Design of Three-Dimensional Missile Guidance Law via Sliding Mode with Saturation Constraint

Amir Torabi*, Amin Adine Ahari and Ali Karsaz

Faculty of Electrical Engineering, Khorasan University, Mashhad, Iran

Abstract

The three-dimensional missile guidance law with saturating actuator is proposed for the pursuit of an unpredictably maneuvering target. The proposed guidance law achieves a satisfactorily robust performance without having to solve a complicated nonlinear Hamilton-Jacobi Partial Differential Inequality (HJPDI). In the proposed guidance scheme, the nonlinear part of the relative motion equations between the missile and target is first cancelled. Then, a tunable sliding mode law is designed to eliminate the effect of external disturbance. The proposed guidance law can achieve reduced interception time than others, yet it is considerably simpler. An example is proposed to confirm it.

Keywords: Missiles; Sliding mode law

Nomenclature

r- Relative distance between the missile and the target

Φ-Pitch line-of-sight angle (PLOS)

θ-Yaw line-of-sight angle (YLOS)

$\ddot{r}$ -Relative acceleration along LOS

$\dot{\Phi}$ -Angular velocity of Φ

$\dot{\theta}$ -Angular velocity of θ

$\ddot{\Phi}$ -Angular acceleration of Φ

$\ddot{\theta}$ -Angular acceleration of θ

$\vec{e}_r$ -Unit vector along LOS

$\vec{e}_\theta$ -Unit vector along YLOS

$\vec{e}_\Phi$ -Unit vector along PLOS

$\vec{a} = w_r \vec{e}_r + w_\theta \vec{e}_\theta + w_\Phi \vec{e}_\Phi$ -Acceleration vector of target

$\vec{a}_M = u_r \vec{e}_r + u_\theta \vec{e}_\theta + u_\Phi \vec{e}_\Phi$ -Acceleration vector of missile

$V_r = \dot{r}$ -Relative velocity along to LOS

$V_\Phi = r\dot{\Phi}$ -Relative velocity normal to PLOS

$V_\theta = r\dot{\theta}\cos\Phi$ -Relative velocity normal to YLOS

Introduction

The missile guidance is a well-known nonlinear control engineering area of research. Since the basic principles were extensively covered by Locke [1] and Lin [2], many technologies have been developed to improve guidance performance and to overcome environmental disturbances.

Various guidance laws have been exploited with different design concepts over the years [2]. Currently, the most popular terminal guidance laws, as defined by Locke [1], involve Line-Of-Sight (LOS) guidance, LOS rate guidance and other advanced guidance such as proportional navigation guidance [1], Augmented Proportional Navigation Guidance (APNG) [3], optimal guidance law based on the linear quadratic regulator theory [4] and linear exponential Gaussian theory [5].

The variable structure control law has been applied to many guidance

problems. For example, Zhou et al. [6] constructed the adaptive sliding-mode guidance using linearised equations. Babu et al. [7] proposed the guidance law against highly maneuvering targets using the sliding surface of the zero LOS rate based on the Lyapunov method. Brierly and Longchamp [8] presented the sliding mode guidance law including rigid-body dynamics and actuator time delay. Moon et al. [9] studied the sliding-mode missile guidance law via variable structure control.

Recently, Chen et al. [10] derived a new nonlinear fuzzy H_∞ guidance law with a saturation of actuators against maneuvering targets. They first approximated the nonlinear missile target pursuit dynamics with the Takagi - Sugeno linear fuzzy models. Then, based on this, they constructed a H_∞ control to handle the approximation error and external disturbances. However, it is a known fact that the approximation error between the fuzzy model and the true one is difficult to obtain exactly [11].

In this paper, sliding mode guidance law based on the three-dimensional (3-D) nonlinear engagement kinematics is proposed. The purpose of the proposed control design is to attenuate the influence of the target's maneuvers on the performance of the guidance law. That is to say that the proposed guidance is not only robust with respect to target maneuvering, but is also guaranteed state boundedness (Lyapunov stability).

Statement of the Problem

For a 3-D interception, relative motion between the missile and target is described by the spherical coordinates system (r , θ, Φ) with the origin fixed on the location of the missile.

The 3-D missile - target engagement is shown in Figure 1, where a missile is attempting to intercept a moving target.

The missile and target are assumed to be point masses in order to
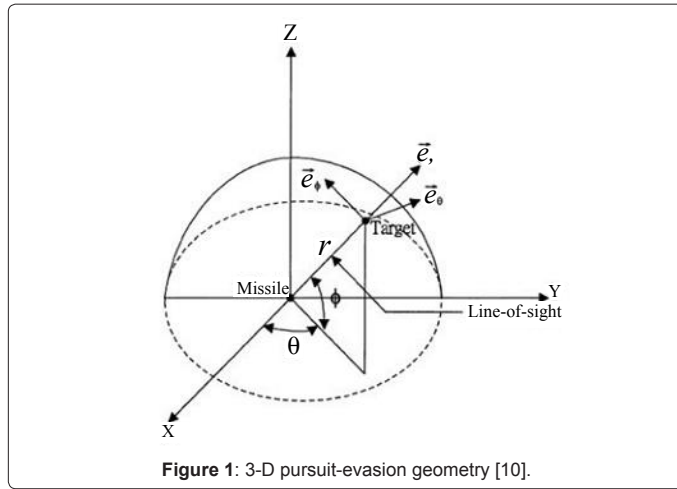

Figure 1: 3-D pursuit-evasion geometry [10].

easily analyze the missile guidance. The governing equations of the relative motion [10] can be derived as:

$$\ddot{r} - r\dot{\varphi}^2 - r\dot{\theta}^2 \cos^2\varphi = w_r - u_r$$

$$r\ddot{\theta}\cos\varphi + 2\dot{r}\dot{\theta}\cos\varphi - 2r\dot{\varphi}\dot{\theta}\sin\varphi = w_\theta - u_\theta$$

$$r\ddot{\varphi} + 2\dot{r}\dot{\varphi} + r\dot{\theta}^2 \cos\varphi\sin\varphi = w_\varphi - w_\varphi \qquad (1)$$

where r is the relative distance between the missile and target; u and f are the aspect angles (Figure 1); w_r, w_θ and w_Φ are the target's acceleration components; u_r, u_θ and u_Φ are the missile's acceleration components, that are to be found, to the actuators with a saturation constraint that is defined as follows:

$$sat\left(u(t)\right) = [sat(u_r) sat(u_\theta) sat(u_\varphi)]^T \qquad (2)$$

$$sat(u_i(t)) = \begin{cases} \underline{u_i} & if u_i < \underline{u_i} \\ u_i & if \underline{u_i} \leq u_i \leq \overline{u_i} \\ \overline{u_i} & if \overline{u_i} < u_i \end{cases} \qquad (3)$$

For $i = r, \theta$ and φ

The kinematics (1) with input saturation constraint can be rewritten as the following nonlinear state-space equation [10].

$$\dot{x}(t) = F(x(t)) + BSat(u(t)) + Dw(t) \qquad (4)$$

$$x(t) = \begin{bmatrix} r \\ \theta \\ \varphi \\ v_r \\ v_\theta \\ v_\varphi \end{bmatrix}, F(x(t)) = \begin{bmatrix} v_r \\ \dfrac{v_\theta}{\cos\varphi} \\ \dfrac{v_\varphi}{r} \\ \dfrac{v_\varphi^2 + v_\theta^2}{r} \\ -\dfrac{v_r v_r}{r} + \dfrac{v_\theta v_\varphi \tan\varphi}{r} \\ -\dfrac{v_r v_\varphi}{r} - \dfrac{v_\theta^2 \tan\varphi}{r} \end{bmatrix}$$

$$u(t) = \begin{bmatrix} u_r \\ u_\theta \\ u_\Phi \end{bmatrix}, w(t) = \begin{bmatrix} w_r \\ w_\theta \\ w_\Phi \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Robust guidance law under unpredictable target acceleration $w=0$

We now prove that the proposed tunable guidance law can robustly stabilize the nonlinear missile-target dynamics (4) whether target acceleration w exists or not. In this section, we discuss the case of non-target acceleration, $w=0$.

A good guidance law must guarantee a decreasing relative distance r and, at the same time, keep the LOS aspect angular rate as small as possible. It is equivalent to satisfying the following conditions:

V_r- radial relative velocity converges to negative value (5)

V_θ and V_Φ- tangential relative velocities, converge to zero (6)

When V_θ and V_Φ converge to zero, it means that the missile and target are on a convergence course. Only when the radial relative velocity, V_r, has decreased to a negative value, can the relative distance r between the missile and target decrease to zero.

Moon et al. [9] designed the guidance law for a two-dimensional air-to-air engagement whose state-space form is shown as

$$\begin{bmatrix} \dot{r} \\ \dot{V_r} \\ \dot{V_\theta} \end{bmatrix} = \begin{bmatrix} V_r \\ \dfrac{V_\theta^2}{r} \\ -\dfrac{V_r V_\varphi}{r} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} W - \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u \qquad (7)$$

They used variable structure control to design the control $\bar{u}$ in the form

$$\bar{u}\begin{bmatrix} \bar{u}_r \\ \bar{u}_\theta \end{bmatrix} = \begin{bmatrix} \dfrac{V_\theta^2}{r} + \cdots \\ -\dfrac{v_r v_\varphi}{r} \end{bmatrix} \qquad (8)$$

From the control law (8) handling two-dimensional dynamics (7), it is reasonable to extend the form similar to (8) for the 3-D dynamics (4). Thus, the completely new control law, a form similar to $\tilde{u}$ in (8), is constructed mixed with some tunable parameters as follows

$$u\begin{bmatrix} u_r \\ u_\theta \\ u_\Phi \end{bmatrix} = \begin{bmatrix} \dfrac{v_\theta^2 + v_\varphi^2}{r} + \alpha_1 V_r + \alpha_2 \\ -\dfrac{v_r v_r}{r} + \dfrac{v_\theta v_\varphi \tan\varphi}{r} + \alpha_3 V_\theta + \alpha_4 \\ -\dfrac{v_r v_\varphi}{r} - \dfrac{v_\theta^2 \tan\varphi}{r} + \alpha_5 V_\Phi + \alpha_6 \end{bmatrix} \qquad (9)$$

Where α_i, $i = 1 - 6$, are positive real constants. The saturation constraint of the actuators is given by (3). Suppose that the controller (9) lies between the upper and lower bound of saturation constraint (3). Substituting (9) into (4) yields the closed-loop system as

$$\begin{bmatrix} \dot{r} \\ \dot{\theta} \\ \dot{\varphi} \\ \dot{v_r} \\ \dot{v_\theta} \\ \dot{v_\varphi} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{1}{r\cos(\varphi)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dfrac{1}{r} \\ 0 & 0 & 0 & -\alpha_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\alpha_3 \end{bmatrix} \begin{bmatrix} r \\ \theta \\ \varphi \\ v_r \\ v_\theta \\ v_\varphi \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\alpha_2 \\ -\alpha_4 \\ -\alpha_6 \end{bmatrix}, \begin{bmatrix} r_0 \\ \theta_0 \\ \varphi_0 \\ v_{r0} \\ v_{\theta 0} \\ v_{\varphi 0} \end{bmatrix} \neq 0 \qquad (10)$$

Now consider a Lyapunov function candidate $V = 0.5(r^2 + \theta^2 + \varphi^2 + V_r^2 + V_\theta^2 + V_\varphi^2)$ for (10). Differentiating V with

respect to time gives

$$\dot{V} = (r - \alpha_1 V_r - \alpha_2)V_r + (\frac{\theta}{r\cos(\varphi)} - \alpha_3 V_\theta - \alpha_4) + (\frac{\Phi}{r} - \alpha_5 V_\varphi - \alpha_6) \quad (11)$$

In order to satisfy Lyapunov stability for (11), $\dot{V} \leq 0$

Next, the parameters (α_i, i=1-6) in (9) are considered. Rewriting the dynamics in (10)

$$\dot{r} = V_r \qquad r_0 \neq 0 (12a)$$
$$\dot{v}_r = -\alpha_1 V_r - \alpha_2 \quad v_{r0} = 0 (12b)$$
$$\dot{v}_\theta = -\alpha_3 V_\theta - \alpha_4 \quad v_{\theta 0} = 0 (12c)$$
$$\dot{v}_\varphi = -\alpha_5 V_\varphi - \alpha_6 \quad v_{\Phi 0} = 0 (12d)$$

where the solutions are shown as

$$r = \frac{-(v_{r0} + (\alpha_2/\alpha_1))}{\alpha_1}e^{-\alpha_1 t} - \frac{\alpha_2}{\alpha_1}t + r_0 \quad (13a)$$

$$V_r = (v_{r0} + \frac{\alpha_2}{\alpha_1})e^{-\alpha_1 t} - \frac{\alpha_2}{\alpha_1} \quad (13b)$$

$$V_\theta = (v_{\theta 0} + \frac{\alpha_2}{\alpha_1})e^{-\alpha_3 t} - \frac{\alpha_4}{\alpha_3} \quad (13c)$$

$$V_\Phi = (v_{\Phi 0} + \frac{\alpha_2}{\alpha_1})e^{-\alpha_5 t} - \frac{\alpha_6}{\alpha_5} \quad (13d)$$

To tune the positive real numbers (α_1, α_3, α_5) is equivalent to adjusting the convergent rate of ($V_r V_\theta$, V_φ) in (13). The tunable parameters (α_1, α_3, α_5) affect the final convergent state ($V_r V_\theta$, V_φ) in (13). From (12b), it is known that the larger parameter α_1 may cause quicker stabilization of V_r and the larger α_2 may cause V_r to converge to a more negative value. From (12a), it is obvious that a more negative V_r would accelerate the convergence of the relative distance, r, between the missile and target, i.e., a good performance (less intercept time) for a non-accelerating target, w=0, is achieved. This is a completely new and original 3-D guidance design for tuning (α_1, α_2) so as to obtain less interception time and to tune (α_3, α_4, α_5, α_6) to handle the convergence of states in (10).

Parameter α_1 is concerned with the convergence rate of V_r in (13b). A larger α_2 may cause V_r to converge to a more negative value. α_3>0, α_5>0 and α_4=α_6=0 in (12) guarantees that V_θ and V_φ decrease to zero. In other words, the guidance law (9) with α_1>0, α_2>0 α_3>0, α_5>0 and α_4=α_6=0 can simultaneously satisfy the good guidance demand (5) and (6). Furthermore, the abrupt and large guidance command during the initial maneuver can also be avoided by tuning the four parameters (α_1, α_2, α_3, α_4). Thus, the proposed control law becomes

$$u = \begin{bmatrix} u_r \\ u_\theta \\ u_\Phi \end{bmatrix} = \begin{bmatrix} \dfrac{v_\theta^2 + v_\theta^2}{r} + \alpha_1 V_r + \alpha_2 \\ -\dfrac{v_r v_r}{r} + \dfrac{v_\theta v_\varphi \tan\varphi}{r} + \alpha_3 V_\theta \\ -\dfrac{v_r v_\varphi}{r} + \dfrac{v_\theta^2 \tan\varphi}{r} + \alpha_5 V_\Phi \end{bmatrix} \quad (14)$$

The parameters (α, i=1, 2, 3, 5) of the control law are tuned by the binary search. This is an off-line tuning algorithm. If the performance demand is acceptable, the parameter's selection is complete. There are many search algorithms that can be used to execute the tuning algorithm, for example, the genetic algorithm, binary search algorithm, and evolution algorithm. This paper adopts the binary search algorithm.

Robust guidance law under unpredictable target acceleration, $w \neq 0$

Section 4.1 has shown that the proposed guidance law (9) can stabilize the states (r, θ, φ, V_r, V_θ, V_φ) in the nonlinear missile - target dynamics (4) with non-target acceleration (w=0).

This section proves in detail that the proposed guidance law (14) can also resist the variations of the target's maneuvers after adding the unpredictable target acceleration w to (4). Because the control law (14) satisfies the Lyapunov stability. Substituting the control law (14) into (4) yields the closed-loop system

$$\begin{bmatrix} \dot{r} \\ \dot{\theta} \\ \dot{\varphi} \\ \dot{v}_r \\ \dot{v}_\theta \\ \dot{v}_\varphi \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{r\cos(\varphi)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & -\alpha_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha_2 & 0 \\ & & & & 0 & -\alpha_3 \end{bmatrix} \begin{bmatrix} r \\ \theta \\ \varphi \\ v_r \\ v_\theta \\ v_\varphi \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\alpha_2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_r \\ w_\theta \\ w_\Phi \end{bmatrix} \quad (15)$$

The state equations including target acceleration w in (24) are

$$\dot{v}_r = \alpha_1 V_r - \alpha_2 + w_r \quad (16)$$
$$\dot{v}_\theta = -\alpha_3 V_\theta + w_\theta \quad (17)$$
$$\dot{v}_\varphi = -\alpha_5 V_\Phi + w_\Phi \quad (18)$$

where the target accelerations w_r, w_θ and w_φ are regarded as disturbances. The robust guidance design objective can be stated as: to find the missile acceleration u=$[u_r, u_\theta, u_\Phi]^T$, such that the L_2 gains of the system states described by(16) - (18) are lower or equal to γ, the prescribed norm bound. That is

$$\frac{\int_0^\infty V_r^T V_r dt}{\int_0^\infty w_r^T w_r dt} = \frac{\|V_r\|_2^2}{\|w_r\|_2^2} \leq \gamma_1^2 \rightarrow \frac{\|V_r\|_2}{\|w_r\|_2} \leq \gamma_1 \quad (19)$$

$$\frac{\int_0^\infty V_\theta^T V_\theta dt}{\int_0^\infty w_\theta^T w_\theta dt} = \frac{\|V_\theta\|_2^2}{\|w_\theta\|_2^2} \leq \gamma_2^2 \rightarrow \frac{\|V_\theta\|_2}{\|w_\theta\|_2} \leq \gamma_2 \quad (20)$$

$$\frac{\int_0^\infty V_\Phi^T V_\Phi dt}{\int_0^\infty w_\Phi^T w_\Phi dt} = \frac{\|V_\Phi\|_2^2}{\|w_\Phi\|_2^2} \leq \gamma_3^2 \rightarrow \frac{\|V_\Phi\|_2}{\|w_\Phi\|_2} \leq \gamma_3 \quad (21)$$

where γ_1, γ_2, and γ_3, are prescribed values that denote the effects of unpredictable acceleration components w_r, w_θ and w_Φ on V_r, V_θ and V_Φ respectively. The following results provide sufficient conditions to satisfy robustness performance bound (19) - (21).

Theorem: Given (25), $\|V_r\|_2 / \|W_r\|_2 > \gamma_1$ holds if the following inequalities

$$(\alpha_1 + \frac{1}{2} + \frac{1}{2\gamma_1^2})V_r \leq \alpha_2 \text{ for } V_r \geq 0 \quad (22a)$$

$$(-\alpha_1 + \frac{1}{2} + \frac{1}{2\gamma_1^2})V_r \geq \alpha_2 \text{ for } V_r \leq 0 \quad (22b)$$

are satisfied.

Sliding Mode Control (SMC)

The Sliding mode control approach is recognized as one of the efficient tool to design robust controllers for complex high-order nonlinear dynamic plant operating under uncertainty conditions.

The phenomenon "Sliding mode" may appear in dynamic systems governed by ordinary differential equations with discontinuous state functions. The conventional example of sliding mode-a second order relay system can be found in any text book on nonlinear control. The control input in the second order system.

$$\ddot{x} + a_2\dot{x} + a_1 x = u$$

$$u = -Msign(s), s = cx + \dot{x}, a_1, a_2, M, c : cst$$

may take only two values, M and -M, and undergoes discontinuities on the straight line $s = 0$ in the state plane $(x, \dot{x})$. It follows from the analysis of the state plane that, in the neighborhood segment mn on the switching line sliding mode along this line. The equation of this line

$$\dot{x} + cx = 0$$

may be interpreted as the sliding mode equation.

Sliding mode becomes the principle operation mode in so-called variable structures systems. A variable structure system consists of a set of continuous subsystems with a proper switching logic and, as a result, control actions are discontinuous functions of the system state, disturbances (if they are accessible for measurement), and reference inputs. The previous example of the relay system with state dependent amplitude of the control variable may serve as an illustration of a variable structure system:

$$u = -k|x|\,sign(s), k \text{ is constant}$$

Now the system with $\alpha_1=0$ and $\alpha_2<0$ consists of two unstable linear structures ($u=kx$ and $u=-kx$) and $u=-kx$ with $x=0$ and $s=0$ as switching lines. As it is clear from the system state plane, the state reaches the switching line $s=0$ for any initial conditions. Then, the sliding mode occurs on this line (Figure 2) with the motion equation $\dot{x} + cx = 0$, while the state vector decays exponentially.

Similarly, to the relay system, after the start of the sliding mode, the motion is governed by a reduced order equation which does not depend on the plant parameters.

Now we demonstrate sliding modes in non-linear affine systems of general from

$$\dot{x} = f(x,t) + B(x,t)u \qquad (23)$$

$$u_i = \begin{cases} u_i^+(x,t) \, if \, s_i(x) > 0 \\ u_i^-(x,t) \, if \, s_i(x) < 0 \end{cases} \quad i = 1,\dots,m \qquad (24)$$

Where $x \in R^m$ is a state vector, $u \in R^m$ is a control vector, $u_i^+(x,t), u_i^-(x,t)$ and $S_i(x)$ are continuous functions of their arguments, $u_i^+(x,t) \neq u_i^-(x,t)$. The control is discontinuities in some surface in the system state space.

Sliding mode control problem

Consider the problem of doing set point control for a system of the form

$$x^{(n)} = f(x) + b(x)u$$

Further, we seek a solution that is robust to uncertainties in $f(x)$ and $b(x)$. Note that the system can be rewritten in vector form. It is then equivalent to
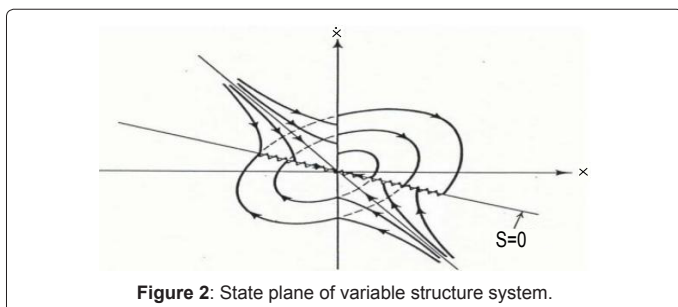

Figure 2: State plane of variable structure system.

$$\begin{pmatrix} \dot{x}_1 \\ \vdots \\ \vdots \\ \dot{x}_{(n-1)} \\ \dot{x}_{(n)} \end{pmatrix} = \begin{pmatrix} x_2 \\ \cdot \\ \vdots \\ \vdots \\ x_n \\ f(x)+b(x)u \end{pmatrix} \text{ with } x = \begin{pmatrix} \dot{x}_1 \\ \vdots \\ \cdot \\ \dot{x}_{(n-1)} \\ \dot{x}_{(n)} \end{pmatrix}$$

The system is a diagonal nonlinear system.

Define x_d (the set-point) and $\tilde{x}$ (the error signal) the difference between x and x_d.

We take a 2 step approach to design the controller:

1. Define the sliding mode: This is a surface that is invariant of the controlled dynamics, where the controlled dynamics are exponentially stable, and where the system tracks the desired set-point.

2. Define the control that drives the state to the sliding mode in finite time.

Define the sliding mode $s(t)$ as follows:

$$s(x) = \{x \,|\, s(x,t) = 0 \qquad (25)$$

where $s(x,t)$ is defined by

$$s(x,t) = \left(\frac{d}{dt} + \lambda\right)^{n-1} \tilde{x}(t), \lambda > 0 \qquad (26)$$

Note that on the surface $s(t)$ the error dynamics are governed by the equation

$$\left(\frac{d}{dt} + \lambda\right)^{n-1} \tilde{x}(t) = 0 \qquad (27)$$

On this surface the error will converge to 0 exponentially.

This implies that if there exists a control input $u(t)$ such that $x(t)$ is in $s(t)$ it follows that $x(t)$ is in $s(t)$ for all $T>t$ and the error will converge exponentially to 0 for this control input.

Remarks

The choice of $s(x,t)$ is somewhat arbitrary. You may choose any error dynamic which leads to exponentially stable behavior

$$s(x,t) = p\left(\frac{d}{dt}\right)\tilde{x} \qquad (28)$$

where $p(s)$ is a polynomial with all zeros in the left half plane, then the error dynamics will converge exponentially.

The strategy to converge to the sliding mode is that we add something to $u(t)$ which will drive us to the sliding mode in finite time.

More formally, we assume that for all $x, b(x) \neq 0$. To prove that the sliding mode exists, we derive a control such that $\dot{s} = 0$. With $s(x,t)$ defined as above:

$$s = \tilde{x}^{(n-1)} + \dots \lambda^{n-1}\tilde{x} \quad (29)$$

Differentiating:

$$\dot{s} = \tilde{x}^{(n)} + \dots + \lambda^{n-1}\dot{\tilde{x}}$$

$$= \tilde{x}^{(n)} - x_d^{(n)} + \dots + \lambda^{n-1}\dot{\tilde{x}}$$

$$= f(x) + b(x)u - x_d^{(n)} + \dots + \lambda^{n-1}\dot{\tilde{x}} \qquad (30)$$

Defining u as:

$$u^* = \frac{1}{b(x)}(-f(x) + x_d^{(n)} + \dots + \lambda^{n-1}\dot{\tilde{x}})$$

Then for $u^* = u$ it is clear that for all time $\dot{s} = 0$. In order to complete

the second control design step, we consider the following control:

$$u = u * \frac{k}{b(x)} \text{sgn}(x) \quad (31)$$

where:

$$\text{sgn}(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases}$$

To prove convergence to the sliding mode, we show that with this control, s will converge to 0 in finite time. Consider the development of $\frac{1}{2}s^2$ over time. Differentiating gives:

$$\frac{\partial}{\partial t} \frac{1}{2} S^2 = \dot{s}s$$

$$= (-k \text{sgn}(s)).s$$

$$= -k(\text{sgn}(s).s)$$

$$= -k|s| \quad (32)$$

Thus $|\dot{s}| = -k$, so we get convergence to the sliding mode in finite time.

In summary, the motion consists of a reaching phase during in which trajectories starting off the manifold $s=0$ move toward it and reach it in finite time, followed by a sliding phase during which the motion is confined to the manifold $s=0$ and the dynamics of the system are represented by a reduced order model with exponentially stable error dynamics. The manifold $s=0$ is called the sliding mode, and the control law

$$u = -\beta(x)\text{sgn}(s)$$ is called sliding control mode.

Sliding mode for u_r, u_θ and u_Φ (where are the missile's acceleration components) shown in Figure 3.

Simulation example

In this section, a numerical simulation justifies the availability and superiority of the proposed method in some scenario.

One maneuvering strategy of targets in 3-D is employed to examine the robustness and tracking performance of the guidance law; that is, the external disturbance $w(t)$ in (4) is generated by the following maneuvering step target to test the robustness in this example.

$$w_r = \lambda_T \vec{e}_t$$

$$w_\theta = \lambda_T \frac{-\dot{\phi}}{\sqrt{\dot{\phi}^2 + \dot{\theta} \cos^2(\phi)}} \vec{e}_\theta$$
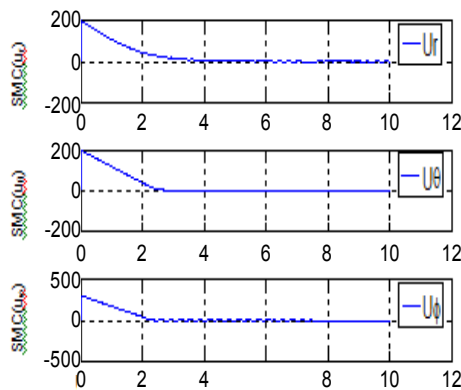
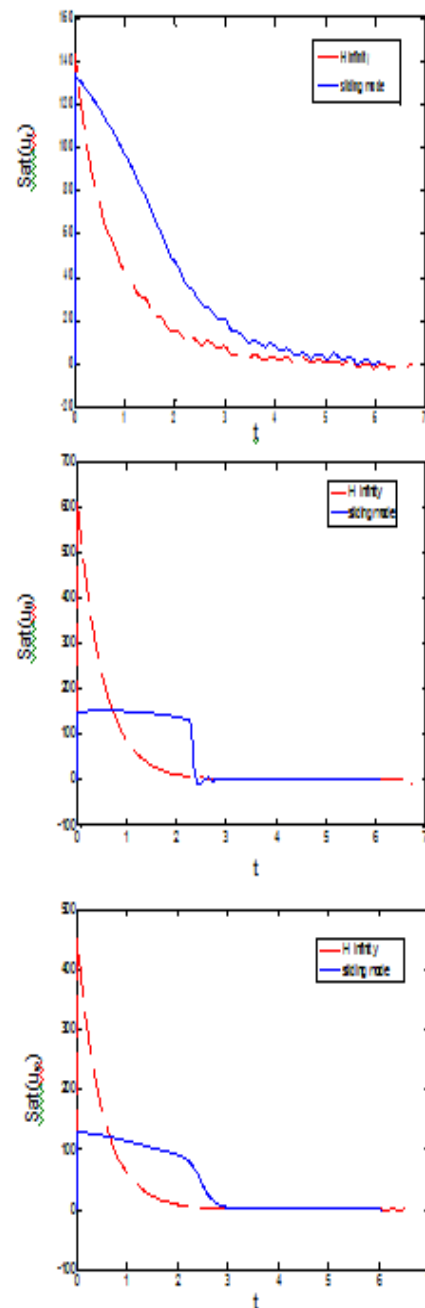**Figure 3**: Illustration of sliding mode for u_r, u_θ and u_Φ

$$w_\theta = \lambda_T \frac{\dot{\theta} \cos(\phi)}{\sqrt{\dot{\phi}^2 + \dot{\theta} \cos^2(\phi)}} \vec{e}_\phi$$

Where λ_T is the target's navigation gain with random value within 0 - 4 G and $\Omega = 20$ rad/s. This scenario is proposed to demonstrate the performance robustness of the proposed method.

Let Target escape from missile ($w_r > 0$)

$r = 4$Km
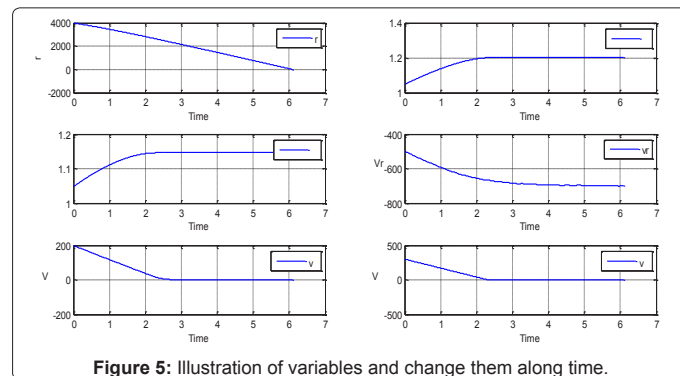
$$\theta = \frac{\pi}{3} \quad \phi = \frac{\pi}{3}$$

Figure 4: Trajectories of relative missile-target distance r and missile acceleration with Coefficients H$_\infty$: $\alpha_1 = 1$, $\alpha_2 = 610$ and $\alpha_3 = \alpha_4 = 2$.

Time & distance	Controller	
	H_∞	Sliding mode
Interception Time (s)	6.74	6.09
Mass Distance r (meter)	0.46	0.65

Table 1: Characteristics controller proposed.

Figure 5: Illustration of variables and change them along time.

$V_r = -500\text{m/s}, V_\theta = 200\text{m/s}, V_\phi = 300\text{m/s}$

Table 1 shows time and distance both as controllers. Also, it represents the characteristics of each method. In Figure 4 two guidance laws are compared to each other. From the view point of short interceptive time, the proposed guidance is more successful than that of Chen et al. [10] and Shieh [11]. It should be emphasized that the target models are only for demonstrating the feasibility of the proposed method. Finally, Figure 5 shows variables and their changes along time.

Conclusion

In this paper, the three-dimensional kinematics equations which describe the moving relationships of the missile and target are established**.**

A completely new robust guidance law with a saturation constraint against target escape is proposed. By regarding the target maneuvers as unpredictable disturbance inputs, the missile-target nonlinear 3-D guidance problems considered as a nonlinear disturbance attenuation control problem. Without solving the complicated Hamilton-Jacobi Partial Differential Inequality (HJPDI), a guidance law is proposed to achieve the design goal with less interruption time.

Finally, some sufficient condition is addressed supporting our guidance law having H_∞ and sliding mode robust performance. An illustrative example has been proposed to demonstrate that the interception time was less than that proposed by other guidance laws.

References

1. Locke AS (1955) Guidance. Princeton, New Jersey, USA.

2. Lin CF (1991) Modern Navigation, Guidance, and Control Processing. Prentice-Hall, USA.

3. Ha IJ, Chong S (1992) Design of CLOS guidance law via feedback linearization. IEEE T Aero Elec Sys 28: 51-63.

4. Zarchan P (1990) Tactical and Strategic Missile Guidance. AIAA, Washington DC, USA.

5. Bryson AE Jr., Ho YC (1969) Applied Optimal Control. Waltham, MA, USA.

6. Potter JE (1964) A guidance-navigation separation theorem. AIAA, Washington DC, USA.

7. Speyer JL, Greenwell WM, Hull DG (1982) Adaptive noise estimation and guidance for homing missiles. J Guid Control Dynam.

8. Nesline FW, Zarchan P (1981) A new look at classical vs. modern homing missile guidance. J Guid Control Dynam 4: 78-85.

9. Vadali SR, Oh HS (1992) Space station attitude control and momentum management: A nonlinear look. J Guid Control Dynam 15: 577-586.

10. Ferreres G, M'Saad M (1996) Parametric robustness evaluation of an H missile autopilot. J Guid Control Dynam 19: 621-627.

11. Shieh C-S (2007) Design of three-dimensional missile guidance law via tunable nonlinear H_∞ control with saturation constraint. IET Control Theory Appl 1: 756-763.