In this paper, the differential transformation method is modified to be easily employed to solve wide kinds of nonlinear initial-value problems. In this approach, the nonlinear term is replaced by its Adomian polynomials for the index $k$, and hence the dependent variable components are replaced in the recurrence relation by their corresponding differential transform components of the same index. Thus the nonlinear initial-value problem can be easily solved with less computational effort. New theorems for product and integrals of nonlinear functions are introduced. In order to show the power and effectiveness of the present modified method and to illustrate the pertinent features of related theorems, several numerical examples with different types of nonlinearities are considered.

**Keywords:** Differential transform method; Adomian polynomials; Nonlinear equations

**AMS Subject Classifications:** 35Gxx, 45Jxx

**Abstract**

In this work, we use lower case letters for the original functions and upper case letters stand for the transformed functions.

\begin{equation}
\hat{y}(x) = \sum_{k=0}^{\infty} Y(k)(x - x_0)^k
\end{equation}

In this work, we use lower case letters for the original functions and upper case letters stand for the transformed functions.

\begin{enumerate}
  \item **Theorem 1.** If $y(x) = f(x) \pm h(x)$, then $Y(k) = F(k) \pm H(k)$.
  \item **Theorem 2.** If $y(x) = cf(x)$, then $Y(k) = cF(k)$, where $c$ is a constant.
  \item **Theorem 3.** If $y(x) = f^{(n)}(x)$, then $Y(k) = \frac{(k+n)!}{k!} F(k+n)$.
  \item **Theorem 4.** If $y(x) = f(x)h(x)$, then $Y(k) = \sum_{l=0}^{k} F(l)H(k-l)$.
  \item **Theorem 5.** If $y(x) = x^m$, then $Y(k) = \delta(k-m)$
\end{enumerate}

The above theorems can be deduced from equations (1) and (2).

**The Modified Differential Transform Method**

In this section, we will introduce a reliable and efficient algorithm to calculate the differential transform of a nonlinear function $g(y(x))$. The Adomian polynomials of this nonlinear function are determined formally as follows [12,13],

\begin{equation}
\tilde{A}_m = \frac{1}{n!} \left[ \frac{d^n}{dx^n} \left( g \left( \sum_{i=0}^{\infty} \tilde{A}_i y_i \right) \right) \right]_{x=a}
\end{equation}

\text{where } a \leq x \leq b.

**References**


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That is, the Adomian polynomials of $g(y(x))$ are
\[
\hat{A}_0 = g(y_0),
\]
\[
\hat{A}_1 = y_1 g^{(1)}(y_0),
\]
\[
\hat{A}_2 = y_2 g^{(2)}(y_0) + \frac{1}{2!} y_1^2 g^{(3)}(y_0),
\]
\[
\hat{A}_3 = y_3 g^{(3)}(y_0) + y_1 y_2 g^{(2)}(y_0) + \frac{1}{3!} y_1^3 g^{(5)}(y_0),
\]
\[
\hat{A}_4 = y_4 g^{(4)}(y_0) + (y_1 y_3 + \frac{1}{2!} y_1^2 y_2) g^{(3)}(y_0) + \frac{1}{2!} y_1^4 g^{(6)}(y_0),
\]
\[
\hat{A}_5 = y_5 g^{(5)}(y_0) + (y_1 y_4 + y_1 y_2 y_2) g^{(4)}(y_0) + \frac{1}{2!} (y_1)^2 y_3 g^{(4)}(y_0) + \frac{1}{3!} (y_1)^3 y_2 g^{(5)}(y_0) + \frac{1}{4!} (y_1)^4 y_1 g^{(6)}(y_0),
\]
and so on.

6.1 Lemma: If $f(x) = g(y(x))$, then $F(k) = A_k$ where $A_k$ are the Adomian polynomials $\hat{A}_k$ but with replacing $y_k$ by $Y(k), k = 0, 1, 2, \cdots$

6.2 Proof: The differential transforms of $f(x)$ are computed by utilizing (1) as
\[
F(0) = \frac{1}{0!} [g(y(x))]_{x=x_0} = g(y(x_0)) = g(Y(0)) = A_0,
\]
\[
F(1) = \frac{1}{1!} \left[ \frac{d}{dx} g(y(x)) \right]_{x=x_0} = y^{(1)}(x_0) g^{(1)}(y(x_0)) = Y(1) g^{(1)}(Y(0)) = A_1,
\]
\[
F(2) = \frac{1}{2!} \left[ \frac{d^2}{dx^2} g(y(x)) \right]_{x=x_0} = \frac{1}{2!} \left[ y^{(2)}(x_0) g^{(3)}(y(x_0)) + (y^{(1)}(x_0))^2 g^{(2)}(y(x_0)) \right] = Y(2) g^{(3)}(Y(0)) + \frac{1}{2!} Y(1)^2 g^{(4)}(Y(0)) = A_2,
\]
\[
F(3) = \frac{1}{3!} \left[ \frac{d^3}{dx^3} g(y(x)) \right]_{x=x_0} = \frac{1}{3!} \left[ y^{(3)}(x_0) g^{(5)}(y(x_0)) + 3 y^{(1)}(x_0) y^{(2)}(y(x_0)) + (y^{(1)}(x_0))^3 g^{(4)}(y(x_0)) \right] = Y(3) g^{(5)}(Y(0)) + \sum_{k=0}^{2} Y(k) Y(2)^{k+1} g^{(4)}(Y(0)) + \frac{1}{3!} Y(1)^3 g^{(6)}(Y(0)) = A_3.
\]
In general we have, $F(k) = A_k$.

Consequently, the inverse transform of the nonlinear function can be written as
\[
f(x) = g(y(x)) = \sum_{k=0}^{\infty} A_k (x - x_0)^k,
\]
where $A_k$ are the Adomian polynomial of $f(x) = g(y(x))$.

The advantage of using this algorithm for computing differential transformation of nonlinear functions comparing with the algorithm suggested in [7], is this algorithm dealing directly with nonlinear function of the problem in hand in its form without any differentiation or algebraic manipulations or even there is no need to compute the differential transform of other functions to obtain the required one. This will be clear throughout the following theorems.

6.3 Theorem 6. If $f(x) = h(x) g(y(x))$, then $F(k) = \sum_{k=0}^{\infty} H(k) A_{k-k}$.

6.4 Proof: By utilizing definition (1), we can get
\[
F(0) = \frac{1}{0!} [h(x) g(y(x))]_{x=x_0} = h(x_0) g(y(x_0)) = H(0) g(Y(0)) = H(0) A_0,
\]
\[
F(1) = \frac{1}{1!} \left[ \frac{d}{dx} \left( h(x) g(y(x)) \right) \right]_{x=x_0} = h^{(1)}(x_0) g(y(x_0)) + h(x_0) y^{(1)}(x_0) g^{(1)}(y(x_0)) + \sum_{k=0}^{1} h^{(1)}(x_0) g^{(k+1)}(y(x_0)) A_k = H(1) A_0 + H(0) A_1,
\]
\[
F(2) = \frac{1}{2!} \left[ \frac{d^2}{dx^2} \left( h(x) g(y(x)) \right) \right]_{x=x_0} = \sum_{k=0}^{2} h^{(2)}(x_0) g(y(x_0)) + \sum_{k=0}^{1} h^{(1)}(x_0)y^{(1)}(x_0) g^{(k+1)}(y(x_0)) + \sum_{k=0}^{1} y^{(1)}(x_0) g^{(k+1)}(y(x_0)) = H(2) A_0 + H(1) A_1 + H(0) A_2,
\]
\[
F(3) = \frac{1}{3!} \left[ \frac{d^3}{dx^3} \left( h(x) g(y(x)) \right) \right]_{x=x_0} = \sum_{k=0}^{3} h^{(3)}(x_0) g^{(k+1)}(y(x_0)) + \sum_{k=0}^{1} h^{(1)}(x_0) y^{(1)}(x_0) g^{(k+1)}(y(x_0)) + \sum_{k=0}^{1} y^{(1)}(x_0) g^{(k+1)}(y(x_0)) = H(3) A_0 + H(2) A_1 + H(1) A_2 + H(0) A_3.
\]
In general we have, $F(k) = \sum_{k=0}^{\infty} H(k) A_{k-k}$.

6.5 Theorem 7. If $f(x) = \int_{x_0}^{x} g(y(t)) dt$, then $F(k) = \frac{A_k - 1}{k}, k \geq 1$.

6.6 Proof: By using (3), the transform of the integral can be found as
\[
f(x) = \sum_{k=0}^{\infty} \int_{x_0}^{x} A_k (t - x_0)^k dt = \sum_{k=0}^{\infty} \int_{x_0}^{x} A_k (t - x_0)^k dt = \sum_{k=0}^{\infty} \frac{A_k - 1}{k} (t - x_0)^k,
\]
Again utilizing (3), we get $F(k) = \frac{A_k - 1}{k}$, where $k \geq 1$ and $F(0) = f(x_0) = 0$.

6.7 Theorem 8. If $f(x) = \int_{x_0}^{x} g(y(t)) dt$, then
\[
F(k) = \sum_{k=0}^{\infty} \int_{x_0}^{x} g(y(t)) dt = \sum_{k=0}^{\infty} \frac{A_k - 1}{k} (t - x_0)^k.
\]
then $F(k) = \sum_{k_1=1}^{k} \frac{1}{k_1!} H(k-k_1) A_{k_1-1}$, \( k \geq 1 \)

6.8 Proof: Utilizing the definition of the transform, we can get

$$F(0) = \frac{1}{0!} \left\{ h(x) \int_{x_0}^{x} g(y(t)) dt \right\} |_{x=x_0} = 0 ,$$

$$F(1) = \frac{1}{1!} \left\{ \frac{d}{dx} \left[ h(x) \int_{x_0}^{x} g(y(t)) dt \right] \right\} |_{x=x_0}$$

$$= \left[ h^{(1)}(x) \int_{x_0}^{x} g(y(t)) dt + h(x) g(y(x)) \right] |_{x=x_0}$$

$$= H(0) A_0 ,$$

$$F(2) = \frac{1}{2!} \left\{ \frac{d^2}{dx^2} \left[ h(x) \int_{x_0}^{x} g(y(t)) dt \right] \right\} |_{x=x_0}$$

$$= \frac{1}{2!} \left\{ 2h^{(2)}(x_0) g(y(x_0)) + h(x_0) y^{(1)}(x_0) g^{(1)}(y(x_0)) \right\}$$

$$= H(1) A_0 + H(0) A_1 / 2 ,$$

$$F(3) = \frac{1}{3!} \left\{ \frac{d^3}{dx^3} \left[ h(x) \int_{x_0}^{x} g(y(t)) dt \right] \right\} |_{x=x_0}$$

$$= \frac{1}{3!} \left\{ 3h^{(3)}(x_0) g(y(x_0)) + 3h^{(2)}(x_0) y^{(1)}(x_0) g^{(1)}(y(x_0)) \right.\right.$$\left.$$+ h(x_0) y^{(2)}(x_0) g^{(1)}(y(x_0)) \right.$$\right.$$+ \left.$$y^{(1)}(x_0)^2 g^{(2)}(y(x_0)) \right\}$$

$$= H(2) A_0 + \frac{1}{2} H(1) A_1 + \frac{1}{3} H(0) A_2 ,$$

In general we have, $F(k) = \sum_{k_1=1}^{k} \frac{1}{k_1!} H(k-k_1) A_{k_1-1}$, where $k \geq 1$.

6.9 Theorem 9. If \( f(x) = \int_{x_0}^{x} g_1(t) g_2(y(t)) dt \), then

$F(k) = \sum_{k_1=1}^{k} \frac{1}{k_1!} H(k-k_1) A_{k_1-1} \cdot k \geq 1 \cdot$

6.10 Proof: Utilizing the definition of the transform, we can get

$$F(0) = \frac{1}{0!} \left\{ \int_{x_0}^{x} g_1(t) g_2(y(t)) dt \right\} |_{x=x_0} = 0 ,$$

$$F(1) = \frac{1}{1!} \left\{ \frac{d}{dx} \left[ \int_{x_0}^{x} g_1(t) g_2(y(t)) dt \right] \right\} |_{x=x_0}$$

$$= g_1(x_0) g_2(y(x_0)) = G_1(0) A_0 ,$$

$$F(2) = \frac{1}{2!} \left\{ \frac{d^2}{dx^2} \left[ \int_{x_0}^{x} g_1(t) g_2(y(t)) dt \right] \right\} |_{x=x_0}$$

$$= \frac{1}{2!} \left\{ g_1^{(1)}(x_0) g_2(y(x_0)) + g_1(x_0) g_2^{(1)}(y(x_0)) \right\}$$

$$= [G_1(1) A_0 + G_1(0) A_1] / 2 ,$$

$$F(3) = \frac{1}{3!} \left\{ \frac{d^3}{dx^3} \left[ \int_{x_0}^{x} g_1(t) g_2(y(t)) dt \right] \right\} |_{x=x_0}$$

$$= \frac{1}{3!} \left\{ g_1^{(2)}(x_0) g_2(y(x_0)) + 2 g_1^{(1)}(x_0) y^{(1)}(x_0) g_2^{(1)}(y(x_0)) \right.$$\right.$$+ g_1(x_0) [y^{(2)}(x_0) g_2^{(1)}(y(x_0)) \right.$$\right.$$+ (y^{(1)}(x_0)^2) g_2^{(2)}(y(x_0))]$$

$$= [G_1(2) A_0 + G_1(1) A_1 + G_1(0) A_2] / 3 ,$$

In general we have, $F(k) = \sum_{k_1=1}^{k} \frac{1}{k_1!} G_1(k_1-1) A_{k_1-1} H(k-k_1)$.
\[ \begin{align*}
1 + & \left( \frac{3}{3!} \right) \left( x_0 x_1^2 \right) \left( y(x_0) \right) + \left( \frac{3}{3!} \right) \left( x_0 x_1^2 \right) \left( y(x_0) \right) + \\
& + \frac{h(x_0)}{2} \left( x_0 x_2^2 \right) \left( y(x_0) \right) + \\
& + \frac{h(x_0)}{2} \left( x_0 x_2^2 \right) \left( y(x_0) \right) + \\
& + \left( \frac{1}{1} \right) \left( x_0 x_2^2 \right) \left( y(x_0) \right) \\
& = H(2)G_1(0)A_0 + \\
& \left[ H(1)G_1(0)A_0 + \frac{1}{2} H(0)G_2(1)A_0 + \frac{1}{2} H(0)G_2(1)A_0 + \frac{1}{2} H(0)G_2(1)A_0 + \frac{1}{2} H(0)G_2(1)A_0 \right]^{1/3},
\end{align*} \]

In general we have, \( F(k) = \sum_{k_1=1}^{k_0} \sum_{k_2=1}^{k_0} \frac{1}{n^2} G_1((k_1-1)A_{k_2-k_1} H(k-k_2)). \)

**Applications and Numerical Results**

In this section, we implement the proposed method on some different examples with different types of nonlinearity.

**7.1 Example 1.** Consider the nonlinear Volterra integro-differential equation

\[ y''(x) + y'(x) y(x) + y(x) = \cos 2x + \int_0^x 1 + \sin 2t \, dt, \quad 0 \leq x \leq 1 \]  

(4)

with the initial conditions \( y(0) = 1 \) and \( y'(0) = 1. \)

The differential transformation of equation (4) and the initial conditions (5) are

\[ \begin{align*}
Y(k+2) &= \frac{k!}{(k+2)!} \sum_{m=0}^{k} \frac{(m+1)(m+2)Y(k-m) - Y(k)}{2}, \\
A_{k-2} &= \frac{1}{k-2} \sum_{m=0}^{k-2} \frac{2^m m!}{(m+1)!} \sin \left( \frac{\pi (m+1)}{2} \right) A_{k-m-2},
\end{align*} \]

where \( [2^k \cos(\pi k/2)]/k! \) and \( [2^k \sin(\pi k/2)]/k! \) are the differential transforms of \( \cos(2x) \) and \( \sin(2x) \), respectively and \( A_k \) are the differential transform of the nonlinear function \( g(y) = y^2 \), and \( Y(0) = Y'(0) = 1. \) Using the Lemma, \( A_0 = g(Y(0)) = 1, \) \( A_1 = -2Y(1), \) \( A_2 = -2Y(2) + 3Y^2(1), \) \( A_3 = -2Y(3) + 6Y(Y(2)) - 4Y^3(1), \) \( A_4 = -2Y(4) - 2Y(1)Y(3) - Y^2(2) + 3Y^2(1)Y(2) + 5Y^4(1), \) \( \ldots \)

Utilizing the recurrence relation, the transformed initial conditions and \( A_k, \ Y(k) \) are evaluated. Hence using the inverse transformation formula, the following series solution up to \( O(x^{10}) \) can be obtained

\[ y(x) = 1 + x - \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!} - \frac{x^7}{7!} + \frac{x^8}{8!} + O(x^{10}). \]

For sufficiently large number of terms, the closed form of the solution is \( y(x) = \sin x + \cos x \), which is the exact solution. Table 1 shows the absolute relative error obtained for three various numbers of terms and at some test points.

**7.2 Example 2.** Consider the nonlinear Volterra integro-differential equation

\[ 6(x^2 + 1)y''(x) = (x^3 + 3x^2 + 6x + 6)e^{-x} + \int_0^x \left[ 6 \cos(2 \pi) \right] dt, \quad 0 \leq x \leq 1, \]  

(6)

with the initial condition \( y(0) = 0. \)

The differential transformation of equation (6) and the initial condition (7) are

\[ \begin{align*}
Y(k+1) &= -\frac{k-1}{k+1} Y(k-1) + \frac{1}{6(k+1)} \left[ (-1)^k (6-11k+6k^2-k^3) + A_{k-4} \right],
\end{align*} \]

(8)

where \( \lambda/k! \) are the differential transforms of \( e^{2x} \), \( A_k \) are the differential transforms the nonlinear function \( g(y) = e^{-\tan y} \) and \( Y(0) = 0. \)

If we put \( x = 0 \) into equation (6), we can get \( y'(0) = 1 \) and hence \( Y(1) = 1. \)

The following system for \( k = 1, 2, 3, \ldots, 8 \) is obtained from (8)

\[ Y(2) = 0, \]  

\[ Y(3) = -\frac{1}{3} Y(1), \]  

\[ Y(4) = -\frac{2}{5} Y(2), \]  

\[ Y(5) = -\frac{3}{5} Y(3) + \frac{1}{6(5)} \left[ \frac{6}{4!} + \frac{A_1}{4} \right], \]  

\[ Y(6) = -\frac{4}{6} Y(4) + \frac{1}{6(6)} \left[ \frac{24}{5!} + \frac{A_2}{5} \right], \]  

\[ Y(7) = -\frac{5}{7} Y(5) + \frac{1}{6(7)} \left[ \frac{60}{6!} + \frac{A_3}{6} \right]. \]

**Table 1:** Numerical comparison of results in example 1.
where differential transform components \( A_k \) are:\n\[
A_0 = e^{-\tan(Y(0))} = 1, \\
A_1 = Y(1), \quad A_2 = Y^2(1) - Y(2), \quad A_3 = -(1/2)Y^3(1) + Y(1) \quad Y(2) - Y(3), \\
A_4 = Y(1)Y(3) + (1/2)Y^2(2) - \\
(3/2)Y^2(1)Y(2) + (3/8)Y^4(1) - Y(4) .
\]

By solving the above systems for \( Y(k) \), the series solution of problem (6) and (7) up to \( O(x^{10}) \) is given by
\[
y(x) = x - x^3/3 + x^5/5 - x^7/7 + x^9/9 + O(x^{10}) .
\]

For sufficiently large terms, the closed form of the solution is \( y(x) = \tan^{-1} x \), which is the exact solution. Table 2 shows the absolute relative error obtained for three various numbers of terms and at some test points.

7.3 Example 3. Let us consider the nonlinear Volterra integro-differential equation
\[
y''(x) - 2y(x)y'(x) = -x + \int_0^x \frac{y(t)}{1 + y^2(t)} dt, \quad 0 \leq x \leq 1 ,
\]
with the initial conditions
\[
y(0) = 0 , \quad \text{and} \quad y'(0) = 1 .
\]

The differential transformation of this equation and its initial conditions are
\[
Y(k + 2) = \frac{k!}{(k + 2)!} \left[ 2 \sum_{m=0}^{k} \left( \frac{(m + 1)Y(m + 1)Y(k - m) - \delta(k - 1) + \sum_{m=0}^{k} (m + 1)Y(m + 1)A_{k-m-1} \right) \right] ,
\]
\[
Y(0) = 0 \quad \text{and} \quad Y(1) = 1 .
\]

\( A_k \) can be obtained by using Lemma as:\n\[
A_0 = (1 + Y^2(0))^{-1} = 1, A_1 = 0, \\
A_2 = -Y^2(1), \quad A_3 = -2Y(1)Y(2), \quad A_4 = -2(1 + Y(3) + Y^2(2)) + Y^4(1), \ldots .
\]

By solving for \( Y(k) \), the series solution of problem (9) and (10) up to \( O(x^{10}) \) is given by
\[
y(x) = x + \frac{x^3}{3} + \frac{x^5}{15} - \frac{x^7}{315} + \frac{62x^9}{2835} + O(x^{10}) .
\]

For sufficiently large number of terms, the closed form of the solution is \( y(x) = \tan^{-1} x \), which is the exact solution. Table 3 shows the absolute relative error obtained for three various numbers of terms and at some test points.

7.4 Example 4. Consider the initial-value problem of Bratu-type [7]
\[
y''(x) - 2e^{y(x)} = 0 , \quad 0 \leq x \leq 1 , (11) \\
y(0) = 0 , \quad \text{and} \quad y'(0) = 0 . (12)
\]

The differential transformation of this equation and its initial conditions are
\[
Y(k + 2) = \frac{1}{(k + 1)(k + 2)} \left[ \sum_{k_1=0}^{k} \sum_{k_2=0}^{k-k_1} 2Y(k + 4) - \delta(k - 1) + \right] , \\
Y(0) = 1 \quad \text{and} \quad Y(1) = 0 .
\]

\( A_k \) are:\n\[
A_0 = \ln Y(0), \quad A_1 = Y(1)Y(0), \\
A_2 = Y(2)/Y(0) - (Y(1)/Y(0))^2 / 2 , \\
A_3 = Y(3)/Y(0) + Y(1)Y(2)/(Y(0))^2 -
(1)/(Y(0))^3 / 3
\]

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Table 2: Numerical comparison of results in example 2.
A_4 = (Y(4)/Y(0) - [2Y(1)] Y(3) + [Y(2)]^2)/Y(0)^2 + \ldots

The following differential transform components are obtained: Y(2) = 1, Y(4) = 1/2, Y(6) = 1/6, Y(8) = 1/24, \ldots The series solution of problem (13) and (14) up to O(x^{10}) is given by

y(x) = 1 + x^2 + \frac{1}{2} x^4 + \frac{1}{6} x^6 + \frac{1}{24} x^8 + O(x^{10}).

This is the same result with that obtained by [7]. The closed form solution of this problem is y(x) = e^x.

7.6 Example 6. Consider the nonlinear initial-value problem

\[ y'(x) = \frac{9}{4} \sqrt{y(x)} + y(x), \quad y(0) = 1, \quad y'(0) = 2. \]

The differential transformation of this problem are

\[ Y(k+2) = \frac{9A_k + 4Y(k)}{4(k+1)(k+2)}, \quad Y(0) = 1 \text{ and } Y(1) = 2. \]

where A_k are: A_0 = \sqrt{Y(0)}, A_1 = (1/2) \sqrt{Y(0)}), A_2 = Y(2)/2Y(0) - (Y(1))^2/(2Y(0)), A_3 = Y(3)/2Y(0) - Y(1)Y(2)/(4Y(0))^{3/2},
\[ (Y(1))^2/(16Y(0))^{5/2} \]
\[ A_4 = Y(4)/2Y(0) - [2Y(1)] Y(3) + [Y(2)]^2)/Y(0)^2 + \ldots
\]

The following differential transform components are obtained:

\[ Y(2) = 13/8, Y(3) = 17/24, Y(4) = 149/768, Y(5) = 77/1920, Y(6) = 641/92160, Y(7) = 317/322560, Y(8) = 2609/20643840, Y(9) = 1277/92897280, \ldots \]

The series solution of problem (15) and (16) up to O(x^{10}) is given by

\[ y(x) = 1 + x^2 + \frac{13}{8} x^4 + \frac{17}{24} x^6 + \frac{149}{768} x^8 + \frac{77}{1920} x^{10} + \frac{641}{92160} x^{12} \]

The exact solution of this example is

\[ y(x) = 9/4 \left[ \frac{3}{2} x^{5/2} + \frac{1}{6} e^{x/2} - 1 \right]^2. \]