}$$

which is exactly $\eta_{L_2}(\eta_{L_2}(x, y, u), \eta_{L_2}(x, y, v), \eta_{L_2}(x, y, w))$. This is the desired conclusion. □

Let $f : (L_1, L_2, \pi) \to (L_1', L_2', \pi')$ be a morphism of the category $\mathcal{B}$. The map $f : L_1 \to L_1'$ is a homomorphism of left quasigroups. Moreover, $h := \pi' \circ f \circ \pi^{-1} : L_2 \to L_2'$ is a homomorphism of ternary systems from G_{L_2} to $G_{L_2'}$, because h is a homomorphism of left quasigroups. As a result, $f : (L_1, L_2, \pi) \to (L_1', L_2', \pi')$ is a morphism of the category $\mathcal{H}$.

We set $J(L_1, L_2, \pi) = (L_1, G_{L_2}, \pi)$ for $(L_1, L_2, \pi) \in \mathcal{B}$ and $J(f) = f$ for a morphism f of $\mathcal{B}$.

Proposition 19. *$J : \mathcal{B} \to \mathcal{H}$ is a functor.*

This functor J is essentially surjective. In fact, for any $(L, G, \pi) \in \mathcal{H}$, we can construct a left quasigroup L_2 such that $(L, L_2, \pi) \in \mathcal{B}$ and $J(L, L_2, \pi) = (L, G, \pi)$. We fix any element $\lambda_0 \in G$. Set $L_2 = G$ as sets and

$$a * b = \eta\big(\lambda_0, b, a\big), \quad a, b \in L_2(= G). \tag{5.4}$$

Due to (3.1) and (4.1), L_2 is a left quasigroup; its left division is defined by

$$a \backslash_{L_2} c = \eta\big(a, c, \lambda_0\big), \quad a, c \in L_2. \tag{5.5}$$

Proposition 20. $(L, L_2, \pi) \in \mathcal{B}$.

Proof. We need only show (5.1). Let $a, a', b, c \in L_2(= G)$. With the aid of (5.4) and (5.5) we have

$$(a * c) \backslash_{L_2} \big((a * b) * c\big) = \eta\big(\eta\big(\lambda_0, c, a\big), \eta\big(\lambda_0, c, \eta\big(\lambda_0, b, a\big)\big), \lambda_0\big). \tag{5.6}$$

From (3.2) and (4.1),

$$\begin{aligned} \eta\big(\lambda_0, c, \eta\big(\lambda_0, b, a\big)\big) &= \eta\big(\lambda_0, c, \eta\big(a', \eta\big(\lambda_0, b, a'\big), a\big)\big) \\ &= \eta\big(\eta\big(\lambda_0, c, a'\big), \eta\big(\lambda_0, c, \eta\big(\lambda_0, b, a'\big)\big), \eta\big(\lambda_0, c, a\big)\big). \end{aligned}$$

By taking into account (4.1) again, the right-hand side of (5.6) is

$$\eta\big(\eta\big(\lambda_0, c, a\big), \eta\big(\lambda_0, c, \eta\big(\lambda_0, b, a\big)\big), \lambda_0\big) = \eta\big(\eta\big(\lambda_0, c, a'\big), \eta\big(\lambda_0, c, \eta\big(\lambda_0, b, a'\big)\big), \lambda_0\big),$$

which is exactly the right-hand side of (5.1) by virtue of (5.4) and (5.5). This proves the proposition. □

It is immediate that $J(L, L_2, \pi) = (L, G, \pi)$, and consequently, the following holds.

Proposition 21. *The functor $J : \mathcal{B} \to \mathcal{H}$ is essentially surjective.*

Corollary 22. *The functor $SF'J : \mathcal{B} \to \mathcal{D}$ is essentially surjective.*

The final task of this section is to construct dynamical Yang-Baxter maps by means of the functor $SF'J : \mathcal{B} \to \mathcal{D}$ and quasigroups of reflection; see [17, Section 1].

Definition 23. A pair $(G, *)$ of a nonempty set G and a binary operation $(*)$ on G is called a quasigroup of reflection if and only if $(G, *)$ is a left quasigroup (Definition 1(1)) satisfying

$$x * x = x, \quad \forall x \in G,$$

$$(x * y) * y = x, \quad \forall x, y \in G, \tag{5.7}$$

$$(x * y) * z = (x * z) * (y * z), \quad \forall x, y, z \in G. \tag{5.8}$$

It follows from (5.7) that $(G, *)$ is a quasigroup (Definition 1(2)).

Remark 24. (1) The above definition is slightly different from that in [17] (see also [26, II.1.1] and [27, Section 1]); the order of the binary operation $(*)$ on G is reversed.

(2) The identity (5.8) is called a right distributive law [30, Section V.2].

(3) The quasigroup of reflection gives an involutory quandle [1,12,37] by reversing the order of the binary operation in Definition 23.

A straightforward computation shows that Nobusawa's quasigroup $(Q, *)$ in Example 2 is a quasigroup of reflection.

Let $(G, *)$ be a quasigroup of reflection, and L a left quasigroup isomorphic to G as sets. We denote by π a set-theoretic bijection from L to G. Because (5.8) immediately induces (5.1),

Proposition 25. $(L, G, \pi) \in \mathcal{B}$.

The quasigroup $G = (G, *)$ of reflection hence produces the dynamical Yang-Baxter map $R(\lambda)$ defined by $(L, R) = SF'J(L, G, \pi) \in \mathcal{D}$.

For example, let $L = (G, \cdot)$ denote the left quasigroup (4.4) and $\pi : L(= G) \to G$ the identity map on G. The above dynamical Yang-Baxter map $R(\lambda)$ induced by $(L, G, \pi) \in \mathcal{B}$ is

$$R(\lambda)(u, v) = \big(v, v * \big(u \backslash_G \lambda\big)\big), \quad \lambda, u, v \in L(= G). \tag{5.9}$$

For Nobusawa's quasigroup $Q = (Q, *)$, the corresponding dynamical Yang-Baxter map (5.9) is really dependent on the parameter λ; in fact,

$$R(1)(1, 1) = (1, 1), \quad R(2)(1, 1) = (1, 2).$$

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